NUMERICAL DISCRETIZATION OF A BRINKMAN–DARCY–FORCHHEIMER MODEL UNDER SINGULAR FORCING*

ALEJANDRO ALLENDES†, GILBERTO CAMPAÑA‡, AND ENRIQUE OTÁROLA§

Abstract. In Lipschitz two-dimensional domains, we study a Brinkman–Darcy–Forchheimer problem on the weighted spaces $\mathbf{H}_0^1(\omega,\Omega) \times L^2(\omega,\Omega)/\mathbb{R}$, where ω belongs to the Muckenhoupt class A_2. Under a suitable smallness assumption, we establish the existence and uniqueness of a solution. We propose a finite element scheme and obtain a quasi-best approximation result in energy norm *à la Céa* under the assumption that Ω is convex. We also devise an a posteriori error estimator and investigate its reliability and efficiency properties. Finally, we design a simple adaptive strategy that yields optimal experimental rates of convergence for the numerical examples that we perform.

Key words. nonlinear equations, a Brinkman–Darcy–Forchheimer problem, Dirac measures, Muckenhoupt weights, finite element approximations, a posteriori error estimates, adaptive loop.

AMS subject classifications. 35Q30, 35Q35, 35R06, 65N12, 65N15, 65N50, 76S05.

1. Introduction.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. The purpose of this work is to study existence and approximation results for a Brinkman–Darcy–Forchheimer problem under *singular forcing*. To be specific, we will study the following system of partial differential equations (PDEs):

$$-\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + |\mathbf{u}|\mathbf{u} + \mathbf{u} + \nabla\mathsf{p} = \mathbf{f} \text{ in } \Omega, \quad \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega. \tag{1}$$

Here, $\mathbf{u}$ and p represent the velocity and the pressure of the fluid, respectively, $\mathbf{f}$ is an externally applied force, and $|\cdot|$ denotes the Euclidean norm. Our main source of novelty and originality here is that we allow $\mathbf{f}$ to be singular, say a Dirac measure, so that the problem cannot be understood with the usual energy setting.

Darcy's law, $\mathbf{u} = -K\nabla p/\mu$, is a *linear* relationship that describes the creeping flow of Newtonian fluids in porous media and is backed by years of experimental data [30]. Since this linear relationship finds several applications in engineering, it is thus no surprise that its analysis and approximation have been investigated by several authors. However, Darcy's law may be inaccurate for modeling fluid flow through porous media with high Reynolds numbers or through media with high porosity [13, 37, 54]. As an alternative model, in such a scenario, it is possible to consider the convective Brinkman–Darcy–Forchheimer equations [32, 48, 52]. These equations incorporate the well-known *convective* term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ [29] and the *nonlinear* modification of Darcy's law $|\mathbf{u}|\mathbf{u}$ suggested by Forchheimer in [26].

While it is fair to say that the study of approximation techniques for (1) and related models in a standard setting is matured and well understood [22, 23, 29, 30, 49], recent applications and models have emerged where the motion of a fluid is described by (1) or variation of it, but due to the singularity of forces $\mathbf{f}$, the problem must be

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†Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile. (alejandro.allendes@usm.cl, http://aallendes.mat.utfsm.cl/).

‡Departamento de Ciencias, Universidad Técnica Federico Santa María, Valparaíso, Chile. (gilberto.campana@usm.cl.

§Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile. (enrique.otarola@usm.cl, http://eotarola.mat.utfsm.cl/).

understood in a completely different setting and rigorous approximation techniques are scarce. For instance, [36] models the motion of active thin structures by using a linear model related to (1), where the right-hand side is a linear combination of Dirac measures. A second example comes from optimal control theory. References [12, 27] set up a problem where the state is governed by the stationary Navier–Stokes equations, but with a forcing (control) that is measure valued. The motivation behind this is what the authors denote as sparsity of the control. We finally mention [11], where the authors study a class of asymptotically Newtonian fluids (Newtonian under large shear rates) under singular forcing. The authors show existence and uniqueness as well as some regularity results; see [42] for extensions to convex polyhedral domains.

In this work, we continue our program aimed at developing numerical methods for models of fluids under singular forces. The guiding principle that we follow is that by introducing a weight, and working in the ensuing weighted function spaces, we can allow for data that is singular, so that (1) fits our theory. In [40] we developed existence and uniqueness for the Stokes problem over a reduced class of weighted spaces. The numerical analysis of such a model is presented in [5, 21], where a priori and a posteriori, respectively, error analyses are discussed. The Navier–Stokes equations are considered in [41], where existence and uniqueness for small data is proved. In the setting of uniqueness, an a priori error analysis for a numerical scheme is also developed. A posteriori error estimates for suitable discretizations of such a nonlinear model can be found in [6]. This brings us to the current work and its contributions. Before presenting the main contributions of our work, we would like to mention references [13, 16, 37, 44], where different solution techniques for problem (1) with $\mathbf{f}$ smooth are discussed. To the best of our knowledge, this is the first work that addresses the numerical approximation of (1) when $\mathbf{f}$ is singular.

Let us elaborate on the main contributions of our work:

• *Existence and uniqueness of a solution:* We introduce a concept of weak solution within $\mathbf{H}_0^1(\omega,\Omega) \times L^2(\omega,\Omega)/\mathbb{R}$ and show, on the basis of a fixed-point argument, that the proposed weak problem admits a unique solution under a suitable smallness assumption on $\mathbf{f}$. To accomplish this task, we first establish the well-posedness of a Brinkman problem on the space $\mathbf{H}_0^1(\omega,\Omega) \times L^2(\omega,\Omega)/\mathbb{R}$ by utilizing the continuity method and the well-posedness of the Stokes problem derived in [40, Theorem 17].

• *Finite element discretization:* We propose a finite element discretization scheme for problem (1) on the basis of the following two classical inf-sup stable pairs: the mini element and the Taylor–Hood element. We show that the proposed finite element scheme admits a unique solution and we obtain a quasi-best approximation result in energy norm *à la Céa*. We must immediately observe that, since $\mathbf{f}$ is very singular, it is not expected for the pair $(\mathbf{u}, \mathsf{p})$ to have any regularity properties beyond those merely needed for the problem to be well-posed. Consequently, rates of convergence in energy norm cannot be obtained from the derived quasi-best approximation result.

• *A posteriori error analysis:* Solutions to (1), because of the singular datum $\mathbf{f}$, are not expected to be smooth, and thus adaptive methods must be developed to efficiently approximate them. We devise a residual-based a posteriori error estimator for the proposed finite element discretization scheme. We prove that the devised error estimator is globally reliable; see Theorem 14. In Theorem 16, we investigate efficiency properties for the proposed local indicators. In addition, we devise an adaptive finite element method based on the proposed error estimator and provide numerical experiments in convex and non-convex domains.

The manuscript is organized as follows. In Section 2, we set notation, recall basic facts about weights, and introduce the weighted spaces we shall work with. In

Section 3, we analyze, as an instrumental step, a Brinkman problem on weighted spaces. In Section 4, we introduce a weak formulation for problem (1) and establish a well-posedness result. A numerical discretization scheme is presented in Section 5. Here, we also obtain a quasi-best approximation result *à la Céa*. Section 6 is one of the highlights of our work. There we propose an a posteriori error estimator for suitable inf-sup stable finite element pairs and introduce a Ritz projection on weighted spaces. We prove that the energy norm of the error can be bounded in terms of the energy norm of the Ritz projection and obtain the global reliability of the proposed estimator. We also explore local efficiency estimates. We conclude the manuscript by presenting, in Section 7, a series of numerical experiments that illustrate the theory.

2. Notation and preliminaries. Let us set notation and describe the setting we shall operate with.

2.1. Notation. We shall use standard notation for Lebesgue and Sobolev spaces. Spaces of vector-valued functions and their elements will be indicated with boldface.

If $\mathcal{W}$ and $\mathcal{Z}$ are Banach function spaces, we write $\mathcal{W} \hookrightarrow \mathcal{Z}$ to denote that $\mathcal{W}$ is continuously embedded in $\mathcal{Z}$. We denote by $\mathcal{W}'$ and $\|\cdot\|_{\mathcal{W}}$ the dual and the norm of $\mathcal{W}$, respectively. Given $\mathfrak{p} \in (1,\infty)$, we denote by $\mathfrak{p}'$ its Hölder conjugate, i.e., the real number such that $1/\mathfrak{p} + 1/\mathfrak{p}' = 1$. The relation $a \lesssim b$ indicates that $a \leq Cb$, with a positive constant C that does not depend on either a, b, or the discretization parameters. The value of C might change at each occurrence.

2.2. Weighted function spaces. A notion that will be fundamental and that will allow us to deal with singular forcing and nonstandard rheologies is that of a weight. A weight is a locally integrable, nonnegative function defined on $\mathbb{R}^2$. Given a weight ω and a measurable set $E \subset \mathbb{R}^2$, we let $\omega(E) = \int_E \omega \mathrm{d}\mathbf{x}$. Given a measurable set $E \subset \mathbb{R}^2$ of positive Lebesgue measure, we define $\fint_E \omega(\mathbf{x})\mathrm{d}\mathbf{x} = (1/|E|)\int_E \omega \mathrm{d}\mathbf{x}$.

Of particular interest to us will be the so-called Muckenhoupt A_p weights: Let $p \in [1,\infty)$. We say that a weight ω belongs to the Muckenhoupt class A_p if [17, 18, 38, 51]

$$
\begin{aligned}
[\omega]_{A_p} &:= \sup_B \left(\fint_B \omega\right)\left(\fint_B \omega^{\frac{1}{1-p}}\right)^{p-1} < \infty, \qquad p \in (1,\infty),\\
[\omega]_{A_1} &:= \sup_B \left(\fint_B \omega\right) \sup_{\mathbf{x}\in B} \frac{1}{\omega(\mathbf{x})} < \infty, \qquad p = 1,
\end{aligned} \tag{2}
$$

where the supremum is taken over all balls $B \in \mathbb{R}^2$. In addition, $A_\infty := \cup_{p<\infty} A_p$. We call $[\omega]_{A_p}$, for $p \in [1,\infty)$, the Muckenhoupt characteristic of ω. We observe that, for $p \in (1,\infty)$, there is a certain symmetry in the A_p classes with respect to Hölder conjugate exponents: $\omega \in A_p$ if and only if $\omega' = \omega^{1/(1-p)} \in A_{p'}$ [51, Remark 1.2.4]. Finally, we note that if $1 \leq p < q < \infty$, then $A_p \subset A_q$ [51, Remark 1.2.4].

The prototypical A_p weights are the power weights: Let $\mathbf{z} \in \Omega$ be an interior point in Ω and $\alpha \in \mathbb{R}$. For $p > 1$, the weight

$$
\mathsf{d}_{\mathbf{z}}^{\alpha}(\mathbf{x}) := |\mathbf{x} - \mathbf{z}|^{\alpha} \tag{3}
$$

belongs to A_p if and only if $\alpha \in (-2, 2(p-1))$ [50, Chapter IX, Corollary 4.4]. We notice that there is a neighborhood of $\partial\Omega$ where $\mathsf{d}_{\mathbf{z}}^{\alpha}$ is strictly positive and continuous. This observation motivates the consideration of a restricted class of Muckenhoupt weights [25, Definition 2.5].

DEFINITION 1 (class $A_p(D)$). *Let $D \subset \mathbb{R}^2$ be a Lipschitz domain and $p \in [1,\infty)$. We say that $\varpi \in A_p$ belongs to $A_p(D)$ if there is an open set $\mathcal{G} \subset \Omega$ and $\epsilon, \varpi_l > 0$ such that: $\{\mathbf{x} \in D : \mathrm{dist}(\mathbf{x}, \partial D) < \epsilon\} \subset \mathcal{G}$, $\varpi \in C(\overline{\mathcal{G}})$, and $\varpi_l \leq \varpi(\mathbf{x})$ for all $\mathbf{x} \in \overline{\mathcal{G}}$.*

Let $p \in [1,\infty)$, $\omega \in A_p$, and $E \subset \mathbb{R}^2$ be an open set. We define $L^p(\omega, E)$ as the space of Lebesgue p-integrable functions with respect to the measure $\omega(\mathbf{x})\mathsf{d}\mathbf{x}$. We define $W^{1,p}(\omega, E)$ as the set of functions $v \in L^p(\omega, E)$ with weak derivatives $D^\alpha v \in \mathbf{L}^p(\omega, E)$ for $|\alpha| \leq 1$. The norm of v in $W^{1,p}(\omega, E)$ is given by [51, Section 2.1]

$$\|v\|_{W^{1,p}(\omega,E)} := \left(\|v\|^p_{L^p(\omega,E)} + \|\nabla v\|^p_{\mathbf{L}^p(\omega,E)}\right)^{\frac{1}{p}}. \tag{4}$$

We also define $W^{1,p}_0(\omega, E)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\omega, E)$. When $p = 2$, we set $H^1(\omega, E) := W^{1,p}(\omega, E)$ and $H^1_0(\omega, E) := W^{1,p}_0(\omega, E)$.

If $1 \leq p < \infty$ and ω belongs to A_p, then $L^p(\omega, E)$, $W^{1,p}(\omega, E)$, and $W^{1,p}_0(\omega, E)$ are Banach spaces [51, Proposition 2.1.2] and smooth functions are dense [51, Corollary 2.1.6]; see also [31, Theorem 1]. In addition, [24, Theorem 1.3] guarantees a weighted Poincaré inequality which, in turn, implies that over $W^{1,p}_0(\omega, E)$ the seminorm $\|\nabla v\|_{L^p(\omega,E)}$ is an equivalent norm to the one defined in (4).

3. A Brinkman problem under singular forcing. In this section, we study the well-posedness of the following Brinkman problem: Find $(\mathbf{u}, \mathsf{p})$ such that

$$-\Delta \mathbf{u} + \mathbf{u} + \nabla \mathsf{p} = \mathbf{f} \text{ in } \Omega, \quad \text{div } \mathbf{u} = g \text{ in } \Omega, \quad \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega, \tag{5}$$

where we allow the data $\mathbf{f}$ and g to be singular. The analysis of problem (5) is a key step to establish the well-posedness of the Brinkman–Darcy–Forchheimer model (1).

We begin our studies by proposing a weak formulation for (5). Given $\omega \in A_2$, $\mathbf{f} \in \mathbf{H}^{-1}(\omega, \Omega)$, and $g \in L^2(\omega, \Omega)/\mathbb{R}$, find $(\mathbf{u}, \mathsf{p}) \in \mathbf{H}^1_0(\omega, \Omega) \times L^2(\omega, \Omega)/\mathbb{R}$ such that

$$\int_\Omega (\nabla \mathbf{u} : \nabla \mathbf{v} + \mathbf{u} \cdot \mathbf{v} - \mathsf{p} \text{ div } \mathbf{v} + \mathsf{q} \text{ div } \mathbf{u}) = \langle \mathbf{f}, \mathbf{v} \rangle + \int_\Omega g\mathsf{q}, \tag{6}$$

for all $\mathbf{v} \in \mathbf{H}^1_0(\omega^{-1}, \Omega)$ and $\mathsf{q} \in L^2(\omega^{-1}, \Omega)/\mathbb{R}$. Here, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathbf{H}^{-1}(\omega, \Omega) := \mathbf{H}^1_0(\omega^{-1}, \Omega)'$ and $\mathbf{H}^1_0(\omega^{-1}, \Omega)$. Notice that, owing to the boundary conditions on $\mathbf{u}$, we must necessarily have the compatibility condition $\int_\Omega g = 0$.

To simplify the presentation of the subsequent material, we define the spaces

$$\mathcal{X} := \mathbf{H}^1_0(\omega, \Omega) \times L^2(\omega, \Omega)/\mathbb{R}, \qquad \mathcal{Y} := \mathbf{H}^1_0(\omega^{-1}, \Omega) \times L^2(\omega^{-1}, \Omega)/\mathbb{R}. \tag{7}$$

The well-posedness of the Brinkman problem is established in the following result.

THEOREM 2 (well-posedness of the Brinkman problem). *Let $d \in \{2, 3\}$ and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $\omega \in A_2(\Omega)$. If $\mathbf{f} \in \mathbf{H}^{-1}(\omega, \Omega)$ and $g \in L^2(\omega, \Omega)/\mathbb{R}$, then there exists a unique solution $(\mathbf{u}, \mathsf{p}) \in \mathbf{H}^1_0(\omega, \Omega) \times L^2(\omega, \Omega)/\mathbb{R}$ of problem (6), which satisfies the estimate*

$$\|\nabla \mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)} + \|\mathsf{p}\|_{L^2(\omega,\Omega)} \leq C_{\mathcal{B}} \left(\|\mathbf{f}\|_{\mathbf{H}^{-1}(\omega,\Omega)} + \|g\|_{L^2(\omega,\Omega)}\right), \qquad C_{\mathcal{B}} > 0. \tag{8}$$

Proof. Inspired by the proof of [28, Theorem 6.8], we proceed on the basis of the *method of continuity* presented in [28, Theorem 5.2]. We split the proof in four steps.

