Approximation of higher-order powers of the spectral fractional Laplacian via polyharmonic extension

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Abstract

We use the polyharmonic extension approach to develop a numerical technique for discretizing higher-order powers of the spectral fractional Laplacian $(-\Delta)^s$ with $s \in (1, 2)$.

Keywords: fractional diffusion, nonlocal operators, spectral fractional Laplacian, finite elements.
2020 MSC: 35R11, 65N12, 65N30.

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1. Introduction

In their seminal 2007 work [8], Caffarelli and Silvestre introduced the harmonic extension approach to localize, in $\mathbb{R}^d$, the fractional Laplacian of order between zero and one. While these ideas were to some extent already known [15], it is difficult to overstate the impact of [8]. Any attempt to provide a comprehensive list of extensions and applications would inevitably overlook important contributions. Therefore, we restrict the discussion to noting that this work opened the door to a PDE-based approach for approximating solutions to certain fractional diffusion problems; see [17, 2].

The original work [8] and the numerical developments [17, 2] were limited to fractional powers of order between zero and one. One variant of the harmonic extension approach is the use of *polyharmonic extensions* to study higher-order fractional powers of the Laplacian, that is, $(-\Delta)^s$ for $s > 1$. This was initiated in [19][1] and further developed in [9, 11, 16, 4].

In this work, we inaugurate the numerical analysis of problems involving higher-order fractional Laplacians. We focus on $s \in (1, 2)$ and provide a numerical counterpart to [9, 11, 16, 4]. Specifically, we develop a PDE-based approach for the numerical resolution of the following problem. Assume $d \in \mathbb{N}$, $s \in (1, 2)$, and let $\Omega \subset \mathbb{R}^d$ be a convex polytope. Given $f : \Omega \to \mathbb{R}$, find $u : \bar{\Omega} \to \mathbb{R}$ such that

$$(-\Delta)^s u = f \text{ in } \Omega. \tag{1}$$

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[1] see, however, the comments in [11] about this work.

Here, $(-\Delta)^s$ denotes the *spectral* fractional Dirichlet Laplacian.

In our presentation, we use the context and notation from [17, 2] and draw on several results from these works.

2. Polyharmonic extensions

Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to \mathcal{H}$ be a linear, unbounded, positive, self-adjoint operator with discrete spectrum; that is, there exists $\{(\lambda_k, \varphi_k)\}_{k\in\mathbb{N}} \subset \mathbb{R}_+ \times \mathcal{H}$ such that $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$ and $\mathcal{L}\varphi_k = \lambda_k\varphi_k$ for every $k \in \mathbb{N}$. For $s \in \mathbb{R}$, we define the s-th power of $\mathcal{L}$ in the sense of *spectral theory* as

$$\mathcal{L}^s w = \sum_{k=1}^{\infty} \lambda_k^s W_k \varphi_k, \qquad W_k = (w, \varphi_k)_{\mathcal{H}}, \qquad k \in \mathbb{N}.$$

For $s > 0$, we define the Hilbert space

$$\mathcal{H}^s_{\mathcal{L}} := \mathcal{D}(\mathcal{L}^{s/2}) = \left\{ w = \sum_{k=1}^{\infty} W_k\varphi_k \in \mathcal{H} \ : \ \|w\|^2_{\mathcal{H}^s_{\mathcal{L}}} := (w,w)_{\mathcal{H}^s_{\mathcal{L}}} = \sum_{k=1}^{\infty} \lambda_k^s |W_k|^2 < \infty \right\},$$

where $(v,w)_{\mathcal{H}^s_{\mathcal{L}}} := (\mathcal{L}^{s/2}v, \mathcal{L}^{s/2}w)_{\mathcal{H}}$ for $v, w \in \mathcal{H}^s_{\mathcal{L}}$. We identify $(\mathcal{H}^s_{\mathcal{L}})'$ with $\mathcal{H}^{-s}_{\mathcal{L}} = \{\mathcal{L}^s v : v \in \mathcal{H}^s_{\mathcal{L}}\}$ using the identity $\langle \mathcal{L}^s v, w\rangle = (\mathcal{L}^{s/2}v, \mathcal{L}^{s/2}w)_{\mathcal{H}}$ for $v, w \in \mathcal{H}^s_{\mathcal{L}}$. Therefore, we can write elements $f \in \mathcal{H}^{-s}_{\mathcal{L}}$ as follows:

