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A locking-free scheme for the LQR control of a Timoshenko beam

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ABSTRACT

In this paper we analyze a locking-free numerical scheme for the LQR control of a Timoshenko beam. We consider a non-conforming finite element discretization of the system dynamics and a control law constant in the spatial dimension. To solve the LQR problem we seek a feedback control which depends on the solution of an algebraic Riccati equation. An optimal error estimate for the feedback operator is proved in the framework of the approximation theory for control of infinite dimensional systems. This estimate is valid with constants that do not depend on the thickness of the beam, which leads to the conclusion that the method is locking-free. In order to assess the performance of the method, numerical tests are reported and discussed.

1. Introduction

This paper concerns the numerical approximation of an optimal control problem and stability properties of flexible beam structures. The concrete engineering problems that motivate this study come from very different fields: aerospace, structures, meteorology, nanotechnology, etc. An overview can be found in the books [1,2], and references therein. To establish efficient numerical methods for solving this problem, it is necessary to perform a complete study of both the control problem and the beam structure model.

The structural behavior of beams has been studied using a variety of approaches. The most commonly used model for thick beams is the Timoshenko model (see [3]), which includes the effect of shear. Recently in [4], it has been concluded that the Timoshenko model is remarkably accurate in comparison with other theories of beams: the Euler–Bernoulli model and a two-dimensional elasticity model. However, it is well-known that standard finite element methods applied to this model produce very unsatisfactory results when the thickness of the beam is small with respect to the other dimension of the structure; this fact is known as the locking phenomenon (see the book of Chapelle and Bathe [5]). To avoid locking, special methods based on reduced integration or mixed formulations have been devised and are typically used. The first mathematical work dealing with numerical locking and how to avoid it is the paper in [6], in which it has been proved that locking arises because of the shear term, and a locking-free method based on a mixed formulation has been proposed and analyzed. Recently, this proposed method has been used and analyzed in application to the problem of the numerical approximation of an active vibration control of a Timoshenko beam (see [7,8]).

On the other hand, for our optimal control problem, we consider a quadratic cost functional and a control law defined by means of a feedback operator acting over the system states. The so-called LQR (linear–quadratic regulator) problem constitutes a cornerstone of modern linear control theory. It was studied originally in a finite dimensional context and also

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became a subject of interest in the framework of control theory for partial differential equations (see [9,10]), which are related to several applications (see [11,1,12–14], for instance).

In general terms, to find the solution of our optimal control problem over infinite dimensional systems we need to perform two main tasks: first, we need to approximate the system dynamics using a locking-free finite element method (see [6]) and after that, we need to solve a finite dimensional algebraic Riccati equation associated with the solution of an LQR problem. We remark that the LQR control strategy for the Timoshenko equations has been studied and computationally implemented (see [13,14]) but, to the best of the author's knowledge, the mathematical analysis and computational validation of the optimal convergence rates for this control problem cannot be found in the literature. Indeed, we obtain convergence and error estimates that do not depend on the thickness of the beam and, consequently, the well-known numerical locking phenomenon is avoided in the approximation of the control and the state variables.

The goal of the problem considered is to compensate the vibrations arising from a set of initial conditions. To achieve this purpose, we seek a control signal represented in a feedback form using the state variables. By following the abstract theory stated in [15], a series of assumptions connected with stability and consistency properties of the approximated solution of the Timoshenko problem must be proved in order to obtain an optimal convergence rate for the control problem. Such a result is stated as an optimal convergence rate for the feedback operator of the LQR problem, and such convergence is analyzed under the functional gain framework developed in [16,10,17,13], among others.

The outline of this paper is as follows. In Section 2 we state the abstract optimal control problem and the conditions needed in order to prove existence and uniqueness of the exact solution. In Section 3 we deal with approximation issues: we state the approximated control problem and we prove the conditions that ensure convergence rates of optimal order. In Section 4 we present complete numerical results that allow us to validate the stabilization and the convergence rates obtained.

2. Problem statement: mathematical foundations

Let us consider an elastic beam of thickness $t \in (0, 1]$, with reference configuration $\mathbf{I} \times (-t/2, t/2)$, where $\mathbf{I} := (0, L)$ with L the length of the beam. The deformation of the beam is described by means of the Timoshenko model in terms of the rotation amplitude θ of its midplane and the transverse displacement amplitude w (see [3]). Assuming that the beam is clamped, its deformation is the solution of the following problem:

Find (w, θ) such that

$$\begin{cases} \rho A \dfrac{\partial^2 w}{\partial \tau^2} - kAG\left(\dfrac{\partial^2 w}{\partial x^2} - \dfrac{\partial \theta}{\partial x}\right) = \bar{u}(x,\tau) & x \in \mathbf{I},\ \tau \geq 0, \\ \rho I \dfrac{\partial^2 \theta}{\partial \tau^2} - EI\dfrac{\partial^2 \theta}{\partial x^2} - kAG\left(\dfrac{\partial w}{\partial x} - \theta\right) = 0 & x \in \mathbf{I},\ \tau \geq 0, \\ w(0,\tau) = w(L,\tau) = \theta(0,\tau) = \theta(L,\tau) = 0 & \tau \geq 0, \\ w(x,0) = f(x), \qquad \dot{w}(x,0) = \zeta(x) & x \in \mathbf{I}, \\ \theta(x,0) = g(x), \qquad \dot{\theta}(x,0) = \eta(x) & x \in \mathbf{I} \end{cases} \tag{1}$$

where x represents the spatial coordinate and τ the time. The coefficients ρ, E and I, that will be assumed constant, represent the mass density, the Young modulus and the inertia moment, respectively. The coefficient k is a correction factor usually taken as $5/6$; A and G represent the sectional area of the beam and elasticity modulus of the shear. The external load $\bar{u}(x,\tau)$ denotes a distributed control acting over an interval $\mathbf{I}_c$ in the following way:

$$\bar{u}(x,\tau) := \begin{cases} \hat{u}(\tau) & x \in \mathbf{I}_c, \\ 0 & x \in \mathbf{I}/\mathbf{I}_c. \end{cases}$$

In order to formulate the optimal control problem and state the existence and uniqueness of the solution, we proceed as in the same framework used in [13] (see also [12], where a mathematical analysis of wave equations has been considered). First of all, according to the LQR theory presented in [15], we are interested in finding a feedback control law in the form

$$\bar{u}(x,\tau) = -\mathbf{G}\bar{y}(x,\tau), \tag{2}$$

for the output regulation problem of the vibrations of the Timoshenko beam over an infinite time horizon, where $\mathbf{G}$ is a gain operator obtained from an algebraic Riccati equation and $\bar{y}$ is the optimal state vector; both will be specified later.

In order to define the optimal control problem we start by writing our equation as a evolutionary system of first order. We set the operator

$$\mathcal{A} = -\begin{bmatrix} \dfrac{kAG}{\rho A}\partial_{xx} & -\dfrac{kAG}{\rho A}\partial_x \\ \dfrac{kAG}{\rho I}\partial_x & \dfrac{EI}{\rho I}\partial_{xx} - \dfrac{kAG}{\rho I} \end{bmatrix}, \tag{3}$$

and the state vector

$$y = \begin{bmatrix} w(x,\tau) & \theta(x,\tau) & \dot{w}(x,\tau) & \dot{\theta}(x,\tau) \end{bmatrix},$$

where ∂_x denotes differentiation with respect to x and $\dot{()}$ stands for time differentiation. Notice that the domain of this operator is $D(\mathcal{A}) = \left[H^2(\mathbf{I}) \cap H_0^1(\mathbf{I})\right]^2$.

In this paper, we introduce a damping operator $\mathcal{D}$ which is proportional to the operator $\mathcal{A}$. In fact we consider $\mathcal{D} := -\alpha\mathcal{A}$, where $\alpha > 0$ represents the damping factor; such a way to describe the damping effect is known as strong damping, as it replicates the elastic operator. Other types of damping have been considered in the literature; for example, in Ref. [13], the author considers the damping operator as $\alpha\mathcal{I}$, where $\mathcal{I}$ represents the identity operator. Such a choice guarantees the existence and uniqueness of the abstract control problem under the theoretical framework shown in [18], as we will see below.

Then, with this setting and using the classical framework stated in [10], we can formally obtain the following first-order state space representation for the problem (1):

$$\begin{cases} \dot{y}(x,\tau) = \mathbf{A}y + \mathbf{B}u, \\ y(x,0) = y_0, \end{cases} \tag{4}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\alpha\mathcal{A} \end{bmatrix} \quad \text{and} \quad \mathbf{B}u = \begin{bmatrix} 0 \\ 0 \\ \mathcal{I}_c u(\tau) \\ 0 \end{bmatrix}. \tag{5}$$

Notice that each equation of (1) has been adequately rescaled and we denote by $u = \frac{1}{\rho A}\hat{u}$ the rescaled distributed control variable. In this setting, 0 and I represent the null and the identity matrix in the space of square matrices of size 2, $\mathbf{B}$ denotes the distributed control operator, with the operator $\mathcal{I}_c : L^2(\mathbf{I}) \to L^2(\mathbf{I})$ defined by $(\mathcal{I}_c v)(x) = \chi_c v(x)$ for all $x \in \mathbf{I}$, where χ_c denotes the characteristic function of the subset $\mathbf{I}_c$. We consider the operator $\mathbf{B}$ from $\mathbb{R}$ onto $[D(\mathbf{A}^*)]'$.