Step 1. A bounded linear map $\mathcal{S}$ associated to a Stokes problem. We define the Stokes operator

$$\mathcal{S} : \mathcal{X} \to \mathcal{Y}', \quad \langle \mathcal{S}(\mathbf{u}, \mathsf{p}), (\mathbf{v}, \mathsf{q}) \rangle := \int_\Omega (\nabla \mathbf{u} : \nabla \mathbf{v} - \mathsf{p} \text{ div } \mathbf{v} + \mathsf{q} \text{ div } \mathbf{u}). \tag{9}$$

We notice that $\mathcal{S}$ is a bounded linear operator. In fact, we have the bound

$$\|\mathcal{S}(\mathbf{u},\mathsf{p})\|_{\mathcal{Y}'} = \sup_{(\mathbf{0},0)\neq(\mathbf{v},\mathsf{q})\in\mathcal{Y}} \frac{\langle \mathcal{S}(\mathbf{u},\mathsf{p}),(\mathbf{v},\mathsf{q})\rangle}{\|(\mathbf{v},\mathsf{q})\|_{\mathcal{Y}}} \lesssim \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)} + \|\mathsf{p}\|_{L^2(\omega,\Omega)}.$$

We now introduce the following weak formulation associated with the Stokes operator $\mathcal{S}$. Given $\mathbf{g}\in\mathbf{H}^{-1}(\omega,\Omega)$ and $h\in L^2(\omega,\Omega)/\mathbb{R}$, find $(\boldsymbol{\varphi},\psi)\in\mathcal{X}$ such that $\langle\mathcal{S}(\boldsymbol{\varphi},\psi),(\mathbf{v},\mathsf{q})\rangle = \langle \mathbf{g},\mathbf{v}\rangle + (h,\mathsf{q})_{L^2(\Omega)}$ for all $(\mathbf{v},\mathsf{q})\in\mathcal{Y}$. The well-posedness of this Stokes system follows from [40, Theorem 17].

Step 2. A bounded linear map $\mathcal{B}$ associated to a Brinkman problem. We define

$$\mathcal{B}:\mathcal{X}\to\mathcal{Y}', \quad \langle\mathcal{B}(\mathbf{u},\mathsf{p}),(\mathbf{v},\mathsf{q})\rangle := \int_\Omega (\nabla\mathbf{u}:\nabla\mathbf{v}+\mathbf{u}\cdot\mathbf{v}-\mathsf{p}\ \text{div}\ \mathbf{v}+\mathsf{q}\ \text{div}\ \mathbf{u}). \tag{10}$$

The map $\mathcal{B}$ is linear and bounded. In particular, we have $\|\mathcal{B}(\mathbf{u},\mathsf{p})\|_{\mathcal{Y}'}\lesssim\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)}+\|\mathsf{p}\|_{L^2(\omega,\Omega)}$. With $\mathcal{B}$ at hand, problem (6) can be equivalently written as follows: Find $(\mathbf{u},\mathsf{p})\in\mathcal{X}$ such that $\langle\mathcal{B}(\mathbf{u},\mathsf{p}),(\mathbf{v},\mathsf{q})\rangle = \langle\mathbf{f},\mathbf{v}\rangle+(g,\mathsf{q})_{L^2(\Omega)}$ for all $(\mathbf{v},\mathsf{q})\in\mathcal{Y}$.

Step 3. The a priori estimate (8). Let us introduce, for $t\in[0,1]$, the operator

$$\mathcal{L}_t:\mathcal{X}\to\mathcal{Y}', \qquad \mathcal{L}_t := (1-t)\mathcal{S}+t\mathcal{B}. \tag{11}$$

Observe that $\mathcal{L}_0=\mathcal{S}$, $\mathcal{L}_1=\mathcal{B}$, and that $\mathcal{L}_t$ is a linear and bounded operator from $\mathcal{X}$ into $\mathcal{Y}'$. Let us consider the following family of equations: Find $(\mathbf{u},\mathsf{p})\in\mathcal{X}$ such that $\langle\mathcal{L}_t(\mathbf{u},\mathsf{p}),(\mathbf{v},\mathsf{q})\rangle = \langle\mathbf{f},\mathbf{v}\rangle+(g,\mathsf{q})_{L^2(\Omega)}$ for all $(\mathbf{v},\mathsf{q})\in\mathcal{Y}$, where $t\in[0,1]$. For $t\in[0,1]$, the solvability of this problem is then equivalent to the invertibility of the map $\mathcal{L}_t$. Let $(\mathbf{u}_t,\mathsf{p}_t)\in\mathcal{X}$ be a solution to such a problem. In what follows, we prove

$$\|\nabla\mathbf{u}_t\|_{\mathbf{L}^2(\omega,\Omega)}+\|\mathsf{p}_t\|_{L^2(\omega,\Omega)}\lesssim\|\mathbf{f}\|_{\mathbf{H}^{-1}(\omega,\Omega)}+\|g\|_{L^2(\omega,\Omega)}, \tag{12}$$

which is equivalent to $\|(\mathbf{u}_t,\mathsf{p}_t)\|_{\mathcal{X}}\lesssim\|\mathcal{L}_t(\mathbf{u},\mathsf{p})\|_{\mathcal{Y}'}$. An important observation is that $(\mathbf{u}_t,\mathsf{p}_t)$ can be seen as a solution to the following Stokes problem: Find $(\mathbf{u}_t,\mathsf{p}_t)\in\mathcal{X}$ such that $\langle\mathcal{S}(\mathbf{u}_t,\mathsf{p}_t),(\mathbf{v},\mathsf{q})\rangle = \langle\mathbf{f},\mathbf{v}\rangle+(g,\mathsf{q})_{L^2(\Omega)}-t(\mathbf{u}_t,\mathbf{v})_{\mathbf{L}^2(\Omega)}$ for all $(\mathbf{v},\mathsf{q})\in\mathcal{Y}$. We can thus apply the estimate in [40, Theorem 17] to arrive at

$$\|\nabla\mathbf{u}_t\|_{\mathbf{L}^2(\omega,\Omega)}+\|\mathsf{p}_t\|_{L^2(\omega,\Omega)}\lesssim\|\mathbf{f}\|_{\mathbf{H}^{-1}(\omega,\Omega)}+\|g\|_{L^2(\omega,\Omega)}+\|\mathbf{u}_t\|_{\mathbf{L}^2(\omega,\Omega)}. \tag{13}$$

To obtain (12), we proceed by a contradiction argument. Assuming that (12) is false, it is possible to find sequences $\{(\mathbf{u}_k,\mathsf{p}_k)\}_{k\in\mathbb{N}}\subset\mathbf{H}_0^1(\omega,\Omega)\times L^2(\omega,\Omega)/\mathbb{R}$ and $\{(\mathbf{f}_k,g_k)\}_{k\in\mathbb{N}}\subset\mathbf{H}^{-1}(\omega,\Omega)\times L^2(\omega,\Omega)/\mathbb{R}$ such that $(\mathbf{u}_k,\mathsf{p}_k,\mathbf{f}_k,g_k)$ satisfies, for $k\in\mathbb{N}$, $\langle\mathcal{L}_t(\mathbf{u}_k,\mathsf{p}_k),(\mathbf{v},\mathsf{q})\rangle = \langle\mathbf{f}_k,\mathbf{v}\rangle+(g_k,\mathsf{q})_{L^2(\Omega)}$ for all $(\mathbf{v},\mathsf{q})\in\mathcal{Y}$ and $\|\nabla\mathbf{u}_k\|_{\mathbf{L}^2(\omega,\Omega)}+\|\mathsf{p}_k\|_{L^2(\omega,\Omega)}=1$, but $\|\mathbf{f}_k\|_{\mathbf{H}^{-1}(\omega,\Omega)}+\|g_k\|_{L^2(\omega,\Omega)}\to 0$ as $k\uparrow\infty$. Since $\{(\mathbf{u}_k,\mathsf{p}_k)\}_{k\in\mathbb{N}}$ is uniformly bounded in $\mathbf{H}_0^1(\omega,\Omega)\times L^2(\omega,\Omega)/\mathbb{R}$, we deduce the existence of a non-relabelared subsequence $\{(\mathbf{u}_k,\mathsf{p}_k)\}_{k\in\mathbb{N}}$ such that $\mathbf{u}_k\rightharpoonup\mathbf{u}$ in $\mathbf{H}_0^1(\omega,\Omega)$ and $\mathsf{p}_k\rightharpoonup\mathsf{p}$ in $L^2(\omega,\Omega)/\mathbb{R}$ as $k\uparrow\infty$. The limit $(\mathbf{u},\mathsf{p})$ satisfies $\mathcal{L}_t(\mathbf{u},\mathsf{p})=\mathbf{0}$ in $\mathcal{Y}'$. Consequently, $(\mathbf{u},\mathsf{p})=(\mathbf{0},0)$. On the other hand, the compact embedding $\mathbf{H}_0^1(\omega,\Omega)\hookrightarrow\mathbf{L}^2(\omega,\Omega)$ [33, Theorem 4.12], [41, Proposition 2] shows that $\mathbf{u}_k\to\mathbf{0}$ in $\mathbf{L}^2(\omega,\Omega)$. We can thus invoke the Gårding-like inequality (13) to deduce that

$$1=\|\nabla\mathbf{u}_k\|_{\mathbf{L}^2(\omega,\Omega)}+\|\mathsf{p}_k\|_{L^2(\omega,\Omega)}\lesssim\|\mathbf{f}_k\|_{\mathbf{H}^{-1}(\omega,\Omega)}+\|g_k\|_{L^2(\omega,\Omega)}+\|\mathbf{u}_k\|_{\mathbf{L}^2(\omega,\Omega)}\to 0,\ \ k\uparrow\infty,$$

which is a contradiction. We have thus obtained the desired estimate (12).

Step 4. The method of continuity and the well-posedness of (6). With the estimate (12) at hand, we invoke [28, Theorem 5.2] and the fact that $\mathcal{L}_0=\mathcal{S}$ maps $\mathcal{X}$ onto $\mathcal{Y}'$ to deduce that $\mathcal{L}_1=\mathcal{B}$ maps $\mathcal{X}$ onto $\mathcal{Y}'$ as well, i.e., problem (6) admits a solution. Since problem (6) is linear, estimate (12) guarantees the uniqueness of solutions. We have thus proved that problem (6) is well-posed. ∎

4. A Brinkman–Darcy–Forchheimer model. We now show the existence of solutions for problem (1). We begin by recalling that, if $\mathbf{v}$ is sufficiently smooth and solenoidal, then the convective term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ can be rewritten as $\mathrm{div}(\mathbf{v}\otimes\mathbf{v})$. This property will be used to propose a weak formulation for problem (1).

4.1. Weak formulation. For a given weight ω in the class A_2, we define the bilinear forms $a_0 : \mathbf{H}_0^1(\omega,\Omega)\times\mathbf{H}_0^1(\omega^{-1},\Omega)\to\mathbb{R}$, $a_1 : \mathbf{L}^2(\omega,\Omega)\times\mathbf{L}^2(\omega^{-1},\Omega)\to\mathbb{R}$, and $b_\pm : \mathbf{H}_0^1(\omega^{\pm1},\Omega)\times L^2(\omega^{\mp1},\Omega)/\mathbb{R}\to\mathbb{R}$ by

$$a_0(\mathbf{w},\mathbf{v}) := \int_\Omega \nabla\mathbf{w} : \nabla\mathbf{v}, \qquad a_1(\mathbf{w},\mathbf{v}) := \int_\Omega \mathbf{w}\cdot\mathbf{v}, \qquad b_\pm(\mathbf{v},\mathsf{q}) := -\int_\Omega \mathsf{q}\ \mathrm{div}\ \mathbf{v},$$

respectively. With a_0 and a_1 at hand, we define $a : \mathbf{H}_0^1(\omega,\Omega)\times\mathbf{H}_0^1(\omega^{-1},\Omega)\to\mathbb{R}$ by $a(\mathbf{w},\mathbf{v}) := a_0(\mathbf{w},\mathbf{v}) + a_1(\mathbf{w},\mathbf{v})$. We now introduce forms associated to the nonlinear terms $(\mathbf{u}\cdot\nabla)\mathbf{u}$ and $|\mathbf{u}|\mathbf{u}$ in (1). We define $c : [\mathbf{H}_0^1(\omega,\Omega)]^2\times\mathbf{H}_0^1(\omega^{-1},\Omega)\to\mathbb{R}$ and $d : [\mathbf{H}_0^1(\omega,\Omega)]^2\times\mathbf{H}_0^1(\omega^{-1},\Omega)\to\mathbb{R}$ by

$$c(\mathbf{u},\mathbf{w};\mathbf{v}) := -\int_\Omega \mathbf{u}\otimes\mathbf{w} : \nabla\mathbf{v}, \qquad d(\mathbf{u},\mathbf{w};\mathbf{v}) := \int_\Omega |\mathbf{u}|\mathbf{w}\cdot\mathbf{v},$$

respectively.

With these ingredients at hand, we consider the following weak formulation for problem (1): Find $(\mathbf{u},\mathsf{p})\in\mathcal{X}$ such that

$$a(\mathbf{u},\mathbf{v}) + b_-(\mathbf{v},\mathsf{p}) + c(\mathbf{u},\mathbf{u};\mathbf{v}) + d(\mathbf{u},\mathbf{u};\mathbf{v}) = \langle\mathbf{f},\mathbf{v}\rangle, \quad b_+(\mathbf{u},\mathsf{q}) = 0, \tag{14}$$

for all $(\mathbf{v},q)\in\mathcal{Y}$. Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathbf{H}^{-1}(\omega,\Omega)$ and $\mathbf{H}_0^1(\omega^{-1},\Omega)$. We recall that the spaces $\mathcal{X}$ and $\mathcal{Y}$ are defined in (7).

In what follows, we will make use of the following inf-sup condition on weighted spaces that immediately follows from the existence of a right inverse of the divergence; see [20, Theorem 3.1], [45, Theorem 1], [1, Theorem 2.8], and [21, Lemma 6.1]:

$$\|\mathsf{p}\|_{L^2(\omega,\Omega)} \lesssim \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{H}_0^1(\omega^{-1},\Omega)} \frac{b_-(\mathbf{v},\mathsf{p})}{\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\omega^{-1},\Omega)}} \quad \forall\mathsf{p}\in L^2(\omega,\Omega)/\mathbb{R}. \tag{15}$$

We will also make use of the following weighted inf-sup condition for a_0 [40]:

$$\begin{aligned}\inf_{\mathbf{0}\neq\mathbf{v}\in\mathbf{H}_0^1(\omega,\Omega)}\ \sup_{\mathbf{0}\neq\mathbf{w}\in\mathbf{H}_0^1(\omega^{-1},\Omega)} & \frac{a_0(\mathbf{v},\mathbf{w})}{\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\omega,\Omega)}\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\omega^{-1},\Omega)}} \\ &= \inf_{\mathbf{0}\neq\mathbf{w}\in\mathbf{H}_0^1(\omega^{-1},\Omega)}\ \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{H}_0^1(\omega,\Omega)} \frac{a_0(\mathbf{v},\mathbf{w})}{\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\omega,\Omega)}\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\omega^{-1},\Omega)}} > 0,\end{aligned} \tag{16}$$

which holds under the further restriction that $\omega\in A_2(\Omega)$.

The following result guarantees the boundedness of the convective and Forchheimer terms on weighted spaces.

Lemma 3 (boundedness of the convective and Forchheimer terms). *If $\omega\in A_2$, $\mathbf{u},\mathbf{w}\in\mathbf{H}_0^1(\omega,\Omega)$, and $\mathbf{v}\in\mathbf{H}_0^1(\omega^{-1},\Omega)$, then*

$$\begin{aligned}|c(\mathbf{u},\mathbf{w};\mathbf{v})| &\leq C_{4\to2}^2\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)}\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\omega,\Omega)}\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\omega^{-1},\Omega)}, \\ |d(\mathbf{u},\mathbf{w};\mathbf{v})| &\leq C_{4\to2}^2 C_{2\to2}\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)}\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\omega,\Omega)}\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\omega^{-1},\Omega)}.\end{aligned} \tag{17}$$

Here, $C_{4\to2}$ and $C_{2\to2}$ denote the best constants in the embeddings $\mathbf{H}_0^1(\omega,\Omega)\hookrightarrow\mathbf{L}^4(\omega,\Omega)$ and $\mathbf{H}_0^1(\omega^{-1},\Omega)\hookrightarrow\mathbf{L}^2(\omega^{-1},\Omega)$, respectively.