$$f = \sum_{k=1}^{\infty} F_k\varphi_k : \qquad \|f\|_{\mathcal{H}^{-s}_{\mathcal{L}}} := \left(\sum_{k=1}^{\infty} \lambda_k^{-s}|F_k|^2\right)^{\frac{1}{2}} < \infty.$$

This extends the definition of the norm $\|\cdot\|_{\mathcal{H}^s_{\mathcal{L}}}$ to $s < 0$. Note that $\mathcal{L}^s : \mathcal{H}^s_{\mathcal{L}} \to \mathcal{H}^{-s}_{\mathcal{L}}$ is an isometry with inverse $\mathcal{L}^{-s}$.

We now follow [16, Theorem 1.2] to describe the polyharmonic extension approach for $s \in (1,2)$. Let

$$\mathfrak{b} := 3 - 2s \in (-1, 1).$$

We introduce the weighted space $L^2(y^{\mathfrak{b}}, \mathbb{R}_+; \mathcal{H})$, with $y \mapsto |y|^{\mathfrak{b}} \in A_2(\mathbb{R})$, and define

$$\begin{aligned} H^1_{\mathcal{L}}(y^{\mathfrak{b}}, \mathbb{R}_+; \mathcal{H}) &:= H^1(y^{\mathfrak{b}}, \mathbb{R}_+; \mathcal{H}) \cap L^2(y^{\mathfrak{b}}, \mathbb{R}_+; \mathcal{H}_{\mathcal{L}}), \\ H^2_{\mathcal{L}}(y^{\mathfrak{b}}, \mathbb{R}_+; \mathcal{H}) &:= H^2(y^{\mathfrak{b}}, \mathbb{R}_+; \mathcal{H}) \cap L^2(y^{\mathfrak{b}}, \mathbb{R}_+; \mathcal{H}^2_{\mathcal{L}}). \end{aligned}$$

On the weighted space $L^2(y^{\mathfrak{b}}, \mathbb{R}_+; \mathcal{H})$, we define the (unbounded) operators

$$\mathbb{D}_{\mathfrak{b}} w := -\frac{1}{y^{\mathfrak{b}}}\, \mathrm{d}_y \left(y^{\mathfrak{b}}\, \mathrm{d}_y w\right) = -\,\mathrm{d}^2_{yy} w - \frac{\mathfrak{b}}{y}\, \mathrm{d}_y w, \qquad \mathbb{L}_{\mathfrak{b}} w := \mathbb{D}_{\mathfrak{b}} w + \mathcal{L} w.$$

Theorem 1 (extension)**.** *Let* $s \in (1,2)$ *and* $f \in \mathcal{H}^{-s}_{\mathcal{L}}$*. If* $\mathfrak{u} \in H^2_{\mathcal{L}}(y^{\mathfrak{b}}, \mathbb{R}_+; \mathcal{H})$ *solves*

$$\mathbb{L}^2_{\mathfrak{b}}\mathfrak{u} = 0 \ \textit{in}\ \mathbb{R}_+ \times \mathcal{H}, \qquad \lim_{y\downarrow 0} y^{\mathfrak{b}}\, \mathrm{d}_y \mathfrak{u} = 0 \ \textit{in}\ \mathcal{H}, \qquad -\lim_{y\downarrow 0} y^{\mathfrak{b}}\, \mathrm{d}_y \mathbb{L}_{\mathfrak{b}}\mathfrak{u} = d_s f \ \textit{in}\ \mathcal{H}^{-s}_{\mathcal{L}}, \tag{2}$$