Proof. Since we are in two dimensions and ω and ω^{-1} belong to A_2, [24, Theorem 1.3] shows that there exists $\zeta > 0$ such that $\mathbf{H}_0^1(\omega^{\pm 1},\Omega) \hookrightarrow \mathbf{L}^{2k}(\omega^{\pm 1},\Omega)$ for every $k \in [1, 2+\zeta]$. Consequently,

$$\begin{aligned} |c(\mathbf{u},\mathbf{w};\mathbf{v})| &\leq \|\mathbf{u}\|_{\mathbf{L}^4(\omega,\Omega)}\|\mathbf{w}\|_{\mathbf{L}^4(\omega,\Omega)}\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\omega^{-1},\Omega)} \\ &\leq C_{4\to 2}^2\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)}\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\omega,\Omega)}\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\omega^{-1},\Omega)}. \end{aligned}$$

Similarly, $|d(\mathbf{u},\mathbf{w};\mathbf{v})| \leq C_{4\to 2}^2 C_{2\to 2}\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)}\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\omega,\Omega)}\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\omega^{-1},\Omega)}$. ∎

4.2. Existence and uniqueness for small data. Let us redefine the mapping $\mathcal{B} : \mathcal{X} \to \mathcal{Y}'$ and define $\mathcal{NL} : \mathcal{X} \to \mathcal{Y}'$ and $\mathcal{F} \in \mathcal{Y}'$ by

$$\begin{aligned} \langle \mathcal{B}(\mathbf{u},\mathsf{p}),(\mathbf{v},\mathsf{q})\rangle &:= a(\mathbf{u},\mathbf{v}) + b_-(\mathbf{v},\mathsf{p}) + b_+(\mathbf{u},\mathsf{q}), \\ \langle \mathcal{NL}(\mathbf{u},\mathsf{p}),(\mathbf{v},\mathsf{q})\rangle &:= c(\mathbf{u},\mathbf{u};\mathbf{v}) + d(\mathbf{u},\mathbf{u};\mathbf{v}), \end{aligned}$$

and $\langle \mathcal{F},(\mathbf{v},\mathsf{q})\rangle := \langle \mathbf{f},\mathbf{v}\rangle$, respectively. Here, $(\mathbf{v},q) \in \mathcal{Y}$. We recall that the spaces $\mathcal{X}$ and $\mathcal{Y}$ are defined in (7). With this functional framework at hand, problem (14) can be equivalently written as the following equation in $\mathcal{Y}'$: $\mathcal{B}(\mathbf{u},\mathsf{p}) + \mathcal{NL}(\mathbf{u},\mathsf{p}) = \mathcal{F}$.

The map $\mathcal{B}$ is linear and bounded; see the proof of Theorem 2 for details. In addition, if Ω is Lipschitz and $\omega \in A_2(\Omega)$, then Theorem 2 guarantees that $\mathcal{B}$ has a bounded inverse. This allows us to define the mapping

$$\mathcal{T} : \mathcal{X} \to \mathcal{X}, \qquad (\mathbf{u},\mathsf{p}) = \mathcal{T}(\mathbf{w},\mathsf{r}) = \mathcal{B}^{-1}[\mathcal{F} - \mathcal{NL}(\mathbf{w},\mathsf{r})]. \tag{18}$$

Therefore, showing the existence of a solution for problem (14) amounts to finding a fixed point of $\mathcal{T}$. In what follows, we prove that, if the datum $\mathbf{f}$ is sufficiently small, then we have the existence and uniqueness of solutions. We begin the analysis with a standard contraction argument; see, for instance, [43, Theorem 3.1], [46, Theorem 5.6], and [41, Proposition 1]. To present such a result, we define $\mathsf{A} := (3C_e\|\mathcal{B}^{-1}\|)^{-1} > 0$ and

$$\mathfrak{B}_\mathsf{A} := \{\mathbf{w} \in \mathbf{H}_0^1(\omega,\Omega) : \operatorname{div}\,\mathbf{w} = 0,\ \|\nabla\mathbf{w}\|_{\mathbf{L}^2(\omega,\Omega)} \leq \mathsf{A}\}, \tag{19}$$

where $C_e := C_{4\to 2}^2(1 + C_{2\to 2})$ and $\|\mathcal{B}^{-1}\|$ denotes the $\mathcal{Y}' \to \mathcal{X}$ norm of $\mathcal{B}^{-1}$. Let us introduce, in addition, the map $\mathcal{T}_1 : \mathbf{H}_0^1(\omega,\Omega) \to \mathbf{H}_0^1(\omega,\Omega)$ defined as $\mathbf{w} \mapsto P_r\mathcal{T}(\mathbf{w},0)$, where $P_r : \mathcal{X} \to \mathbf{H}_0^1(\omega,\Omega)$ corresponds to the projection onto the velocity component.

Proposition 4 (contraction). *Let Ω be a bounded Lipschitz domain and $\omega \in A_2(\Omega)$. If the forcing term $\mathbf{f}$ is sufficiently small, so that*

$$C_e\|\mathcal{B}^{-1}\|^2\|\mathbf{f}\|_{\mathbf{H}^{-1}(\omega,\Omega)} < \tfrac{1}{6}, \tag{20}$$

then $\mathcal{T}_1$ maps $\mathfrak{B}_\mathsf{A}$ to itself and $\mathcal{T}_1$ is a contraction in $\mathfrak{B}_\mathsf{A}$.

Proof. We begin the proof by noticing that, by assumption, we have the bound $\|\mathcal{B}^{-1}\|\|\mathbf{f}\|_{\mathbf{H}^{-1}(\omega,\Omega)} < \mathsf{A}/2$. Let $\mathbf{w} \in \mathfrak{B}_\mathsf{A}$. If $\mathbf{v} = \mathcal{T}_1(\mathbf{w})$, then div $\mathbf{v} = 0$ and

$$\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\omega,\Omega)} \leq \|\mathcal{B}^{-1}\|\|\mathbf{f}\|_{\mathbf{H}^{-1}(\omega,\Omega)} + C_e\|\mathcal{B}^{-1}\|\|\nabla\mathbf{w}\|^2_{\mathbf{L}^2(\omega,\Omega)} < \frac{\mathsf{A}}{2} + \frac{\mathsf{A}}{3} = \frac{5\mathsf{A}}{6}.$$

This shows that $\mathcal{T}_1$ maps $\mathcal{B}_\mathsf{A}$ into itself. We now show that $\mathcal{T}_1$ is a contraction. Let $\mathbf{w}_1, \mathbf{w}_2 \in \mathfrak{B}_\mathsf{A}$. Define $\mathbf{v}_i = \mathcal{T}_1(\mathbf{w}_i)$ for $i \in \{1,2\}$. We now invoke Hölder's inequality, the definition of $\mathfrak{B}_\mathsf{A}$, and the fact that $\omega \in A_2(\Omega)$ to arrive at

$$\begin{aligned} \|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|_{\mathbf{L}^2(\omega,\Omega)} &\leq C_e\|\mathcal{B}^{-1}\|\left[\|\nabla\mathbf{w}_1\|_{\mathbf{L}^2(\omega,\Omega)} + \|\nabla\mathbf{w}_2\|_{\mathbf{L}^2(\omega,\Omega)}\right]\|\nabla(\mathbf{w}_1 - \mathbf{w}_2)\|_{\mathbf{L}^2(\omega,\Omega)} \\ &\leq \frac{2}{3}\|\nabla(\mathbf{w}_1 - \mathbf{w}_2)\|_{\mathbf{L}^2(\omega,\Omega)}. \end{aligned}$$

This proves that $\mathcal{T}_1$ is a contraction and concludes the proof. ∎

The existence and uniqueness of solutions for small data is as follows.

THEOREM 5 (well-posedness for small data). *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $\omega \in A_2(\Omega)$. If $\mathbf{f}$ is sufficiently small so that (20) holds, then there exists a unique solution $(\mathbf{u}, \mathsf{p})$ of problem (14). Moreover, $(\mathbf{u}, \mathsf{p})$ satisfies the estimates*

$$\|\nabla \mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)} \leq \tfrac{3}{2}\|\mathcal{B}^{-1}\|\|\mathbf{f}\|_{\mathbf{H}^{-1}(\omega,\Omega)}, \tag{21}$$

$$\|\mathsf{p}\|_{L^2(\omega,\Omega)} \lesssim \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)} + \|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\omega,\Omega)} + \|\mathbf{f}\|_{\mathbf{H}^{-1}(\omega,\Omega)}, \tag{22}$$

where the hidden constants are independent of $\mathbf{u}$, p, and $\mathbf{f}$.

Proof. In view of of Proposition 4, we have the existence of a unique fixed point $\mathbf{u} \in \mathfrak{B}_{\mathsf{A}}$ of $\mathcal{T}_1$. We now invoke the existence of a right inverse of the divergence operator over A_2-weighted spaces [20, Theorem 3.1] to deduce the existence and uniqueness of the pressure p. To obtain (21) we use the fact that $\mathbf{u}$ is the unique fixed point of $\mathcal{T}_1$:

$$\begin{aligned} \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)} &\leq \|\mathcal{B}^{-1}\|\|\mathbf{f}\|_{\mathbf{H}^{-1}(\omega,\Omega)} + \mathsf{A}C_e\|\mathcal{B}^{-1}\|\|\nabla \mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)} \\ &\leq \|\mathcal{B}^{-1}\|\|\mathbf{f}\|_{\mathbf{H}^{-1}(\omega,\Omega)} + \frac{1}{3}\|\nabla \mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)}. \end{aligned}$$

Finally, to obtain (22) we utilize the weighted inf-sup condition (15). ∎

5. Finite element approximation: a priori error estimates.

In this section, we analyze a finite element solution technique that approximates solutions to (14). To accomplish this task, we will begin the section by introducing some terminology and a few basic ingredients [10, 14, 22]. In what follows, we operate under the assumption that Ω is a Lipschitz polytope so that it can be triangulated exactly.

5.1. Basic ingredients and assumptions.

We denote by $\mathscr{T} = \{K\}$ a conforming partition, or mesh, of $\bar{\Omega}$ into closed simplices K with size $h_K = \text{diam}(K)$. We define $h_{\mathscr{T}} := \max\{h_K : K \in \mathscr{T}\}$. We denote by $\mathbb{T}$ a collection of conforming and shape regular meshes that are refinements of an initial mesh $\mathscr{T}_0$ [14, 22]. We define $\mathscr{S}$ as the set of internal interelement boundaries γ of $\mathscr{T}$. For $\gamma \in \mathscr{S}$, we define h_γ as the length of γ. If $K \in \mathscr{T}$, we define $\mathscr{S}_K$ as the subset of $\mathscr{S}$ that contains the sides of K. For $\gamma \in \mathscr{S}$, we denote by $\mathcal{N}_\gamma$ the subset of $\mathscr{T}$ that contains the two elements that have γ as a side. For $K \in \mathscr{T}$, we define the *stars* or *patches* associated with K:

$$\mathcal{N}_K = \{K' \in \mathscr{T} : \mathscr{S}_K \cap \mathscr{S}_{K'} \neq \emptyset\}, \quad \mathcal{N}_K^* = \{K' \in \mathscr{T} : K \cap K' \neq \emptyset\}. \tag{23}$$

In an abuse of notation, below we will indistinctively denote by $\mathcal{N}_K$, $\mathcal{N}_K^*$, and $\mathcal{N}_\gamma$ either the sets themselves or the union of the elements that comprise them.

Given a mesh $\mathscr{T} \in \mathbb{T}$, we denote by $\mathbf{V}(\mathscr{T})$ and $\mathcal{P}(\mathscr{T})$ the finite element spaces that approximate the velocity field and the pressure, respectively. The following choices are classical.

(a) The *mini element*, which is considered in [22, Section 4.2.4] and is defined by

$$\mathbf{V}(\mathscr{T}) = \{\mathbf{v}_{\mathscr{T}} \in \mathbf{C}(\overline{\Omega}) : \forall K \in \mathscr{T}, \mathbf{v}_{\mathscr{T}}|_K \in [\mathbb{W}(K)]^2\} \cap \mathbf{H}_0^1(\Omega), \tag{24}$$

$$\mathcal{P}(\mathscr{T}) = \{\mathsf{q}_{\mathscr{T}} \in L_0^2(\Omega) \cap C(\overline{\Omega}) : \forall K \in \mathscr{T}, \mathsf{q}_{\mathscr{T}}|_K \in \mathbb{P}_1(K)\}, \tag{25}$$

where $\mathbb{W}(K) := \mathbb{P}_1(K) \oplus \mathbb{B}(K)$ and $\mathbb{B}(K)$ denotes the space spanned by a local bubble function.

(b) The lowest order *Taylor–Hood pair* [22, Section 4.2.5], which is defined by

$$\mathbf{V}(\mathscr{T}) = \{\mathbf{v}_{\mathscr{T}} \in \mathbf{C}(\overline{\Omega}) : \forall K \in \mathscr{T}, \mathbf{v}_{\mathscr{T}}|_K \in [\mathbb{P}_2(K)]^2\} \cap \mathbf{H}_0^1(\Omega), \tag{26}$$

$$\mathcal{P}(\mathscr{T}) = \{\mathsf{q}_{\mathscr{T}} \in L_0^2(\Omega) \cap C(\overline{\Omega}) : \forall K \in \mathscr{T}, \mathsf{q}_{\mathscr{T}}|_K \in \mathbb{P}_1(K)\}. \tag{27}$$

We must immediately notice that if $\omega \in A_2$, we have that the previously defined spaces are such that: $\mathbf{V}(\mathscr{T}) \subset \mathbf{W}_0^{1,\infty}(\Omega) \subset \mathbf{H}_0^1(\omega,\Omega)$ and $\mathcal{P}(\mathscr{T}) \subset L^\infty(\Omega) \subset L^2(\omega,\Omega)/\mathbb{R}$. In addition, these pairs of spaces satisfy the following compatibility condition [21, Theorems 6.2 and 6.4]: There exists $\beta > 0$ such that

$$\beta \|\mathsf{q}_{\mathscr{T}}\|_{L^2(\omega^{\pm 1},\Omega)} \leq \sup_{\mathbf{0}\neq \mathbf{v}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T})} \frac{b_{\mp}(\mathbf{v}_{\mathscr{T}}, \mathsf{q}_{\mathscr{T}})}{\|\nabla \mathbf{v}_{\mathscr{T}}\|_{\mathbf{L}^2(\omega^{\mp 1},\Omega)}} \quad \forall \mathsf{q}_{\mathscr{T}} \in \mathcal{P}(\mathscr{T}). \tag{28}$$

As a final ingredient, we define the *jump* or *interelement residual* of a discrete tensor valued function $\mathbf{w}_{\mathscr{T}}$ on an internal side $\gamma \in \mathscr{S}$ by

$$[\![\mathbf{w}_{\mathscr{T}} \cdot \nu]\!] := \mathbf{w}_{\mathscr{T}} \cdot \nu^+|_{K^+} + \mathbf{w}_{\mathscr{T}} \cdot \nu^-|_{K^-}, \tag{29}$$

where ν^+ and ν^- denote the unit normals on γ pointing towards K^+ and K^-, respectively; K^+, $K^- \in \mathscr{T}$ are such that $K^+ \neq K^-$ and $\partial K^+ \cap \partial K^- = \gamma$.

5.2. The scheme. Let $\omega \in A_2(\Omega)$ and $\mathbf{f} \in \mathbf{H}^{-1}(\omega,\Omega)$. We introduce the following discrete approximation of (14): Find $(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ such that.

$$\begin{aligned} a(\mathbf{u}_{\mathscr{T}}, \mathbf{v}_{\mathscr{T}}) + b_-(\mathbf{v}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}) + c(\mathbf{u}_{\mathscr{T}}, \mathbf{u}_{\mathscr{T}}; \mathbf{v}_{\mathscr{T}}) + d(\mathbf{u}_{\mathscr{T}}, \mathbf{u}_{\mathscr{T}}; \mathbf{v}_{\mathscr{T}}) &= \langle \mathbf{f}, \mathbf{v}_{\mathscr{T}} \rangle, \\ b_+(\mathbf{u}_{\mathscr{T}}, \mathsf{q}_{\mathscr{T}}) &= 0, \end{aligned} \tag{30}$$

for all $\mathbf{v}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T})$ and $\mathsf{q}_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})$, respectively.

Let us denote by $\mathcal{B}_{\mathscr{T}}$ the discrete version of $\mathcal{B}$ induced by the discretization (30). The results that follow are based on the following assumption.

Assumption 6. The operator $\mathcal{B}_{\mathscr{T}}$ is a bounded linear operator whose inverse $\mathcal{B}_{\mathscr{T}}^{-1}$ is bounded uniformly over all partitions $\mathscr{T}$.

The fact that the operator $\mathcal{B}_{\mathscr{T}}^{-1}$ exists is not an issue. Notice that, since we are in finite dimensions, the existence and uniqueness of solutions for the discrete problem (30) are guaranteed by the compatibility condition (28). The main point in assumption 6 is that $\mathcal{B}_{\mathscr{T}}^{-1}$ satisfies a suitable estimate in terms of the problem data which is uniform with respect to discretization.

The existence of a unique discrete solution is the content of the following result.

THEOREM 7 (well-posedness for small data). *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz polytope, and let $\omega \in A_2(\Omega)$. If $\mathbf{f}$ is such that (20) holds with $\mathcal{B}^{-1}$ replaced by $\mathcal{B}_{\mathscr{T}}^{-1}$, then there exists a unique pair $(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ that solves (30). In addition, we have the stability estimates*

$$\|\nabla \mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^2(\omega,\Omega)} \leq \tfrac{3}{2} \|\mathcal{B}_{\mathscr{T}}^{-1}\| \|\mathbf{f}\|_{\mathbf{H}^{-1}(\omega,\Omega)}, \tag{31}$$

$$\|\mathsf{p}_{\mathscr{T}}\|_{L^2(\omega,\Omega)} \lesssim \|\nabla \mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^2(\omega,\Omega)} + \|\nabla \mathbf{u}_{\mathscr{T}}\|^2_{\mathbf{L}^2(\omega,\Omega)} + \|\mathbf{f}\|_{\mathbf{H}^{-1}(\omega,\Omega)}, \tag{32}$$

where the hidden constant are independent of $\mathbf{u}_{\mathscr{T}}$, $\mathsf{p}_{\mathscr{T}}$, and $\mathbf{f}$.

Proof. The proof follows from the arguments developed in the proof of Theorem 5. Succinctly, we mention that instead of $\mathcal{B}^{-1}$ we use the fact that $\mathcal{B}_{\mathscr{T}}^{-1}$ is bounded uniformly with respect to discretization. ∎

To present the auxiliary estimate of Lemma 8 and the quasi-best approximation result of Theorem 9, we will operate under the following assumption: Let Ω be a convex polytope, $\omega \in A_2(\Omega)$, and $(\mathbf{u}, \mathsf{p}) \in \mathbf{H}_0^1(\omega,\Omega) \times L^2(\omega,\Omega)/\mathbb{R}$ with $\mathbf{u}$ solenoidal. Let $(B_{\mathscr{T}}\mathbf{u}, B_{\mathscr{T}}\mathsf{p}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ be the *Brinkman projection* of $(\mathbf{u}, \mathsf{p})$, i.e., the pair $(B_{\mathscr{T}}\mathbf{u}, B_{\mathscr{T}}\mathsf{p})$ is such that

$$\begin{aligned} a(B_{\mathscr{T}}\mathbf{u}, \mathbf{v}_{\mathscr{T}}) + b_-(\mathbf{v}_{\mathscr{T}}, B_{\mathscr{T}}\mathsf{p}) &= a(\mathbf{u}, \mathbf{v}_{\mathscr{T}}) + b_-(\mathbf{v}_{\mathscr{T}}, \mathsf{p}), \\ b_+(B_{\mathscr{T}}\mathbf{u}, \mathsf{q}_{\mathscr{T}}) &= 0, \end{aligned} \tag{33}$$

for all $\mathbf{v}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T})$ and $\mathsf{q}_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})$. Then, we have

$$\|\nabla B_{\mathscr{T}}\mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)} + \|B_{\mathscr{T}}\mathsf{p}\|_{L^2(\omega,\Omega)} \lesssim \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)} + \|\mathsf{p}\|_{L^2(\omega,\Omega)}, \tag{34}$$

where the hidden constant is independent of $(\mathbf{u},\mathsf{p})$, $(B_{\mathscr{T}}\mathbf{u}, B_{\mathscr{T}}\mathsf{p})$, and $h_{\mathscr{T}}$. When the Brinkman operator in (33) is replaced by the Stokes operator, the desired estimate is available in [21, Theorem 4.1]. We notice that, in view of the arguments developed in the proof of [21, Theorem 4.1] the only missing ingredient to obtain (34) is the error estimate [21, estimate (3.9)] for a regularized Green's function. If this estimate were available for the Brinkman operator and the considered finite element paris (24)–(25) and (26)–(27), then the desired estimate (34) would follow immediately.

Lemma 8 (auxiliary estimate). *Let $\Omega \subset \mathbb{R}^2$ be a convex polytope, and let $\omega \in A_2(\Omega)$. Assume that $\mathbf{f}$ is sufficiently small so that (20) holds. Then, we have*

$$\begin{aligned} &\|\nabla(\mathbf{u} - B_{\mathscr{T}}\mathbf{u})\|_{\mathbf{L}^2(\omega,\Omega)} + \|\mathsf{p} - B_{\mathscr{T}}\mathsf{p}\|_{L^2(\omega,\Omega)} \\ &\qquad \lesssim \inf_{\mathbf{w}_{\mathscr{T}}\in\mathbf{V}(\mathscr{T})} \|\nabla(\mathbf{u} - \mathbf{w}_{\mathscr{T}})\|_{\mathbf{L}^2(\omega,\Omega)} + \inf_{\mathsf{q}_{\mathscr{T}}\in\mathcal{P}(\mathscr{T})} \|\mathsf{p} - \mathsf{q}_{\mathscr{T}}\|_{L^2(\omega,\Omega)}, \end{aligned} \tag{35}$$

where the hidden constant is independent of $(\mathbf{u},\mathsf{p})$, $(B_{\mathscr{T}}\mathbf{u}, B_{\mathscr{T}}\mathsf{p})$, and $h_{\mathscr{T}}$.

Proof. The proof is rather standard; it follows, for instance, from the arguments developed in the proof of [21, Corollary 4.2]. ∎

Since it will be useful for the analysis that follows, we define, for $\mathbf{v} \in \mathbf{H}_0^1(\omega^{-1},\Omega)$,

$$\Theta(\mathbf{u},\mathbf{u}_{\mathscr{T}};\mathbf{v}) := c(\mathbf{u},\mathbf{u};\mathbf{v}) - c(\mathbf{u}_{\mathscr{T}},\mathbf{u}_{\mathscr{T}};\mathbf{v}) + d(\mathbf{u},\mathbf{u};\mathbf{v}) - d(\mathbf{u}_{\mathscr{T}},\mathbf{u}_{\mathscr{T}};\mathbf{v}). \tag{36}$$

Notice that $\Theta(\mathbf{u},\mathbf{u}_{\mathscr{T}};\mathbf{v}) = c(\mathbf{u},\mathbf{e}_{\mathbf{u}};\mathbf{v}) + c(\mathbf{e}_{\mathbf{u}},\mathbf{u}_{\mathscr{T}};\mathbf{v}) + d(\mathbf{u},\mathbf{u};\mathbf{v}) - d(\mathbf{u}_{\mathscr{T}},\mathbf{u}_{\mathscr{T}};\mathbf{v})$.

We now obtain the following quasi-best approximation result.

Theorem 9 (quasi-best approximation result). *Let $\Omega \subset \mathbb{R}^2$ be a convex polytope, and let $\omega \in A_2(\Omega)$. Assume that $\mathbf{f}$ is sufficiently small so that problems (14) and (30) admit a unique solution, with sufficiently small norms. Then, we have the following quasi-best approximation result:*

$$\begin{aligned} &\|\nabla(\mathbf{u}-\mathbf{u}_{\mathscr{T}})\|_{\mathbf{L}^2(\omega,\Omega)} + \|\mathsf{p} - \mathsf{p}_{\mathscr{T}}\|_{L^2(\omega,\Omega)} \\ &\qquad \lesssim \inf_{\mathbf{w}_{\mathscr{T}}\in\mathbf{V}(\mathscr{T})} \|\nabla(\mathbf{u}-\mathbf{w}_{\mathscr{T}})\|_{\mathbf{L}^2(\omega,\Omega)} + \inf_{\mathsf{q}_{\mathscr{T}}\in\mathcal{P}(\mathscr{T})} \|\mathsf{p}-\mathsf{q}_{\mathscr{T}}\|_{L^2(\omega,\Omega)}, \end{aligned}$$

where the hidden constant may depend on $\mathbf{f}$ and $\mathbf{u}$, but is independent of $h_{\mathscr{T}}$.

Proof. Define $\mathbf{e}_{\mathscr{T}} := B_{\mathscr{T}}\mathbf{u} - \mathbf{u}_{\mathscr{T}}$ and $\varepsilon_{\mathscr{T}} := B_{\mathscr{T}}\mathsf{p} - \mathsf{p}_{\mathscr{T}}$, where $(B_{\mathscr{T}}\mathbf{u}, B_{\mathscr{T}}\mathsf{p})$ corresponds to the *Brinkman projection* of $(\mathbf{u},\mathsf{p})$. Invoke the definition of the *Brinkman projection* to infer that

$$\begin{aligned} a(\mathbf{e}_{\mathscr{T}},\mathbf{v}_{\mathscr{T}}) + b_{-}(\mathbf{v}_{\mathscr{T}},\varepsilon_{\mathscr{T}}) &= -\Theta(\mathbf{u},\mathbf{u}_{\mathscr{T}};\mathbf{v}_{\mathscr{T}}) \quad \forall \mathbf{v}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T}) \\ b_{+}(\mathbf{e}_{\mathscr{T}},\mathsf{q}_{\mathscr{T}}) &= 0 \quad \forall \mathsf{q}_{\mathscr{T}} \in \mathcal{P}(\mathscr{T}). \end{aligned} \tag{37}$$

We now utilize the stability bound (34) of the *Brinkman projection* to arrive at

$$\|\nabla \mathbf{e}_{\mathscr{T}}\|_{\mathbf{L}^2(\omega,\Omega)} + \|\varepsilon_{\mathscr{T}}\|_{L^2(\omega,\Omega)} \lesssim \left(\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)} + \|\nabla\mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^2(\omega,\Omega)}\right) \|\nabla(\mathbf{u} - \mathbf{u}_{\mathscr{T}})\|_{\mathbf{L}^2(\omega,\Omega)}.$$

This bound and the one in Lemma 8 allow us to arrive at

$$\begin{aligned} &\|\nabla(\mathbf{u}-\mathbf{u}_{\mathscr{T}})\|_{\mathbf{L}^2(\omega,\Omega)} + \|\mathsf{p} - \mathsf{p}_{\mathscr{T}}\|_{L^2(\omega,\Omega)} \lesssim \left(\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)} + \|\nabla\mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^2(\omega,\Omega)}\right) \\ &\cdot \|\nabla(\mathbf{u} - \mathbf{u}_{\mathscr{T}})\|_{\mathbf{L}^2(\omega,\Omega)} + \inf_{\mathbf{w}_{\mathscr{T}}\in\mathbf{V}(\mathscr{T})} \|\nabla(\mathbf{u}-\mathbf{w}_{\mathscr{T}})\|_{\mathbf{L}^2(\omega,\Omega)} + \inf_{\mathsf{q}_{\mathscr{T}}\in\mathcal{P}(\mathscr{T})} \|\mathsf{p}-\mathsf{q}_{\mathscr{T}}\|_{L^2(\omega,\Omega)}. \end{aligned}$$

The desired estimate thus follows from utilizing that $\|\nabla \mathbf{u}\|_{\mathbf{L}^2(\omega,\Omega)}$ and $\|\nabla \mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^2(\omega,\Omega)}$ are sufficiently small so that the term $\|\nabla(\mathbf{u}-\mathbf{u}_{\mathscr{T}})\|_{\mathbf{L}^2(\omega,\Omega)}$ appearing on the right-hand side of the previous estimate can be absorbed into the left. This concludes the proof. ∎

6. Finite element approximation: a posteriori error estimates. In this section, we design and analyze an a posteriori error estimator for the finite-dimensional approximation (30) of the system (14). To accomplish this task, we will assume that the external density force $\mathbf{f}$ has a particular structure, that is, $\mathbf{f} := \mathbf{F}\delta_{\mathbf{z}}$, where $\delta_{\mathbf{z}}$ corresponds to the Dirac delta supported at the interior point $\mathbf{z} \in \Omega$ and $\mathbf{F} \in \mathbb{R}^2$. To handle such a singular forcing term, we consider the weight $\mathsf{d}_{\mathbf{z}}^{\alpha}$ defined in (3) with $\alpha \in (0,2)$. We must immediately notice the following two important properties: First, $\mathsf{d}_{\mathbf{z}}^{\pm\alpha} \in A_2$ and $\mathsf{d}_{\mathbf{z}}^{\alpha} \in A_2(\Omega)$. Second, $\delta_{\mathbf{z}} \in H_0^1(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega)'$; see [35, Lemma 7.1.3] and [34, Remark 21.19].

In what follows, we will assume that $\mathbf{F} \in \mathbb{R}^2$ is such that (20) holds. In addition, we will also assume that $\mathbf{F} \in \mathbb{R}^2$ is such that (20) holds with $\mathcal{B}^{-1}$ replaced by $\mathcal{B}_{\mathscr{T}}^{-1}$. We notice that, within this setting, problems (14) and (30) are well-posed; see Theorems 5 and 7.

Let us begin our studies by redefining the spaces $\mathcal{X}$ and $\mathcal{Y}$ as follows: $\mathcal{X} = \mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega) \times L^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)/\mathbb{R}$ and $\mathcal{Y} = \mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega) \times L^2(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega)/\mathbb{R}$. Let us define the velocity error $\mathbf{e}_{\mathbf{u}}$ and the pressure error e_{p} by

$$\mathbf{e}_{\mathbf{u}} := \mathbf{u} - \mathbf{u}_{\mathscr{T}} \in \mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega), \qquad e_{\mathsf{p}} := \mathsf{p} - \mathsf{p}_{\mathscr{T}} \in L^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)/\mathbb{R}, \tag{38}$$

respectively.

6.1. Ritz projection. As an instrumental step to perform a global reliability analysis, we introduce a suitable Ritz projection $(\mathbf{\Phi},\psi)$ of the residuals. The pair $(\mathbf{\Phi},\psi)$ is defined as the solution to the problem: Find $(\mathbf{\Phi},\psi) \in \mathcal{X}$ such that

$$\begin{aligned} (\nabla\mathbf{\Phi},\nabla\mathbf{v})_{\mathbf{L}^2(\Omega)} &= a(\mathbf{e}_{\mathbf{u}},\mathbf{v}) + b_-(\mathbf{v},e_{\mathsf{p}}) + \Theta(\mathbf{u},\mathbf{u}_{\mathscr{T}};\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega), \\ (\psi,\mathsf{q})_{L^2(\Omega)} &= b_+(\mathbf{e}_{\mathbf{u}},\mathsf{q}) \quad \forall \mathsf{q} \in L^2(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega)/\mathbb{R}. \end{aligned} \tag{39}$$

Here, $\Theta(\mathbf{u},\mathbf{u}_{\mathscr{T}};\mathbf{v})$ is defined as in (36).

The following result establishes the well-posedness of problem (39).

THEOREM 10 (Ritz projection). *Problem (39) admits a unique solution $(\mathbf{\Phi},\psi) \in \mathcal{X}$. In addition, we have the estimate*

$$\begin{aligned} \|\nabla\mathbf{\Phi}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} + \|\psi\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} \lesssim \|\nabla\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} + \|\mathbf{e}_{\mathsf{p}}\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} \\ + \|\nabla\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} \left(\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} + \|\nabla\mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)}\right), \end{aligned} \tag{40}$$

where the hidden constant is independent of $(\mathbf{\Phi},\psi)$, $(\mathbf{u},\mathsf{p})$, and $(\mathbf{u}_{\mathscr{T}},\mathsf{p}_{\mathscr{T}})$.