then $u := \mathfrak{u}(0) \in \mathcal{H}^s_{\mathcal{L}}$ *satisfies* $\mathcal{L}^s u = f$*. Here,* $d_s := 2^{\mathfrak{b}}\frac{\Gamma(2-s)}{\Gamma(s)}$*. Moreover, if*

$$c_s := \frac{2^{1-s}}{\Gamma(s)}, \qquad \psi(z) := c_s z^s K_s(z), \qquad \psi_k(y) := \psi(\sqrt{\lambda_k}y), \tag{3}$$

where K_s denotes the modified Bessel function of the second kind of order s, then

$$\mathfrak{u}(y) = \sum_{k=1}^{\infty} U_k \psi_k(y) \varphi_k, \qquad U_k = (u, \varphi_k)_{\mathcal{H}}. \tag{4}$$

2.1. Higher powers of the spectral fractional Laplacian

We now specialize Theorem 1 to the spectral fractional Laplacian. Let $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ be a bounded convex polytope. Set $\mathcal{H} = L^2(\Omega)$ and $\mathcal{L} = -\Delta_x$, the Dirichlet Laplacian. Then, $\mathcal{H}^s_{\mathcal{L}} = \mathbb{H}^s(\Omega)$ and, recalling that $\mathcal{C} \coloneqq \Omega \times (0, \infty)$,

$$L^2(y^{\mathfrak{b}}, \mathbb{R}_+; \mathcal{H}) = L^2(y^{\mathfrak{b}}, \mathbb{R}_+; L^2(\Omega)) = L^2(y^{\mathfrak{b}}, \mathcal{C}), \quad \|w\|^2_{L^2(y^{\mathfrak{b}}, \mathcal{C})} \coloneqq \int_0^\infty y^{\mathfrak{b}} \int_\Omega |w|^2 \, \mathrm{d}x \, \mathrm{d}y.$$

The spaces $\mathcal{H}_{\mathcal{L}}$ and $\mathcal{H}^2_{\mathcal{L}}$ are as follows: $\mathcal{H}_{\mathcal{L}} = H^1_0(\Omega)$ and $\mathcal{H}^2_{\mathcal{L}} = H^2(\Omega) \cap H^1_0(\Omega)$. Define $\partial_L \mathcal{C} \coloneqq \partial\Omega \times (0, \infty)$. Then, [16, Section 5.3]

$$\begin{aligned} H^1_{\mathcal{L}}(y^{\mathfrak{b}}, \mathbb{R}_+; \mathcal{H}) &= \left\{ w \in H^1(y^{\mathfrak{b}}, \mathcal{C}) \ : \ w_{|\partial_L \mathcal{C}} = 0 \right\}, \\ H^2_{\mathcal{L}}(y^{\mathfrak{b}}, \mathbb{R}_+; \mathcal{H}) &= \left\{ w \in H^1_{\mathcal{L}}(y^{\mathfrak{b}}, \mathbb{R}_+; \mathcal{H}) \ : \ y^{\mathfrak{b}} \nabla w \in H^1(y^{-\mathfrak{b}}, \mathcal{C}) \right\}. \end{aligned}$$

Note that $\mathbb{L}_{\mathfrak{b}} w = \mathbb{D}_{\mathfrak{b}} w + \mathcal{L} w = -\,\mathrm{d}^2_{yy} w - \mathfrak{b} y^{-1} \,\mathrm{d}_y w - \Delta_x w = -y^{-\mathfrak{b}} \mathrm{div}(y^{\mathfrak{b}} \nabla w)$. Define

$$H^2_{Lb}(y^{\mathfrak{b}}, \mathcal{C}) \coloneqq \left\{ w \in H^2(y^{\mathfrak{b}}, \mathcal{C}) \ : \ \lim_{y \downarrow 0} y^{\mathfrak{b}} \partial_y w = 0, \ w_{|\partial_L \mathcal{C}} = \partial_{\mathbf{n}} w_{|\partial_L \mathcal{C}} = 0 \right\}.$$

We endow this space with the norm [16, Section 5.3]

$$\|w\|_{H^2_{Lb}(y^{\mathfrak{b}}, \mathcal{C})} \coloneqq \|\mathbb{L}_{\mathfrak{b}} w\|_{L^2(y^{\mathfrak{b}}, \mathcal{C})} = \left(\int_0^\infty \int_\Omega y^{-\mathfrak{b}} |\mathrm{div}(y^{\mathfrak{b}} \nabla w)|^2 \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{2}}.$$

A weak formulation for (2) is as follows: find $\mathfrak{u} \in H^2_{Lb}(y^{\mathfrak{b}}, \mathcal{C})$ such that

$$\int_{\mathcal{C}} y^{\mathfrak{b}} \mathbb{L}_{\mathfrak{b}} \mathfrak{u} \mathbb{L}_{\mathfrak{b}} v \, \mathrm{d}x \, \mathrm{d}y = d_s \langle f, \mathrm{tr}_\Omega \, v \rangle_{\mathcal{H}^{-s}_{\mathcal{L}}, \mathcal{H}^s_{\mathcal{L}}} \quad \forall v \in H^2_{Lb}(y^{\mathfrak{b}}, \mathcal{C}). \tag{5}$$

Theorem 1 then shows that $u \coloneqq \mathrm{tr}_\Omega \, \mathfrak{u} = (-\Delta)^{-s} f$.

3. Regularity and truncation

We now justify truncating $\mathcal{C}$ to $\mathcal{C}_{\mathcal{Y}} \coloneqq \Omega \times (0, \mathcal{Y})$ for $\mathcal{Y} > 0$. This relies on (4), which coincides with [17, (2.24)] and [2, (4.1)], taking into account that now $s \in (1, 2)$.

Proposition 1 (exponential decay)**.** *For every $a, b > 0$, $a < b$, the function* (4) *satisfies*

$$\|\mathbb{L}_{\mathfrak{b}} \mathfrak{u}\|_{L^2(y^{\mathfrak{b}}, \Omega \times (a,b))} \le \sup_{k=1}^{\infty} I_k(a,b) \|f\|_{\mathbb{H}^{-s}(\Omega)}, \quad I_k(a,b)^2 \coloneqq 4 c_s^2 \int_{\sqrt{\lambda_k} a}^{\sqrt{\lambda_k} b} z K^2_{s-1}(z) \, \mathrm{d}z. \tag{6}$$

In particular, for every $\mathcal{Y} \ge \frac{1}{\sqrt{\lambda_1}}$, we have

$$\|\mathbb{L}_{\mathfrak{b}} \mathfrak{u}\|_{L^2(y^{\mathfrak{b}}, \Omega \times (\mathcal{Y}, \infty))} \lesssim \exp\left(-\frac{\sqrt{\lambda_1} \mathcal{Y}}{2} \right) \|f\|_{\mathbb{H}^{-s}(\Omega)}. \tag{7}$$

Proof. Owing to (4) and the orthogonality properties of $\{\varphi_k\}_{k=1}^{\infty}$, we have

$$\int_a^b y^{\mathfrak{b}} \int_\Omega |\mathbb{L}_{\mathfrak{b}}\mathfrak{u}|^2 \,\mathrm{d}x\,\mathrm{d}y = \int_a^b y^{\mathfrak{b}} \int_\Omega \left| \sum_{k=1}^{\infty} U_k \left[\mathbb{D}_{\mathfrak{b}}\psi_k(y) + \lambda_k \psi_k(y)\right] \varphi_k(x) \right|^2 \mathrm{d}x\,\mathrm{d}y$$
$$= \sum_{k=1}^{\infty} |U_k|^2 \int_a^b y^{\mathfrak{b}} \left|(\mathbb{D}_{\mathfrak{b}} + \lambda_k)\psi_k\right|^2 \mathrm{d}y = \sum_{k=1}^{\infty} \lambda_k^s |U_k|^2 J_k(a,b)^2,$$