Proof. We begin the proof by introducing the linear functional $\mathfrak{G}$ as follows:

$$\mathfrak{G} : \mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega) \to \mathbb{R}, \qquad \mathfrak{G}(\mathbf{v}) := a(\mathbf{e}_{\mathbf{u}},\mathbf{v}) + b_-(\mathbf{v},e_{\mathsf{p}}) + \Theta(\mathbf{u},\mathbf{u}_{\mathscr{T}};\mathbf{v}). \tag{41}$$

Let us show that $\mathfrak{G}$ belongs to $\mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega)'$. To accomplish this task, we first control the nonlinear term $\Theta(\mathbf{u},\mathbf{u}_{\mathscr{T}};\cdot)$ defined in (36). Owing to the estimates of Lemma 3, we obtain the bound

$$\|\Theta(\mathbf{u},\mathbf{u}_{\mathscr{T}};\cdot)\|_{\mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega)'} \leq C_{4\to2}^2(1+C_{2\to2})\|\nabla\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)}\Lambda(\mathbf{u},\mathbf{u}_{\mathscr{T}}), \tag{42}$$

where $\Lambda(\mathbf{u},\mathbf{u}_{\mathscr{T}}) := \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} + \|\nabla \mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)}$. Consequently,

$$\|\mathfrak{G}\|_{\mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega)'} \leq (1+C_{2\to 2})\|\nabla \mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} + \|e_{\mathsf{p}}\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} \\ + C_{4\to 2}^2(1+C_{2\to 2})\|\nabla \mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)}\Lambda(\mathbf{u},\mathbf{u}_{\mathscr{T}}) =: \Gamma(\mathbf{u},\mathbf{u}_{\mathscr{T}},p).$$

Since $\mathsf{d}_{\mathbf{z}}^{\alpha} \in A_2(\Omega)$ and $\mathfrak{G} \in (\mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega))'$, we are thus in position to invoke the results of [40] to deduce the existence and uniqueness of $\mathbf{\Phi} \in \mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)$ satisfying the first equation of problem (39) and the estimate

$$\|\nabla \mathbf{\Phi}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} \lesssim \Gamma(\mathbf{u},\mathbf{u}_{\mathscr{T}},p). \tag{43}$$

Finally, since $\mathbf{e}_{\mathbf{u}} \in \mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)$, $b_+(\mathbf{e}_{\mathbf{u}},\cdot)$ defines a linear and continuous functional in the space $L^2(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega)/\mathbb{R}$. As a consequence, we deduce the existence and uniqueness of $\psi \in L^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)/\mathbb{R}$ satisfying the second equation in (39) and the estimate

$$\|\psi\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} \lesssim \|\mathrm{div}\ \mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)}. \tag{44}$$

The desired estimate (40) thus follows from collecting the bounds (43) and (44). This concludes the proof. ∎

6.2. An upper bound for the error. We now prove that the energy norm of the error can be bounded in terms of the energy norm of the Ritz projection. This key step will allow us to provide a computable upper bound for the error.

Let us begin by introducing the map $\mathfrak{F} : \mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega) \to \mathbb{R}$ as

$$\mathfrak{F}(\mathbf{v}) := (\nabla \mathbf{\Phi}, \nabla \mathbf{v})_{\mathbf{L}^2(\Omega)} - \Theta(\mathbf{u},\mathbf{u}_{\mathscr{T}};\mathbf{v}), \tag{45}$$

where $\Theta(\mathbf{u},\mathbf{u}_{\mathscr{T}};\mathbf{v})$ is defined in (36). It is clear that, for $\mathbf{u}$ and $\mathbf{u}_{\mathscr{T}}$ given, the map $\mathfrak{F}$ is linear. In addition, in view of (42), $\mathfrak{F}$ satisfies the estimate

$$\|\mathfrak{F}\|_{\mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega)'} \leq \|\nabla \mathbf{\Phi}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} + C_{4\to 2}^2(1+C_{2\to 2})\|\nabla \mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)}\Lambda(\mathbf{u},\mathbf{u}_{\mathscr{T}}), \tag{46}$$

where $\Lambda(\mathbf{u},\mathbf{u}_{\mathscr{T}}) := \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} + \|\nabla \mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)}$.

Having introduced the linear map $\mathfrak{F}$, we observe that, in view of the equations in problem (39), the pair $(\mathbf{e}_{\mathbf{u}}, e_{\mathsf{p}})$ can be seen as a solution to the following problem: Find $(\mathbf{e}_{\mathbf{u}}, e_{\mathsf{p}}) \in \mathcal{X}$ such that, for every $\mathbf{v} \in \mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega)$ and $\mathsf{q} \in L^2(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega)/\mathbb{R}$,

$$a(\mathbf{e}_{\mathbf{u}},\mathbf{v}) + b_-(\mathbf{v},e_{\mathsf{p}}) = \mathfrak{F}(\mathbf{v}), \qquad b_+(\mathbf{e}_{\mathbf{u}},\mathsf{q}) = (\psi,\mathsf{q})_{L^2(\Omega)}. \tag{47}$$

With all these ingredients at hand, we present the following result.

PROPOSITION 11 (upper bound for the error). *Let $\mathbf{F} \in \mathbb{R}^2$ be such that*

$$1 - C_{\mathcal{B}}C_{4\to 2}^2(1+C_{2\to 2})(\|\nabla \mathbf{u}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} + \|\nabla \mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)}) \geq \lambda > 0, \tag{48}$$

where $\lambda < 1$. Then, we have the following upper bound for the error $(\mathbf{e}_{\mathbf{u}}, e_{\mathsf{p}})$:

$$\|\nabla \mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} + \|e_{\mathsf{p}}\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} \lesssim \|\nabla \mathbf{\Phi}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} + \|\psi\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)}, \tag{49}$$

where the hidden constant is independent of $(\mathbf{u},\mathsf{p})$, $(\mathbf{u}_{\mathscr{T}},\mathsf{p}_{\mathscr{T}})$, and $(\mathbf{\Phi},\psi)$.

Proof. Invoke the estimate in Theorem 2 and the bound (46) to deduce that

$$\|\nabla \mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} + \|e_{\mathsf{p}}\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} \leq C_{\mathcal{B}}\left(\|\mathfrak{F}\|_{\mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{-\alpha},\Omega)'} + \|\psi\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)}\right) \\ \leq C_{\mathcal{B}}\left(\|\nabla \mathbf{\Phi}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)} + C_{4\to 2}^2(1+C_{2\to 2})\|\nabla \mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)}\Lambda(\mathbf{u},\mathbf{u}_{\mathscr{T}}) + \|\psi\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)}\right).$$

With this bound at hand, we invoke the smallness assumption (48) to conclude. ∎

6.3. A residual-type error estimator. In this section, we introduce an a posteriori error estimator for the finite element approximation (30) of problem (14) on the basis of the discrete pairs $(\mathbf{V}(\mathscr{T}), \mathcal{P}(\mathscr{T}))$ given as in (24)–(25) or (26)–(27). To present it, we first introduce, for $K \in \mathscr{T}$, the local distance

$$D_K := \max_{\mathbf{x}\in K} |\mathbf{x} - \mathbf{z}|. \tag{50}$$

We thus define, for $K \in \mathscr{T}$ and $\gamma \in \mathscr{S}$, the *element residual* $\mathcal{R}_K$ and the *interelement residual* $\mathcal{J}_\gamma$ as

$$\mathcal{R}_K := (\Delta \mathbf{u}_\mathscr{T} - \mathbf{u}_\mathscr{T} - (\mathbf{u}_\mathscr{T} \cdot \nabla)\mathbf{u}_\mathscr{T} - \mathbf{u}_\mathscr{T} \mathrm{div}\, \mathbf{u}_\mathscr{T} - |\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T} - \nabla \mathsf{p}_\mathscr{T})|_K. \tag{51}$$

$$\mathcal{J}_\gamma := [\![(\nabla \mathbf{u}_\mathscr{T} - \mathsf{p}_\mathscr{T}\mathbf{I}) \cdot \nu]\!], \tag{52}$$

where $(\mathbf{u}_\mathscr{T}, \mathsf{p}_\mathscr{T})$ denotes the solution to the discrete problem (30) and $\mathbf{I} \in \mathbb{R}^{2\times 2}$ denotes the identity matrix. The jump $[\![(\nabla \mathbf{u}_\mathscr{T} - \mathsf{p}_\mathscr{T}\mathbf{I}) \cdot \nu]\!]$ of the discrete tensor valued function $\nabla \mathbf{u}_\mathscr{T} - \mathsf{p}_\mathscr{T}\mathbf{I}$ is defined as in (29). With $\mathcal{R}_K$ and $\mathcal{J}_\gamma$ at hand, we define, for $\alpha \in (0, 2)$ and $K \in \mathscr{T}$, the *element error indicator*

$$\mathcal{E}_\alpha(\mathbf{u}_\mathscr{T}, \mathsf{p}_\mathscr{T}; K) := \Big(h_K^2 D_K^\alpha \|\mathcal{R}_K\|^2_{\mathbf{L}^2(K)} + \|\mathrm{div}\, \mathbf{u}_\mathscr{T}\|^2_{L^2(\mathsf{d}_\mathbf{z}^\alpha, K)} + h_K D_K^\alpha \|\mathcal{J}_\gamma\|^2_{\mathbf{L}^2(\partial K \setminus \partial\Omega)} + h_K^\alpha |\mathbf{F}|^2 \#(\{\mathbf{z}\} \cap K) \Big)^{\frac{1}{2}}. \tag{53}$$

For a set E, by $\#(E)$ we mean its cardinality. Here, we must recall that we consider our elements K to be closed sets. The a posteriori *error estimator* is thus defined as

$$\mathcal{E}_\alpha(\mathbf{u}_\mathscr{T}, \mathsf{p}_\mathscr{T}; \mathscr{T}) := \left(\sum_{K\in\mathscr{T}} \mathcal{E}_\alpha^2(\mathbf{u}_\mathscr{T}, \mathsf{p}_\mathscr{T}; K) \right)^{\frac{1}{2}}. \tag{54}$$

6.4. A quasi-interpolation operator. As it is customary in a posteriori error analysis, in order to derive reliability properties for a proposed a posteriori error estimator it is useful to have at hand a suitable quasi-interpolation operator with optimal approximation properties [53]. We consider the operator $\Pi_\mathscr{T} : \mathbf{L}^1(\Omega) \to \mathbf{V}(\mathscr{T})$ analyzed in [39]. The construction of $\Pi_\mathscr{T}$ is inspired in the ideas developed in [15, 19, 47]: it is built on local averages over stars and thus is well-defined for functions in $\mathbf{L}^1(\Omega)$; it also exhibits optimal approximation properties. In what follows, we shall make use of the following estimates of the local interpolation error [5, 39].

PROPOSITION 12 (stability and interpolation estimates). *Let $\alpha \in (-2, 2)$ and $K \in \mathscr{T}$. Then, for every $\mathbf{v} \in \mathbf{H}^1(\mathsf{d}_\mathbf{z}^{\pm\alpha}, \mathcal{N}_K^*)$, we have the local stability bound*

$$\|\nabla \Pi_\mathscr{T} \mathbf{v}\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^{\pm\alpha}, K)} \lesssim \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^{\pm\alpha}, \mathcal{N}_K^*)} \tag{55}$$

and the interpolation error estimate

$$\|\mathbf{v} - \Pi_\mathscr{T} \mathbf{v}\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^{\pm\alpha}, K)} \lesssim h_K \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^{\pm\alpha}, \mathcal{N}_K^*)}. \tag{56}$$

In addition, if $\alpha \in (0, 2)$, then we have

$$\|\mathbf{v} - \Pi_\mathscr{T} \mathbf{v}\|_{\mathbf{L}^2(K)} \lesssim h_K D_K^{\frac{\alpha}{2}} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^{-\alpha}, \mathcal{N}_K^*)}. \tag{57}$$

The hidden constants in the previous estimates are independent of $\mathbf{v}$, K, and $\mathscr{T}$.

Proof. See [5, Proposition 4]. ∎

PROPOSITION 13 (trace interpolation estimate). *Let $\alpha \in (0,2)$, $K \in \mathscr{T}$, $\gamma \subset \mathscr{S}_K$, and $\mathbf{v} \in \mathbf{H}^1(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)$. Then we have the following interpolation error estimate*

$$\|\mathbf{v} - \Pi_{\mathscr{T}}\mathbf{v}\|_{\mathbf{L}^2(\gamma)} \lesssim h_K^{\frac{1}{2}} D_K^{\frac{\alpha}{2}} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)}, \tag{58}$$

where the hidden constant is independent of $\mathbf{v}$, K, and the mesh $\mathscr{T}$.

Proof. See [5, Proposition 5]. ∎

6.5. Reliablity. In what follows, we obtain a global reliability property for the a posteriori error estimator $\mathcal{E}_\alpha$ defined in (54).

THEOREM 14 (global reliability). *Let $\alpha \in (0,2)$, $(\mathbf{u}, \mathsf{p}) \in \mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{\alpha}, \Omega) \times L^2(\mathsf{d}_{\mathbf{z}}^{\alpha}, \Omega)/\mathbb{R}$ be the solution to the continuous problem (14), and $(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ be its finite element approximation obtained as the solution to (30). Assume that the intensity of the forcing term $\mathbf{F}$ is sufficiently small so that (48) holds. Then*

$$\|\nabla \mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha}, \Omega)} + \|e_{\mathsf{p}}\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha}, \Omega)} \lesssim \mathcal{E}_\alpha(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}; \mathscr{T}), \tag{59}$$

where the hidden constant is independent of $(\mathbf{u}, \mathsf{p})$ and $(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}})$, the size of the elements in the mesh $\mathscr{T}$, and $\#\mathscr{T}$.

Proof. To provide the computable upper bound (59), we will utilize the fact that the energy norm of the error can be bounded in terms of the energy norm of the Ritz projection and proceed in three steps.

Step 1: Let $\mathbf{v} \in \mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{\alpha}, \Omega)$ be arbitrary. We utilize the first equation of problems (39) and (14) to conclude that

$$\begin{aligned}(\nabla \mathbf{\Phi}, \nabla \mathbf{v})_{\mathbf{L}^2(\Omega)} = \langle \mathbf{F}\delta_{\mathbf{z}}, \mathbf{v} \rangle - \sum_{K \in \mathscr{T}} \int_K (\nabla \mathbf{u}_{\mathscr{T}} : \nabla \mathbf{v} + \mathbf{u}_{\mathscr{T}} \cdot \mathbf{v} \\ - \mathbf{u}_{\mathscr{T}} \otimes \mathbf{u}_{\mathscr{T}} : \nabla \mathbf{v} + |\mathbf{u}_{\mathscr{T}}| \mathbf{u}_{\mathscr{T}} \cdot \mathbf{v} - \mathsf{p}_{\mathscr{T}} \text{div } \mathbf{v}).\end{aligned} \tag{60}$$

Applying a standard integration by parts argument, on the basis of the fact that, for $\gamma \in \mathscr{S}$, $([\![\mathbf{u}_{\mathscr{T}} \otimes \mathbf{u}_{\mathscr{T}} \cdot \nu]\!], \mathbf{v})_{L^2(\gamma)} = 0$, yields the identity

$$(\nabla \mathbf{\Phi}, \nabla \mathbf{v})_{\mathbf{L}^2(\Omega)} = \langle \mathbf{F}\delta_{\mathbf{z}}, \mathbf{v} \rangle + \sum_{\gamma \in \mathscr{S}} \int_\gamma \mathcal{J}_\gamma \cdot \mathbf{v} + \sum_{K \in \mathscr{T}} \int_K \mathcal{R}_K \cdot \mathbf{v}. \tag{61}$$

We recall that the element residual $\mathcal{R}_K$ and the interelement residual $\mathcal{J}_\gamma$ are defined as in (51) and (52), respectively.