where $J_k(a,b)^2 := \lambda_k^{-s} \int_a^b y^{\mathfrak{b}} |(\mathbb{D}_{\mathfrak{b}} + \lambda_k)\psi_k|^2 \,\mathrm{d}y$. We now employ the change of variable $z = \sqrt{\lambda_k} y$, formula [16, (3.2)], denote by $\tilde{\mathbb{D}}_{\mathfrak{b}}$ the same operator as $\mathbb{D}_{\mathfrak{b}}$ but with respect to the variable z, use [12, (10.29.1)], or alternatively [16, (3.3)], and finally recall that ψ is defined in (3) to obtain

$$J_k(a,b)^2 = \int_{\sqrt{\lambda_k} a}^{\sqrt{\lambda_k} b} z^{\mathfrak{b}} \left| (\tilde{\mathbb{D}}_{\mathfrak{b}} + 1)\psi \right|^2 \mathrm{d}z = 4c_s^2 \int_{\sqrt{\lambda_k} a}^{\sqrt{\lambda_k} b} z K_{s-1}^2(z) \,\mathrm{d}z = I_k(a,b)^2. \tag{8}$$

Finally, if we set $a = \mathcal{Y}$, $b = \infty$, and use the exponential decay of K_s [17, §2.5], we obtain

$$I_k(\mathcal{Y},\infty)^2 \le I_1(\mathcal{Y},\infty)^2 \le 4c_s^2 \exp(-\sqrt{\lambda_1}\mathcal{Y}) \int_0^\infty z^{\max\{3-2s,0\}} \exp(-z) \,\mathrm{d}z \lesssim \exp(-\sqrt{\lambda_1}\mathcal{Y}),$$

which implies (7). ☐

Define

$$H^2_{Lb,\mathcal{Y}}(y^{\mathfrak{b}}, \mathcal{C}_{\mathcal{Y}}) := \left\{ w \in H^2(y^{\mathfrak{b}}, \mathcal{C}_{\mathcal{Y}}) \ : \ \begin{matrix} \lim_{y\downarrow 0} y^{\mathfrak{b}} \partial_y w = 0, & w_{|\partial_L \mathcal{C}} = \partial_{\mathbf{n}} w_{|\partial_L \mathcal{C}} = 0 \\ w_{|y=\mathcal{Y}} = \partial_y w|_{y=\mathcal{Y}} = 0 & \end{matrix} \right\}.$$

If $\mathfrak{u}_{\mathcal{Y}} \in H^2_{Lb,\mathcal{Y}}(y^{\mathfrak{b}}, \mathcal{C}_{\mathcal{Y}})$ solves the problem

$$\int_{\mathcal{C}_{\mathcal{Y}}} y^{\mathfrak{b}} \mathbb{L}_{\mathfrak{b}} \mathfrak{u}_{\mathcal{Y}} \mathbb{L}_{\mathfrak{b}} v \,\mathrm{d}x\,\mathrm{d}y = d_s \langle f, \operatorname{tr}_\Omega v \rangle_{\mathcal{H}_{\mathcal{L}}^{-s}, \mathcal{H}_{\mathcal{L}}^{s}} \quad \forall v \in H^2_{Lb,\mathcal{Y}}(y^{\mathfrak{b}}, \mathcal{C}_{\mathcal{Y}}), \tag{9}$$

then $u_{\mathcal{Y}} := \operatorname{tr}_\Omega \mathfrak{u}_{\mathcal{Y}}$ is an approximation of u, the soluton to (1), in the following sense.

Proposition 2 (exponential approximation). *Let* $\mathcal{Y} \ge \frac{1}{\sqrt{\lambda_1}}$. *If* $\mathfrak{u}$ *and* $\mathfrak{u}_{\mathcal{Y}}$ *solve* (5) *and* (9), *respectively, then we have*

$$\|u - u_{\mathcal{Y}}\|_{\mathbb{H}^s(\Omega)} = \|\operatorname{tr}_\Omega \mathfrak{u} - \operatorname{tr}_\Omega \mathfrak{u}_{\mathcal{Y}}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\mathfrak{u} - \mathfrak{u}_{\mathcal{Y}}\|_{H^2(y^{\mathfrak{b}}, \mathcal{C})} \lesssim \exp\left(-\frac{\sqrt{\lambda_1}\mathcal{Y}}{4}\right) \|f\|_{\mathbb{H}^{-s}(\Omega)}.$$

Proof. The trace estimate follows from [16, (1.5)], and the second estimate follows from Proposition 1 and an adaptation of [17, Lemma 3.3] to compare (5) and (9). ☐

4. Discretization

The finite element discretization of (9) is as follows. Fix $\mathcal{Y} \ge \lambda_1^{-1/2}$, let $\{\mathscr{T}_h\}_{h>0}$ be a quasiuniform family of conforming triangulations of Ω, and let $\{\mathcal{M}_M\}_{M\in\mathbb{N}}$ be a, possibly graded, family of partitions of $[0,\mathcal{Y}]$ with $M = \#\mathcal{M}_M$. In this setting, we define

$$
\begin{aligned}
\mathcal{S}(\mathcal{M}_M) &\coloneqq \left\{p \in C^1([0,\mathcal{Y}]) \ : \ p_{|I_m} \in \mathbb{P}_3, \ m = 1, \ldots, M, \ p(\mathcal{Y}) = p'(0) = p'(\mathcal{Y}) = 0\right\}, \\
W(\mathscr{T}_h) &\coloneqq \left\{v_h \in C^1(\bar{\Omega}) \ : v_{h|T} \in \mathcal{P} \ \ \forall T \in \mathscr{T}_h, \ v_{h|\partial\Omega} = \partial_{\mathbf{n}} v_{h|\partial\Omega} = 0\right\}, \\
V_{\mathcal{Y},h,M} &\coloneqq W(\mathscr{T}_h) \otimes \mathcal{S}(\mathcal{M}_M),
\end{aligned}
$$

where $\mathcal{P}$ is any H^2–conforming finite element space, for example, the Hermite element [6, Example 3.2.6], the Argyris element [6, Example 3.2.10], the composite HCT element [10, Chapter VII, Section 46] [14], etc. [18]. Since $\mathfrak{b} \in (-1,1)$ we have, for $p \in \mathcal{S}(\mathcal{M}_M)$, $y^{\mathfrak{b}} p'(y) \to 0$ as $y \downarrow 0$. Thus, $V_{\mathcal{Y},h,M} \subset H^2_{Lb,\mathcal{Y}}(y^{\mathfrak{b}}, \mathcal{C}_\mathcal{Y})$. Let $\mathfrak{u}_{\mathcal{Y},h,M} \in V_{\mathcal{Y},h,M}$ be the solution to

$$
\int_{\mathcal{C}_\mathcal{Y}} y^{\mathfrak{b}} \mathbb{L}_{\mathfrak{b}} \mathfrak{u}_{\mathcal{Y},h,M} \mathbb{L}_{\mathfrak{b}} v_{h,M} \, \mathrm{d}x \, \mathrm{d}y = d_s \langle f, \operatorname{tr}_\Omega v_{h,M} \rangle, \qquad \forall v_{h,M} \in V_{\mathcal{Y},h,M}. \tag{10}
$$

Our numerical approximation is then $u_{\mathcal{Y},h,M} \coloneqq \operatorname{tr}_\Omega \mathfrak{u}_{\mathcal{Y},h,M}$.Conformity implies that a Céa-type result is immediate. In addition, we can use the Cartesian product structure of $\mathcal{C}_\mathcal{Y}$ as in [2, Lemma 7] to obtain the next result.