Let us now observe that, for every $(\mathbf{v}_{\mathscr{T}}, \mathsf{q}_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$, we have

$$\langle \mathbf{F}\delta_{\mathbf{z}}, \mathbf{v}_{\mathscr{T}} \rangle - a(\mathbf{u}_{\mathscr{T}}, \mathbf{v}_{\mathscr{T}}) - b_-(\mathbf{v}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}) - c(\mathbf{u}_{\mathscr{T}}, \mathbf{u}_{\mathscr{T}}; \mathbf{v}_{\mathscr{T}}) - d(\mathbf{u}_{\mathscr{T}}, \mathbf{u}_{\mathscr{T}}; \mathbf{v}_{\mathscr{T}}) = 0,$$

which follows from rewriting the first equation in (30). Set $\mathbf{v}_{\mathscr{T}} = \Pi_{\mathscr{T}}\mathbf{v}$ into the previous relation, apply again an integration by parts formula, and invoke the relation (61) to arrive at

$$\begin{aligned}(\nabla \mathbf{\Phi}, \nabla \mathbf{v})_{\mathbf{L}^2(\Omega)} = \langle \mathbf{F}\delta_{\mathbf{z}}, \mathbf{v} - \Pi_{\mathscr{T}}\mathbf{v} \rangle + \sum_{K \in \mathscr{T}} \int_K \mathcal{R}_K \cdot (\mathbf{v} - \Pi_{\mathscr{T}}\mathbf{v}) \\ + \sum_{\gamma \in \mathscr{S}} \int_\gamma \mathcal{J}_\gamma \cdot (\mathbf{v} - \Pi_{\mathscr{T}}\mathbf{v}) =: \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned} \tag{62}$$

In what follows, we control the terms I, II, and III following the arguments developed in [6]. Let us begin with the control of the term I. To accomplish this task, we invoke the local bound of [2, Theorem 4.7], the interpolation error estimate (56), and the stability bound (55) as follows: If $K \in \mathscr{T}$ is such that $\mathbf{z} \in K$, then

$$\begin{aligned} \mathrm{I} &\lesssim |\mathbf{F}| \left(h_K^{\frac{\alpha}{2}-1} \|\mathbf{v} - \Pi_{\mathscr{T}} \mathbf{v}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, K)} + h_K^{\frac{\alpha}{2}} \|\nabla(\mathbf{v} - \Pi_{\mathscr{T}} \mathbf{v})\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, K)} \right) \\ &\lesssim |\mathbf{F}| h_K^{\frac{\alpha}{2}} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)}. \end{aligned}$$

To bound the terms II and III, we invoke Hölder's inequality and the interpolation error estimates (57) and (58) to obtain

$$\begin{aligned} \mathrm{II} &\lesssim \sum_{K \in \mathscr{T}} h_K D_K^{\frac{\alpha}{2}} \|\mathcal{R}_K\|_{\mathbf{L}^2(K)} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)}, \\ \mathrm{III} &\lesssim \sum_{\gamma \in \mathscr{S}} h_K^{\frac{1}{2}} D_K^{\frac{\alpha}{2}} \|\mathcal{J}_\gamma\|_{\mathbf{L}^2(\gamma)} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)}. \end{aligned}$$

Having bounded the terms I, II, and III, we invoke the inf-sup condition (16) and the identity (62) to obtain an estimate for $\|\nabla \mathbf{\Phi}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha}, \Omega)}$:

$$\begin{aligned} \|\nabla \mathbf{\Phi}\|^2_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha}, \Omega)} &\lesssim \left[\sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \Omega)} \frac{(\nabla \mathbf{\Phi}, \nabla \mathbf{v})}{\|\nabla \mathbf{v}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \Omega)}} \right]^2 \\ &\lesssim \sum_{K \in \mathscr{T}} \left(h_K D_K^{\alpha} \|\mathcal{J}_\gamma\|^2_{\mathbf{L}^2(\partial K \setminus \partial \Omega)} + h_K^2 D_K^{\alpha} \|\mathcal{R}_K\|^2_{\mathbf{L}^2(K)} + h_K^{\alpha} |\mathbf{F}|^2 \#(\{\mathbf{z}\} \cap K) \right), \end{aligned} \tag{63}$$

upon utilizing a finite overlapping property of stars, which guarantees that

$$\left[\sum_{K \in \mathscr{T}} \|\nabla \mathbf{v}\|^2_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)} \right]^{\frac{1}{2}} \lesssim \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \Omega)}.$$

Consequently, we have $\|\nabla \mathbf{\Phi}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha}, \Omega)} \lesssim \mathcal{E}_\alpha(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}; \mathscr{T})$.

Step 2: Let $\psi \in L^2(\mathsf{d}_{\mathbf{z}}^{\alpha}, \Omega)$. A basic computation reveals that the function $\tilde{q} := \mathsf{d}_{\mathbf{z}}^{\alpha} \psi \in L^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \Omega)$. Define $q = \tilde{q} + c$, where $c \in \mathbb{R}$ is such that $q \in L^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \Omega)/\mathbb{R}$. Setting q in the second equation of problem (39) yields

$$\begin{aligned} \|\psi\|^2_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha}, \Omega)} &= (\psi, q)_{L^2(\Omega)} = b_+(\mathbf{e_u}, q) \\ &= b_+(\mathbf{e_u}, \mathsf{d}_{\mathbf{z}}^{\alpha} \psi) = -b_+(\mathbf{u}_{\mathscr{T}}, \mathsf{d}_{\mathbf{z}}^{\alpha} \psi) \leq \|\mathrm{div}\ \mathbf{u}_{\mathscr{T}}\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha}, \Omega)} \|\psi\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha}, \Omega)}, \end{aligned}$$

upon utilizing that $\int_\Omega \psi = 0$ and $\int_\Omega \mathrm{div}\ \mathbf{e_u} = 0$. We have thus obtained the estimate $\|\psi\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha}, \Omega)} \leq \|\mathrm{div}\ \mathbf{u}_{\mathscr{T}}\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha}, \Omega)}$

Step 3: The desired estimate (59) thus follows from (49) and the estimates derived in steps 1 and 2. This concludes the proof. ∎

6.6. Local efficiency bounds. To derive efficiency bounds for the local indicator $\mathcal{E}_\alpha(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}; K)$, defined in (53), we utilize standard residual estimation techniques but on the basis of suitable bubble functions, whose construction we owe to [2].

Given $K \in \mathscr{T}$, we introduce an element bubble function φ_K which satisfies the following properties: $0 \leq \varphi_K \leq 1$,

$$\varphi_K(\mathbf{z}) = 0, \qquad |K| \lesssim \int_K \varphi_K, \qquad \|\nabla \varphi_K\|_{\mathbf{L}^\infty(R_K)} \lesssim h_K^{-1}, \tag{64}$$

and there exists a simplex $K^* \subset K$ such that $R_K := \text{supp}(\varphi_K) \subset K^*$. Notice that, since φ_K satisfies (64), we have the bound

$$\|\theta\|_{L^2(R_K)} \lesssim \|\varphi_K^{\frac{1}{2}}\theta\|_{L^2(R_K)} \quad \forall \theta \in \mathbb{P}_5(R_K). \tag{65}$$

Second, given $\gamma \in \mathscr{S}$, we introduce a bubble function φ_γ that satisfies the following properties: $0 \leq \varphi_\gamma \leq 1$,

$$\varphi_\gamma(\mathbf{z}) = 0, \qquad |\gamma| \lesssim \int_\gamma \varphi_\gamma, \qquad \|\nabla \varphi_\gamma\|_{\mathbf{L}^\infty(R_\gamma)} \lesssim h_\gamma^{-1}, \tag{66}$$

and $R_\gamma := \text{supp}(\varphi_\gamma)$ is such that, if $\mathcal{N}_\gamma = \{K, K'\}$, there are simplices $K_* \subset K$ and $K'_* \subset K'$ such that $R_\gamma \subset K_* \cup K'_* \subset K \cup K'$.

The following estimates are instrumental in the efficiency analysis that we will perform.

Proposition 15 (estimates for bubble functions). *Let $K \in \mathscr{T}$ and φ_K be the bubble function that satisfies (64). If $\alpha \in (0,2)$, then*

$$h_K \|\nabla(\theta \varphi_K)\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, K)} \lesssim D_K^{-\frac{\alpha}{2}} \|\theta\|_{L^2(K)} \qquad \forall \theta \in \mathbb{P}_5(K). \tag{67}$$

Let $\gamma \in \mathscr{S}$ and φ_γ be the bubble function that satisfies (66). If $\alpha \in (0,2)$, then

$$h_K^{\frac{1}{2}} \|\nabla(\theta \varphi_\gamma)\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_\gamma)} \lesssim D_K^{-\frac{\alpha}{2}} \|\theta\|_{L^2(\gamma)} \qquad \forall \theta \in \mathbb{P}_3(\gamma), \tag{68}$$

where θ is extended to the elements that comprise $\mathcal{N}_\gamma$ as a constant along the direction normal to γ.

Proof. See [2, Lemma 5.2]. ∎

We are now ready to derive efficiency properties for the local error indicator $\mathcal{E}_\alpha(\mathbf{u}_\mathscr{T}, \mathsf{p}_\mathscr{T}; K)$ defined in (53).

Theorem 16 (local efficiency). *Let $(\mathbf{u}, \mathsf{p}) \in \mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^\alpha, \Omega) \times L^2(\mathsf{d}_{\mathbf{z}}^\alpha, \Omega)/\mathbb{R}$ be the solution to the continuous problem (14), and let $(\mathbf{u}_\mathscr{T}, \mathsf{p}_\mathscr{T}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ be its finite element approximation obtained as the solution to (30). Assume that the intensity of the forcing term $\mathbf{F}$ is sufficiently small so that (48) holds. Then*

$$\begin{aligned} \mathcal{E}_\alpha(\mathbf{u}_\mathscr{T}, \mathsf{p}_\mathscr{T}; K)^2 &\lesssim \|\nabla \mathbf{e}_\mathbf{u}\|^2_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K)} + \|e_\mathsf{p}\|^2_{L^2(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K)} + h_K^2 \|\mathbf{e}_\mathbf{u}\|^2_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K)} \\ &+ (1 + h_K^2)\|\mathbf{e}_\mathbf{u}\|^2_{\mathbf{L}^4(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K)} + \sum_{K' \in \mathcal{N}_K^*} h_{K'}^2 D_{K'}^\alpha \||\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T} - \mathbf{\Pi}_{K'}(|\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T})\|^2_{\mathbf{L}^2(K')}, \end{aligned} \tag{69}$$

where $\mathcal{N}_K^*$ is defined in (23), $\mathbf{\Pi}_K$ is the orthogonal projection operator onto $[\mathbb{P}_0(K)]^2$, and the hidden constant is independent of $(\mathbf{u}, \mathsf{p})$ and $(\mathbf{u}_\mathscr{T}, \mathsf{p}_\mathscr{T})$, the size of the elements in the mesh $\mathscr{T}$, and $\#\mathscr{T}$.

Proof. We estimate each contribution in (53) separately. We proceed in five steps.

Step 1: We begin the proof by bounding, for $K \in \mathscr{T}$, the term $h_K^2 D_K^\alpha \|\mathcal{R}_K\|^2_{\mathbf{L}^2(K)}$. To accomplish this task, we define

$$\tilde{\mathcal{R}}_K := (\Delta \mathbf{u}_\mathscr{T} - \mathbf{u}_\mathscr{T} - (\mathbf{u}_\mathscr{T} \cdot \nabla)\mathbf{u}_\mathscr{T} - \mathbf{u}_\mathscr{T} \text{div } \mathbf{u}_\mathscr{T} - \Pi_K(|\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T}) - \nabla \mathsf{p}_\mathscr{T})|_K.$$

Notice that $\tilde{\mathcal{R}}_K = \mathcal{R}_K + |\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T} - \mathbf{\Pi}_K(|\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T})$. A simple application of the triangle inequality yields a first estimate for $\|\mathcal{R}_K\|_{\mathbf{L}^2(K)}$:

$$\|\mathcal{R}_K\|_{\mathbf{L}^2(K)} \leq \|\tilde{\mathcal{R}}_K\|_{\mathbf{L}^2(K)} + \|\mathbf{\Pi}_K(|\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T}) - |\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T}\|_{\mathbf{L}^2(K)}. \tag{70}$$

It thus suffices to control $\|\tilde{\mathcal{R}}_K\|_{\mathbf{L}^2(K)}$. To do this, we define $\boldsymbol{\phi}_K := \varphi_K \tilde{\mathcal{R}}_K$, and observe that (65) guarantees the bound

$$\|\tilde{\mathcal{R}}_K\|^2_{\mathbf{L}^2(K)} \lesssim \int_{R_K} |\tilde{\mathcal{R}}_K|^2 \varphi_K = \int_K \tilde{\mathcal{R}}_K \cdot \boldsymbol{\phi}_K. \tag{71}$$

Let us now utilize that $\varphi_K(\mathbf{z}) = 0$ to immediately deduce the relations $\boldsymbol{\phi}_K(\mathbf{z}) = \varphi_K(\mathbf{z})\tilde{\mathcal{R}}_K(\mathbf{z}) = \mathbf{0}$. We thus set $\mathbf{v} = \boldsymbol{\phi}_K$ as a test function in identity (61) and utilize that $\boldsymbol{\phi}_K|_\gamma = \mathbf{0}$, for every $\gamma \in \mathscr{S}_K$, to arrive at

$$\int_K \tilde{\mathcal{R}}_K \cdot \boldsymbol{\phi}_K = (\nabla \boldsymbol{\Phi}, \nabla \boldsymbol{\phi}_K)_{\mathbf{L}^2(\Omega)} + \int_K (|\mathbf{u}_{\mathscr{T}}|\mathbf{u}_{\mathscr{T}} - \boldsymbol{\Pi}_K(|\mathbf{u}_{\mathscr{T}}|\mathbf{u}_{\mathscr{T}})) \cdot \boldsymbol{\phi}_K. \tag{72}$$

We now control $|(\nabla \boldsymbol{\Phi}, \nabla \boldsymbol{\phi}_K)_{\mathbf{L}^2(K)}|$. To accomplish this task, we set $\mathbf{v} = \boldsymbol{\phi}_K$ as a test function in the first equation of problem (39) and utilize the property $\operatorname{supp} \boldsymbol{\phi}_K \subset K$ and Hölder's inequality to obtain the bound

$$\begin{aligned} |(\nabla \boldsymbol{\Phi}, \nabla \boldsymbol{\phi}_K)_{\mathbf{L}^2(K)}| &\lesssim \|\nabla \mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},K)} \|\nabla \boldsymbol{\phi}_K\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha},K)} \\ + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},K)} \|\boldsymbol{\phi}_K\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha},K)} &+ \|e_{\mathsf{p}}\|_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha},K)} \|\nabla \boldsymbol{\phi}_K\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha},K)} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^4(\mathsf{d}_{\mathbf{z}}^{\alpha},K)} \\ \cdot \left[\|\mathbf{u}\|_{\mathbf{L}^4(\mathsf{d}_{\mathbf{z}}^{\alpha},K)} + \|\mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^4(\mathsf{d}_{\mathbf{z}}^{\alpha},K)}\right] &\left[\|\nabla \boldsymbol{\phi}_K\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha},K)} + \|\boldsymbol{\phi}_K\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha},K)}\right]. \end{aligned} \tag{73}$$

We now notice that, in view of [24, Theorem 1.3], we have the bounds: $\|\mathbf{u}\|_{\mathbf{L}^4(\mathsf{d}_{\mathbf{z}}^{\alpha},K)} \lesssim \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)}$ and $\|\mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^4(\mathsf{d}_{\mathbf{z}}^{\alpha},K)} \lesssim \|\nabla \mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},\Omega)}$. On the other hand, the estimate (67) and the estimate (5.6) in [2] allow us to conclude

$$\|\nabla \boldsymbol{\phi}_K\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha},K)} \lesssim h_K^{-1} D_K^{-\frac{\alpha}{2}} \|\tilde{\mathcal{R}}_K\|_{\mathbf{L}^2(K)}, \qquad \|\boldsymbol{\phi}_K\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha},K)} \lesssim D_K^{-\frac{\alpha}{2}} \|\tilde{\mathcal{R}}_K\|_{\mathbf{L}^2(K)},$$

respectively. With these bounds at hand, estimates (71) and (73) combined with the the relation (72) and the smallness assumption (48) allow us to deduce the local a posteriori bound

$$\begin{aligned} h_K^2 D_K^{\alpha} \|\tilde{\mathcal{R}}_K\|^2_{\mathbf{L}^2(K)} &\lesssim \|\nabla \mathbf{e}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},K)} + \|e_{\mathsf{p}}\|^2_{L^2(\mathsf{d}_{\mathbf{z}}^{\alpha},K)} \\ + h_K^2 \|\mathbf{e}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{\alpha},K)} + (1 + h_K^2) \|\mathbf{e}_{\mathbf{u}}\|^2_{\mathbf{L}^4(\mathsf{d}_{\mathbf{z}}^{\alpha},K)} &+ h_K^2 D_K^{\alpha} \||\mathbf{u}_{\mathscr{T}}|\mathbf{u}_{\mathscr{T}} - \boldsymbol{\Pi}_K(|\mathbf{u}_{\mathscr{T}}|\mathbf{u}_{\mathscr{T}})\|^2_{\mathbf{L}^2(K)}. \end{aligned} \tag{74}$$

The desired estimate for the term $h_K^2 D_K^{\alpha} \|\mathcal{R}_K\|^2_{\mathbf{L}^2(K)}$ thus follows from (70) and (74).