Theorem 2 (best approximation)**.** *Let* $\mathfrak{u}_\mathcal{Y}$ *solve* (9) *and* $\mathfrak{u}_{\mathcal{Y},h,M}$ *solve* (10)*, respectively. Let* $\Pi_x : H^2_0(\Omega) \to W(\mathscr{T}_h)$ *denote a linear projection that is stable in the sense that*

$$
\|\Pi_x w\|_{L^2(\Omega)} \lesssim \|w\|_{L^2(\Omega)}, \qquad \|\Delta \Pi_x w\|_{L^2(\Omega)} \lesssim \|\Delta w\|_{L^2(\Omega)} \quad \forall w \in H^2_0(\Omega).
$$

Let also $\Pi_y : H^2(y^{\mathfrak{b}}, (0,\mathcal{Y})) \to \mathcal{S}(\mathcal{M}_M)$ *be a linear projection. Then,*

$$
\begin{aligned}
\|\operatorname{tr}_\Omega (\mathfrak{u}_\mathcal{Y} - \mathfrak{u}_{\mathcal{Y},h,M})\|_{\mathbb{H}^s(\Omega)} &\lesssim \|\mathfrak{u}_\mathcal{Y} - \mathfrak{u}_{\mathcal{Y},h,M}\|_{H^2_{Lb}(y^{\mathfrak{b}},\mathcal{C}_\mathcal{Y})} \\
&= \min_{v_{h,M} \in V_{\mathcal{Y},h,M}} \|\mathfrak{u}_\mathcal{Y} - v_{h,M}\|_{H^2_{Lb}(y^{\mathfrak{b}},\mathcal{C}_\mathcal{Y})} \\
&\lesssim \|(\mathrm{id} - \Pi_x)\mathbb{D}_{\mathfrak{b}} \mathfrak{u}_\mathcal{Y}\|_{L^2(y^{\mathfrak{b}},\mathcal{C}_\mathcal{Y})} + \|\Delta_x (\mathrm{id} - \Pi_x)\mathfrak{u}_\mathcal{Y}\|_{L^2(y^{\mathfrak{b}},\mathcal{C}_\mathcal{Y})} \\
&\quad + \|\mathbb{D}_{\mathfrak{b}} (\mathrm{id} - \Pi_y)\mathfrak{u}_\mathcal{Y}\|_{L^2(y^{\mathfrak{b}},\mathcal{C}_\mathcal{Y})} + \|(\mathrm{id} - \Pi_y)\Delta_x \mathfrak{u}_\mathcal{Y}\|_{L^2(y^{\mathfrak{b}},\mathcal{C}_\mathcal{Y})}
\end{aligned}
$$

Proof. We first use a trace estimate. The minimum is Céa's lemma. After that, it suffices to set $v_{h,M} = \Pi_x \otimes \Pi_y \mathfrak{u}_\mathcal{Y}$, add and subtract $\Pi_x \mathfrak{u}_\mathcal{Y}$, and use the stability of Π_x. ☐

Remark 1 (regularity and rate of approximation)**.** The derivation of error estimates for the solution of (5) requires estimating the interpolation errors described in Theorem 2. To do this, one uses the explicit representation (4) of the solution as it has been done in [17, 2]. This process is further complicated by the fact that fourth-order equations have a limited regularity shift. For instance, [13, Corollary 7.3.2.5] shows that, if $d = 2$, Ω is a convex polygon, and $\Delta^2 \Phi \in L^2(\Omega)$, then $\Phi \in H^3(\Omega) \cap H^2_0(\Omega)$. This, however, does not imply that $u \in H^4(\Omega)$.

Remark 2 (nonconforming methods)**.** The implementation of H^2-conforming methods in general domains is a complicated endeavor, and not every finite element library offers ready-made elements for this purpose. One possible solution is to use nonconforming methods such as discrete Kirchhoff triangles [3, §8.2.1], virtual elements [5], interior penalty methods [1, 7], or other approaches. Obtaining analogues to Theorem 2 for these methods should follow standard techniques and encounter the usual complications associated with fourth–order problems. One must simply take advantage of the Cartesian product structure of the domain and the problem.

Funding

EO: ANID grant FONDECYT-1220156 and by USM through USM project 2025 PI LII 25 12. AJS: NSF grant DMS-2409918.

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