Step 2: Let $K \in \mathscr{T}$ and $\gamma \in \mathscr{S}_K$. In what follows we bound $h_K D_K^{\alpha} \|\mathcal{J}_\gamma\|^2_{\mathbf{L}^2(\gamma)}$. To accomplish this task, we proceed by using similar arguments to the ones that lead to (74) but now utilizing the bubble function φ_γ. Define the function $\boldsymbol{\Lambda}_\gamma = \varphi_\gamma \mathcal{J}_\gamma$, where $\mathcal{J}_\gamma$ and φ_γ are defined in (52) and (66), respectively. We utilize the construction of the bubble function φ_γ to deduce the bound

$$\|\mathcal{J}_\gamma\|^2_{\mathbf{L}^2(\gamma)} \lesssim \int_\gamma |\mathcal{J}_\gamma|^2 \varphi_\gamma = \int_\gamma \mathcal{J}_\gamma \cdot \boldsymbol{\Lambda}_\gamma. \tag{75}$$

Now, set $\mathbf{v} = \boldsymbol{\Lambda}_\gamma$ in the identity (61) and use that $\boldsymbol{\Lambda}_\gamma(\mathbf{z}) = 0$ and that $\operatorname{supp}(\boldsymbol{\Lambda}_\gamma) \subseteq R_\gamma = \operatorname{supp}(\varphi_\gamma) \subset K_* \cup K'_* \subset \cup\{K' : K' \in \mathcal{N}_\gamma\}$ to arrive at

$$\begin{aligned} \int_\gamma \mathcal{J}_\gamma \cdot \boldsymbol{\Lambda}_\gamma = (\nabla \boldsymbol{\Phi}, \nabla \boldsymbol{\Lambda}_\gamma)_{\mathbf{L}^2(\Omega)} &- \sum_{K' \in \mathcal{N}_\gamma} \int_{K'} \tilde{\mathcal{R}}_{K'} \cdot \boldsymbol{\Lambda}_\gamma \\ &+ \sum_{K' \in \mathcal{N}_\gamma} \int_{K'} (|\mathbf{u}_{\mathscr{T}}|\mathbf{u}_{\mathscr{T}} - \boldsymbol{\Pi}_{K'}(|\mathbf{u}_{\mathscr{T}}|\mathbf{u}_{\mathscr{T}})) \cdot \boldsymbol{\Lambda}_\gamma. \end{aligned} \tag{76}$$

In view of this identity, similar arguments to those developed to obtain (73) yield

$$\begin{aligned}(77)\quad \int_\gamma \mathcal{J}_\gamma \cdot \mathbf{\Lambda}_\gamma \leq& \left|(\nabla \mathbf{\Phi}, \nabla \mathbf{\Lambda}_\gamma)_{\mathbf{L}^2(\mathcal{N}_\gamma)}\right| \\ &+ \sum_{K'\in\mathcal{N}_\gamma} \left(\|\tilde{\mathcal{R}}_{K'}\|_{\mathbf{L}^2(K')} + \||\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T} - \mathbf{\Pi}_{K'}(|\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T})\|_{\mathbf{L}^2(K')}\right)\|\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(K')} \\ \lesssim& \sum_{K'\in\mathcal{N}_\gamma} \left(\|\nabla \mathbf{e_u}\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^\alpha,K')} + \|e_\mathsf{p}\|_{L^2(\mathsf{d}_\mathbf{z}^\alpha,K')}\right)\|\nabla\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^{-\alpha},K')} + \|\mathbf{e_u}\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^\alpha,K')} \\ &\cdot \|\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^{-\alpha},K')} + \|\mathbf{e_u}\|_{\mathbf{L}^4(\mathsf{d}_\mathbf{z}^\alpha,K')}\left(\|\nabla\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^{-\alpha},K')} + \|\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^{-\alpha},K')}\right) \\ &+ \sum_{K'\in\mathcal{N}_\gamma} \left(\|\tilde{\mathcal{R}}_{K'}\|_{\mathbf{L}^2(K')} + \||\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T} - \mathbf{\Pi}_{K'}(|\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T})\|_{\mathbf{L}^2(K')}\right)\|\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(K')}.\end{aligned}$$

The terms $\|\nabla\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^{-\alpha},K')}$ and $\|\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^{-\alpha},K')}$ can be controlled in view of (68) and [2, estimate (5.8)], respectively. In fact, we have

$$(78)\quad \|\nabla\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^{-\alpha},K')} \lesssim h_{K'}^{-\frac{1}{2}} D_{K'}^{-\frac{\alpha}{2}}\|\mathcal{J}_\gamma\|_{\mathbf{L}^2(\gamma)},\; \|\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^{-\alpha},K')} \lesssim h_{K'}^{\frac{1}{2}} D_{K'}^{-\frac{\alpha}{2}}\|\mathcal{J}_\gamma\|_{\mathbf{L}^2(\gamma)}.$$

We also observe that $\|\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(K')} \approx |K'|^{\frac{1}{2}}|\gamma|^{-\frac{1}{2}}\|\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(\gamma)} \approx h_{K'}^{\frac{1}{2}}\|\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(\gamma)}$, as a consequence of $|K'| \approx h_{K'}^2$, $|\gamma| \approx h_{K'}$, and standard arguments. With these ingredients at hand, the inequalities in (77) allow us to arrive at

$$\begin{aligned}\int_\gamma \mathcal{J}_\gamma \cdot \mathbf{\Lambda}_\gamma \lesssim& \sum_{K'\in\mathcal{N}_\gamma} \left(\|\nabla \mathbf{e_u}\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^\alpha,K')} + \|e_\mathsf{p}\|_{L^2(\mathsf{d}_\mathbf{z}^\alpha,K')}\right) h_{K'}^{-\frac{1}{2}} D_{K'}^{-\frac{\alpha}{2}}\|\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(\gamma)} \\ &+ \sum_{K'\in\mathcal{N}_\gamma}\left[\|\mathbf{e_u}\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^\alpha,K')} h_{K'}^{\frac{1}{2}} + \|\mathbf{e_u}\|_{\mathbf{L}^4(\mathsf{d}_\mathbf{z}^\alpha,K')}(h_{K'}^{-\frac{1}{2}} + h_{K'}^{\frac{1}{2}})\right] D_{K'}^{-\frac{\alpha}{2}}\|\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(\gamma)} \\ &+ \sum_{K'\in\mathcal{N}_\gamma} h_{K'}^{\frac{1}{2}}(\|\tilde{\mathcal{R}}_{K'}\|_{\mathbf{L}^2(K')} + \||\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T} - \mathbf{\Pi}_{K'}(|\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T})\|_{\mathbf{L}^2(K')})\|\mathbf{\Lambda}_\gamma\|_{\mathbf{L}^2(\gamma)}.\end{aligned}$$

The desired control for the term $h_K D_K^\alpha \|\mathcal{J}_\gamma\|^2_{\mathbf{L}^2(\gamma)}$ follows from replacing the previous estimate in (75):

$$\begin{aligned}(79)\quad h_K D_K^\alpha \|\mathcal{J}_\gamma\|^2_{\mathbf{L}^2(\gamma)} \lesssim& \sum_{K'\in\mathcal{N}_\gamma}\Big(\|\nabla \mathbf{e_u}\|^2_{\mathbf{L}(\mathsf{d}_\mathbf{z}^\alpha,K')} + \|e_\mathsf{p}\|^2_{L^2(\mathsf{d}_\mathbf{z}^\alpha,K')} + h_{K'}^2\|\mathbf{e_u}\|^2_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^\alpha,K')} \\ &+ (1+h_{K'}^2)\|\mathbf{e_u}\|^2_{\mathbf{L}^4(\mathsf{d}_\mathbf{z}^\alpha,K')} + h_{K'}^2 D_{K'}^\alpha \||\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T} - \mathbf{\Pi}_{K'}(|\mathbf{u}_\mathscr{T}|\mathbf{u}_\mathscr{T})\|^2_{\mathbf{L}^2(K')}\Big).\end{aligned}$$

Step 3: Let $K \in \mathscr{T}$. The control of $\|\text{div}\, \mathbf{u}_\mathscr{T}\|_{L^2(\mathsf{d}_\mathbf{z}^\alpha,K)}$ follows easily from the mass conservation equation $\text{div}\, \mathbf{u} = 0$. In fact,

$$(80)\qquad \|\text{div}\, \mathbf{u}_\mathscr{T}\|_{L^2(\mathsf{d}_\mathbf{z}^\alpha,K)} = \|\text{div}\, \mathbf{e_u}\|_{L^2(\mathsf{d}_\mathbf{z}^\alpha,K)} \lesssim \|\nabla \mathbf{e_u}\|_{\mathbf{L}^2(\mathsf{d}_\mathbf{z}^\alpha,K)}.$$

Step 4: Let $K \in \mathscr{T}$. We now control the term associated with the singular source $\delta_\mathbf{z}$. Let us first notice that if $K \cap \{\mathbf{z}\} = \emptyset$, then the desired estimate (69) follows directly from the estimates derived in the previous three steps. If, on the other hand, $K \cap \{\mathbf{z}\} = \{\mathbf{z}\}$, we must obtain a bound for $h_K^\alpha |\mathbf{F}|^2$ in (53). To accomplish this task, we invoke the smooth function μ introduced in the proof of [2, Theorem 5.3], which is such that

$$\mu(\mathbf{z}) = 1, \qquad \|\mu\|_{L^\infty(\Omega)} = 1, \qquad \|\nabla \eta\|_{\mathbf{L}^\infty(\Omega)} \lesssim h_K^{-1}, \qquad \text{supp}(\mu) \subset \mathcal{N}_K^*. \tag{81}$$

With μ at hand, we define $\mathbf{v}_\mu := \mathbf{F}\mu \in \mathbf{H}_0^1(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \Omega)$. Let us now invoke the fact that $(\mathbf{u}, \mathsf{p})$ and $(\mathbf{\Phi}, \psi)$ solve problems (14) and (39), respectively, to obtain

$$\begin{aligned}|\mathbf{F}|^2 &= \langle \mathbf{F}\delta_{\mathbf{z}}, \mathbf{v}_\mu\rangle = a(\mathbf{u}, \mathbf{v}_\mu) + b_-(\mathbf{v}_\mu, \mathsf{p}) + c(\mathbf{u}, \mathbf{u}; \mathbf{v}_\mu) + d(\mathbf{u}, \mathbf{u}; \mathbf{v}_\mu) \\ &= (\nabla \mathbf{\Phi}, \nabla \mathbf{v}_\mu)_{\mathbf{L}^2(\Omega)} + a(\mathbf{u}_{\mathscr{T}}, \mathbf{v}_\mu) + b_-(\mathbf{v}_\mu, \mathsf{p}_{\mathscr{T}}) + c(\mathbf{u}_{\mathscr{T}}, \mathbf{u}_{\mathscr{T}}; \mathbf{v}_\mu) + d(\mathbf{u}_{\mathscr{T}}, \mathbf{u}_{\mathscr{T}}; \mathbf{v}_\mu).\end{aligned} \tag{82}$$

Since $\text{supp}(\mu) \subset \mathcal{N}_K^*$, similar arguments to the ones utilized to obtain (73) yields

$$\begin{aligned}|(\nabla \mathbf{\Phi}, \nabla \mathbf{v}_\mu)_{\mathbf{L}^2(\Omega)}| &\lesssim \|\mathbf{e_u}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K^*)} \|\mathbf{v}_\mu\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)} \\ &+ \left[\|\nabla \mathbf{e_u}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K^*)} + \|e_{\mathsf{p}}\|_{L^2(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K^*)}\right] \|\nabla \mathbf{v}_\mu\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)} \\ &\qquad \|\mathbf{e_u}\|_{\mathbf{L}^4(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K^*)} \left[\|\nabla \mathbf{v}_\mu\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)} + \|\mathbf{v}_\mu\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)}\right].\end{aligned} \tag{83}$$

In view of the identity (82), the bound (83), and basic estimates on the basis of an integrations by parts arguments, we obtain

$$\begin{aligned}|\mathbf{F}|^2 &\lesssim \left[\|\nabla \mathbf{e_u}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K^*)} + \|e_{\mathsf{p}}\|_{L^2(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K^*)}\right] \|\nabla \mathbf{v}_\mu\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)} + \|\mathbf{e_u}\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K^*)} \\ &\quad \cdot \|\mathbf{v}_\mu\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)} + \|\mathbf{e_u}\|_{\mathbf{L}^4(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K^*)} \left[\|\nabla \mathbf{v}_\mu\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)} + \|\mathbf{v}_\mu\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)}\right] \\ &+ \sum_{K' \in \mathscr{T} : K' \subset \mathcal{N}_K^*} \left(\|\tilde{\mathcal{R}}_{K'}\|_{\mathbf{L}^2(K')} + \||\mathbf{u}_{\mathscr{T}}|\mathbf{u}_{\mathscr{T}} - \mathbf{\Pi}_{K'}(|\mathbf{u}_{\mathscr{T}}|\mathbf{u}_{\mathscr{T}})\|_{\mathbf{L}^2(K')}\right) \|\mathbf{v}_\mu\|_{\mathbf{L}^2(K')} \\ &\qquad + \sum_{K' \in \mathscr{T} : K' \subset \mathcal{N}_K^*} \sum_{\gamma \in \mathscr{S}_{K'} : \gamma \not\subset \partial \mathcal{N}_K} \|\mathcal{J}_\gamma\|_{\mathbf{L}^2(\gamma)} \|\mathbf{v}_\mu\|_{\mathbf{L}^2(\gamma)}.\end{aligned}$$

We now use the estimates

$$\|\mu\|_{L^2(\gamma)} \lesssim h_K^{\frac{1}{2}} \quad \|\mu\|_{L^2(\mathcal{N}_K^*)} \lesssim h_K, \quad \|\mu\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)} \lesssim h_K^{1-\frac{\alpha}{2}},$$

and $\|\nabla \mu\|_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^{-\alpha}, \mathcal{N}_K^*)} \lesssim h_K^{-\frac{\alpha}{2}}$ together with the fact that, since $\mathbf{z} \in K$, we have that $h_K \approx D_K$, to conclude that

$$\begin{aligned}|\mathbf{F}|^2 &\lesssim h_K^{-\frac{\alpha}{2}} |\mathbf{F}| \Bigg[\|\nabla \mathbf{e_u}\|^2_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K^*)} + \|e_{\mathsf{p}}\|^2_{L^2(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K^*)} + h_K^2 \|\mathbf{e_u}\|^2_{\mathbf{L}^2(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K^*)} \\ &+ (1 + h_K^2)\|\mathbf{e_u}\|^2_{\mathbf{L}^4(\mathsf{d}_{\mathbf{z}}^\alpha, \mathcal{N}_K^*)}\Bigg]^{\frac{1}{2}} + h_K^{-\frac{\alpha}{2}} |\mathbf{F}| \Bigg[\sum_{K' \in \mathscr{T} : K' \subset \mathcal{N}_K^*} \sum_{\gamma \in \mathscr{S}_{K'} : \gamma \not\subset \partial \mathcal{N}_K^*} h_{K'}^{\frac{1}{2}} D_{K'}^{\frac{\alpha}{2}} \|\mathcal{J}_\gamma\|_{\mathbf{L}^2(\gamma)} \\ &\quad + \sum_{K' \in \mathscr{T} : K' \subset \mathcal{N}_K^*} h_{K'} D_{K'}^{\frac{\alpha}{2}} \left(\|\tilde{\mathcal{R}}_{K'}\|_{\mathbf{L}^2(K')} + \||\mathbf{u}_{\mathscr{T}}|\mathbf{u}_{\mathscr{T}} - \mathbf{\Pi}_{K'}(|\mathbf{u}_{\mathscr{T}}|\mathbf{u}_{\mathscr{T}})\|_{\mathbf{L}^2(K')}\right)\Bigg].\end{aligned} \tag{84}$$

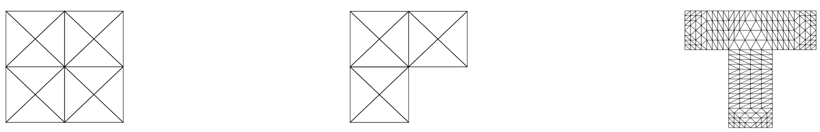
Replacing the estimates (74) and (79) in the previous bound allows us to conclude.

Step 5: Collect the estimates derived in the previous steps, i.e., estimates (74), (79), (80) and (84) to arrive at the desired local efficiency estimate (69). This concludes the proof. ∎

7. Numerical experiments. In this section, we present a series of numerical examples that illustrate the performance of the devised error estimator $\mathcal{E}_\alpha$.

The numerical examples that will be presented have been carried out with the help of a code that we implemented using `C++`. All matrices have been assembled exactly, and the global linear systems were solved using the multifrontal massively parallel sparse direct solver (MUMPS) [7, 8]. The right-hand sides, local indicators, and the error estimator were computed by a quadrature formula which is exact for polynomials of degree 19. To visualize finite element approximations, we have used the open-source application ParaView [3, 9].

For a given partition $\mathscr{T}$, we solve the discrete system (30), within the Taylor–Hood discrete setting (26)–(27), with the help of the iterative strategy described in **Algorithm 1**. Once a discrete solution is obtained, we compute, for each $K \in \mathscr{T}$, the local error indicator $\mathcal{E}_\alpha(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}; K)$, defined in (53), to drive the adaptive procedure described in **Algorithm 2**. A sequence of adaptively refined meshes is thus generated from the initial meshes shown in Figure 1.



(A.1) (A.2) (A.3)

FIG. 1. *The initial meshes used in the adaptive algorithm,* Algorithm *2, when (A.1)* $\Omega = (0,1)^2$*, (A.2)* $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$*, and (A.3)* $\Omega = ((-1.5,1.5) \times (0,1)) \cup ((-0.5,0.5) \times (-2,1))$.

Finally, we define the total number of degrees of freedom as $\mathrm{Ndof} := \dim \mathbf{V}(\mathscr{T}) + \dim \mathcal{P}(\mathscr{T})$.

Algorithm 1 Iterative Scheme.

Input: Initial guess $(\mathbf{u}^0_{\mathscr{T}}, \mathsf{p}^0_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$, interior point $\mathbf{z} \in \Omega$, $\mathbf{F} \in \mathbb{R}^2$, and tol=$10^{-8}$. Set i=1;

1: Find $(\mathbf{u}^i_{\mathscr{T}}, \mathsf{p}^i_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ such that

$$
\begin{aligned}
a(\mathbf{u}^i_{\mathscr{T}}, \mathbf{v}_{\mathscr{T}}) + b_-(\mathbf{v}_{\mathscr{T}}, \mathsf{p}^i_{\mathscr{T}}) + c(\mathbf{u}^{i-1}_{\mathscr{T}}, \mathbf{u}^i_{\mathscr{T}}; \mathbf{v}_{\mathscr{T}}) + d(\mathbf{u}^{i-1}_{\mathscr{T}}, \mathbf{u}^i_{\mathscr{T}}; \mathbf{v}_{\mathscr{T}}) &= \langle \mathbf{F}\delta_{\mathbf{z}}, \mathbf{v}_{\mathscr{T}} \rangle, \\
b_+(\mathbf{u}^i_{\mathscr{T}}, \mathsf{q}_{\mathscr{T}}) &= 0,
\end{aligned}
$$

for all $\mathbf{v}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T})$ and $\mathsf{q}_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})$, respectively.

2: If $|(\mathbf{u}^i_{\mathscr{T}}, \mathsf{p}^i_{\mathscr{T}}) - (\mathbf{u}^{i-1}_{\mathscr{T}}, \mathsf{p}^{i-1}_{\mathscr{T}})| > \mathrm{tol}$, set $i \leftarrow i+1$ and go to step **1**. Otherwise, **return** $(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}) = (\mathbf{u}^i_{\mathscr{T}}, \mathsf{p}^i_{\mathscr{T}})$. Here, $|\cdot|$ denotes the Euclidean norm.

7.1. Convex and non-convex domains. We explore the performance of the devised a posteriori error estimator in problems with homogeneous Dirichlet boundary conditions on convex and non-convex domains.

7.1.1. Convex domain. Let $\Omega = (0,1)^2$, $\mathbf{z} = (0.5,0.5)^{\intercal}$, and $\mathbf{F} = (1,1)^{\intercal}$. We explore the performance of the a posteriori error estimator $\mathcal{E}_\alpha(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}; \mathscr{T})$ when driving the adaptive procedure of **Algorithm 2**. In particular, we investigate the effect of varying the exponent α in the Muckenhoupt weight. To accomplish this task, we consider $\alpha = \{0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75\}$.

Algorithm 2 Adaptive Algorithm.

Input: Initial mesh $\mathscr{T}_0$, interior point $\mathbf{z} \in \Omega$, $\alpha \in (0, 2)$, and $\mathbf{F} \in \mathbb{R}^2$;
1: Solve the discrete problem (30) by using **Algorithm** 1;
2: For each $K \in \mathscr{T}$ compute the local error indicator $\mathcal{E}_\alpha(\mathbf{u}_\mathscr{T}, \mathsf{p}_\mathscr{T}; K)$ defined in (53);
3: Mark an element $K \in \mathscr{T}$ for refinement if;

$$\mathcal{E}_\alpha(\mathbf{u}_\mathscr{T}, \mathsf{p}_\mathscr{T}; K) > \frac{1}{2} \max_{K' \in \mathscr{T}} \mathcal{E}_\alpha(\mathbf{u}_\mathscr{T}, \mathsf{p}_\mathscr{T}; K');$$

4: From step **3**, construct a new mesh, using a longest edge bisection algorithm. Set $i \leftarrow i + 1$ and go to step **1**.

In Figure 2, we report the results obtained for Example 1. We observe that the devised a posteriori error estimator $\mathcal{E}_\alpha$ attains optimal experimental rates of convergence for all the values of the parameter α that we have considered. We also observe that most of the refinement is concentrated around the singular source point.

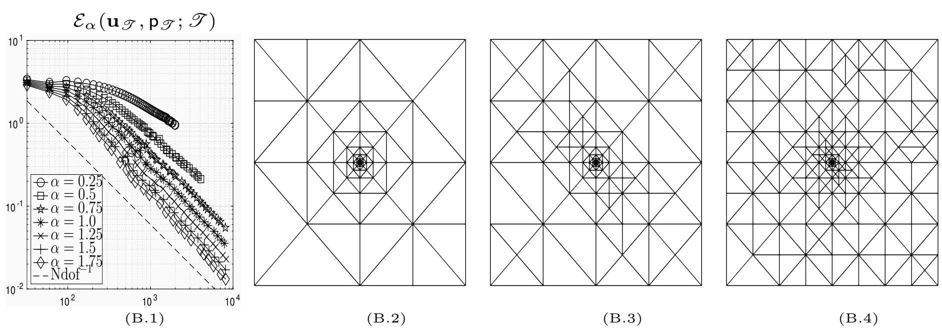


FIG. 2. *Example 1: Experimental rates of convergence for the error estimator* $\mathcal{E}_\alpha(\mathbf{u}_\mathscr{T}, \mathsf{p}_\mathscr{T}; \mathscr{T})$ *considering* $\alpha \in \{0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75\}$ *(B.1) and the meshes obtained after 20 adaptive refinements for* $\alpha = 0.5$ *(156 elements and 85 vertices) (B.2);* $\alpha = 1.0$ *(192 elements and 105 vertices) (B.3); and* $\alpha = 1.5$ *(304 elements and 167 vertices) (B.4).*

7.1.2. Non-convex domain. Let $\Omega = (-1, 1)^2 \setminus [0, 1) \times (-1, 0]$ and $\mathbf{F} = (1, 1)^\intercal$.

In Figure 3, we report the results obtained for Example 2. We observe that optimal experimental rates of convergence are attained for all the values of the parameter α that we have considered: $\alpha \in \{0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75\}$. We also observe that most of the refinement is concentrated around the singular source point and that the geometric singularity is rapidly noticed for values of α such that $\alpha \geq 1$.

7.2. A series of Dirac delta points. We consider $\Omega = ((-1.5, 1.5) \times (0, 1)) \cup ((-0.5, 0.5) \times (-2, 1))$ and go beyond the present theory and consider nonhomogeneous Dirichlet boundary conditions and a series of Dirac delta sources on the right-hand side of the momentum equation:

$$-\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + |\mathbf{u}|\mathbf{u} + \mathbf{u} + \nabla\mathsf{p} = \sum_{\mathbf{z} \in \mathcal{Z}} \mathbf{F}_\mathbf{z}\delta_\mathbf{z} \text{ in } \Omega, \tag{85}$$

where $\mathcal{Z} \subset \Omega$ denotes a finite set with cardinality $\#\mathcal{Z} > 1$ and $\{\mathbf{F}_\mathbf{z}\}_{\mathbf{z} \in \mathcal{Z}} \subset \mathbb{R}^2$. In particular, we consider $\mathbf{F}_\mathbf{z} = (1, 1)^\intercal$ for all $\mathbf{z} \in \mathcal{Z}$. Let us introduce the weight

$$\rho(\mathbf{x}) = \begin{cases} \mathsf{d}_\mathbf{z}^\alpha, & \exists\ \mathbf{z} \in \mathcal{Z} : \ |\mathbf{x} - \mathbf{z}| < \frac{d_\mathcal{Z}}{2}, \\ 1, & |\mathbf{x} - \mathbf{z}| \geq \frac{d_\mathcal{Z}}{2}, \ \forall\ \mathbf{z} \in \mathcal{Z}, \end{cases} \tag{86}$$

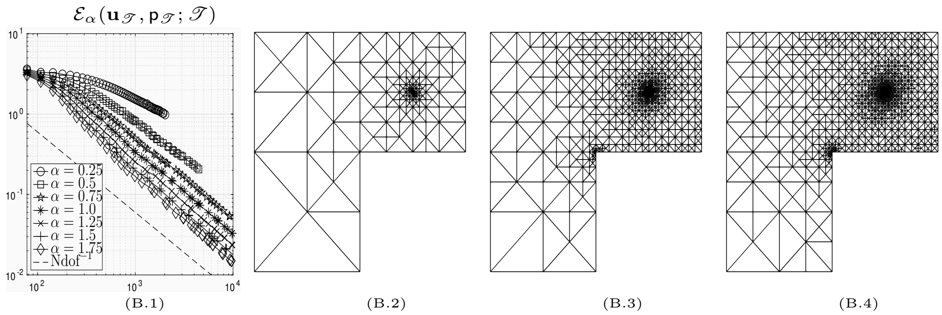


FIG. 3. *Example 2: Experimental rates of convergence for the error estimator* $\mathcal{E}_\alpha(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}; \mathscr{T})$ *considering* $\alpha \in \{0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75\}$ *(B.1) and the meshes obtained after 40 adaptive refinements for* $\alpha = 0.5$ *(534 elements and 280 vertices) (B.2);* $\alpha = 1.0$ *(1917 elements and 994 vertices) (B.3); and* $\alpha = 1.5$ *(2401 elements and 1247 vertices) (B.4).*

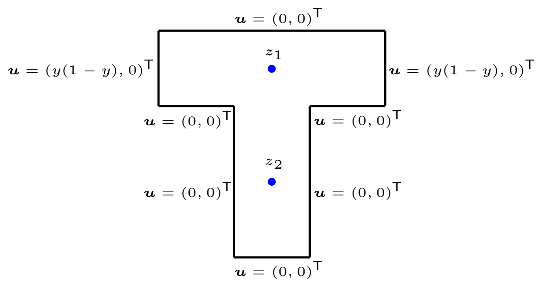


FIG. 4. *Example 3: T–shaped domain with Dirac delta source points located at* $\mathbf{z}_1 = (0, 0.5)$ *and* $\mathbf{z}_2 = (0, -1)$.

where $d_{\mathcal{Z}} = \min\{\text{dist}(\mathcal{Z}, \partial\Omega), \min\{|\mathbf{z} - \mathbf{z}'| : \mathbf{z}, \mathbf{z}' \in \mathcal{Z}, \mathbf{z} \neq \mathbf{z}'\}\}$. With this weight at hand, we modify the definition of the spaces $\mathcal{X}$ and $\mathcal{Y}$ as follows:

$$\mathcal{X} = \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega) \setminus \mathbb{R}, \qquad \mathcal{Y} = \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega) \setminus \mathbb{R}.$$

It can be proved that the weight ρ belongs to the Muckenhoupt class A_2 (see [4, Theorem 6]) and to the restricted class $A_2(\Omega)$. Define $D_{K,\mathcal{Z}} := \min_{\mathbf{z}\in\mathcal{Z}} \{\max_{\mathbf{x}\in K} |\mathbf{x} - \mathbf{z}|\}$. We thus propose the following a posteriori error estimator when the Taylor–Hood scheme is considered:

$$\mathcal{D}_\alpha(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}; \mathscr{T}) := \left(\sum_{K\in\mathscr{T}} \mathcal{D}_\alpha^2(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}; K) \right)^{\frac{1}{2}}, \tag{87}$$

where the local errors indicators are such that

$$\mathcal{D}_\alpha^2(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}; K) := h_K^2 D_{K,\mathcal{Z}}^\alpha \|\mathcal{R}_K\|_{\mathbf{L}^2(K)}^2 + h_K D_{K,\mathcal{Z}}^\alpha \|\mathcal{J}_\gamma\|_{\mathbf{L}^2(\partial K\setminus\partial\Omega)}^2 + \|\text{div } \mathbf{u}_{\mathscr{T}}\|_{L^2(\rho,K)}^2 + \sum_{\mathbf{z}\in\mathcal{Z}\cap K} h_K^\alpha |\mathbf{F}_\mathbf{z}|^2. \tag{88}$$

In Figure 5, we report the results obtained for Example 3. We present the adaptive mesh obtained after 60 adaptive iterations, the streamlines associated with the velocity field $\mathbf{u}_{\mathscr{T}}$, pressure contours, and velocity and pressure elevations. It can be observed that the devised a posteriori error estimator attains an optimal experimental rate of convergence and that most of the refinement is concentrated around the singular sources and the involved geometric singularities.

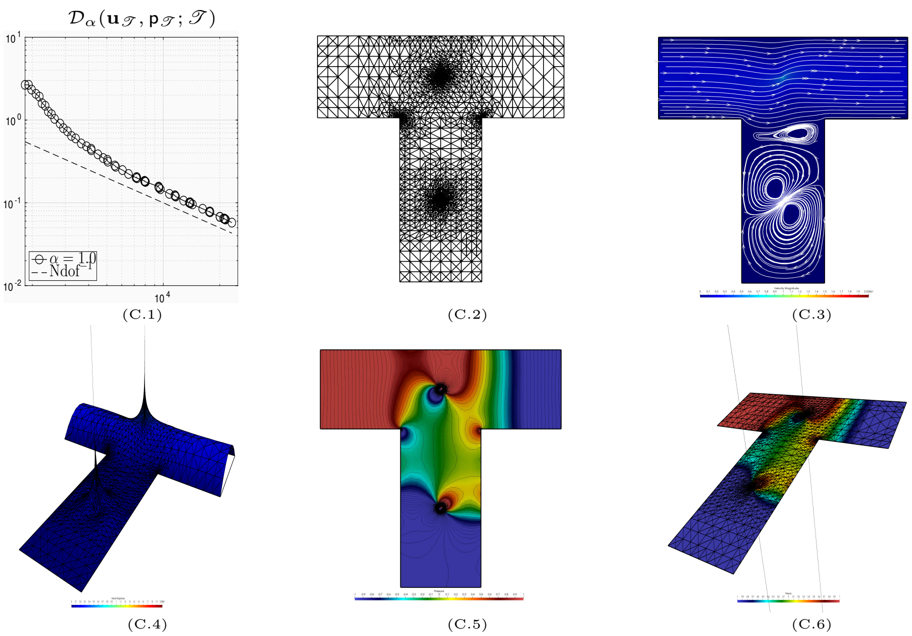


FIG. 5. *Example 3: Experimental rates of convergence for the error estimator* $\mathcal{D}_{1.0}(\mathbf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}}; \mathscr{T})$ *(C.1); the mesh obtained after 60 adaptive refinements (4378 elements and 2263 vertices) (C.2); streamlines for* $|\mathbf{u}_{\mathscr{T}}|$ *(C.3); elevation for* $|\mathbf{u}_{\mathscr{T}}|$ *(C.4); pressure contour (C.5); and elevation for the pressure* $\mathsf{p}_{\mathscr{T}}$ *(C.6).*

Declaration of Competing Interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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