Since we are interested in reducing the vertical displacement and its velocity, following the theory given in [15], we consider a state space $Y = H_0^1(\mathbf{I})^2 \times L^2(\mathbf{I})^2$, such that $\mathcal{D}(\mathbf{A}) \subset Y$, a control space $U = \mathbb{R}$ and a cost functional

$$\mathcal{J}(y,u) = \frac{1}{2}\int_0^\infty \left\{\|\mathbf{R}(w,\theta)\|_Z^2 + |u(\tau)|^2\right\} \mathrm{d}\tau, \tag{6}$$

where the operator $\mathbf{R} : Y \to Z$ is defined as $\mathbf{R}(\phi,\varphi,\varsigma,\psi) = (\phi,\varsigma)$ and $Z := H_0^1(\mathbf{I}) \times L^2(\mathbf{I})$ is the output space. In the associated cost functional, the operator $\mathbf{R}$ is defined such that we only consider the vertical displacement and its velocity; this choice is more suitable when devising control strategies oriented to real applications is considered. Notice that, with this setting, the cost functional (6) can be rewritten as follows:

$$\mathcal{J}(y,u) = \frac{1}{2}\int_0^\infty \left\{\|w(\cdot,\tau)\|_{H^1(\mathbf{I})}^2 + \|\dot{w}(\cdot,\tau)\|_{L^2(\mathbf{I})}^2 + |u(\tau)|^2\right\} \mathrm{d}\tau. \tag{7}$$

The corresponding optimal control problem is:

$$\text{Minimize } \mathcal{J}(y,u) \text{ over all } u \in L^2(0,\infty;\mathbb{R}), \text{ where } y \text{is the solution of (4) due to } u. \tag{8}$$

We state that the control laws are optimal in the sense that they are solutions of the above problem.

Here and herein, for Z a function space, $z \in L^2(Z)$ stands for a function $z(\cdot,t) \in Z$ and $z(\xi,\cdot) \in L^2([0,+\infty[)$.

The existence and uniqueness for the solution of the abstract control problem (7)–(8) is presented in the next lemma.

Lemma 2.1. *For each $y_0 \in H_0^1(\mathbf{I})^2 \times L^2(\mathbf{I})^2$, there exists a unique optimal solution $(\bar{u},\bar{y})$ of the abstract optimal control problem (7)–(8) for the dynamics (4).*

Proof. Considering the theory presented in Section 2 of [12], we only need to prove the following assumptions.

(H.1) $\mathbf{A}$ is the infinitesimal generator of a strongly continuous, analytic semigroup, denoted by $e^{\mathbf{A}t}$, on $H_0^1(\mathbf{I})^2 \times L^2(\mathbf{I})^2$.
(H.2) $\mathbf{B} : \mathbb{R} \to [D(\mathbf{A}^*)]'$ is a linear operator such that $\mathbf{A}^{-\gamma}\mathbf{B} \in \mathcal{L}(\mathbb{R}, H_0^1(\mathbf{I})^2 \times L^2(\mathbf{I})^2)$ for a fixed constant $\gamma \in [0,1)$.
(H.3) $\mathbf{R} : H_0^1(\mathbf{I})^2 \times L^2(\mathbf{I})^2 \to H_0^1(\mathbf{I}) \times L^2(\mathbf{I})$ is a bounded linear operator.
(H.4) *Finite cost condition*: Given $y_0 \in H_0^1(\mathbf{I})^2 \times L^2(\mathbf{I})^2$, there exists $\bar{u} \in L^2(0,\infty;\mathbb{R})$ such that $\mathcal{J}(\bar{u},\bar{y}) < \infty$.

First, from [18], we have that the damping operator can be identified with a fractional power β of the elastic operator, with $1/2 \le \beta \le 1$; our case holds with $\beta = 1$, and then we obtain that $\mathbf{A}$ is the infinitesimal generator of a strongly continuous analytic semigroup on $H^1(\mathbf{I})^2 \times L^2(\mathbf{I})^2$. According to the definition of the operator $\mathbf{B}$, it is clear that $\mathbf{B} \in \mathcal{L}(\mathbb{R}, D(\mathbf{A}^*))$, i.e., (H.2) is satisfied with $\gamma = 0$. Moreover, $\mathbf{R}$ lives in $\mathcal{L}(H_0^1(\mathbf{I})^2 \times L^2(\mathbf{I})^2, H_0^1(\mathbf{I}) \times L^2(\mathbf{I}))$, i.e., (H.3) is satisfied.

Finally, as a consequence of the stability condition stated in Theorem 3.B.1 of [15], the finite cost condition is satisfied: the damping factor allows us to always take the control law $\bar{u} \equiv 0$ such that $J(0,\bar{y}) < \infty$, where $\bar{y}$, which satisfies $\mathbf{R}\bar{y} \in L^2(0,\infty;Z)$, is the solution to (4) due to $\bar{u}$. □

It is well-known that the feedback control law given in (2) is related to the output by

$$\bar{y}(x,\tau) = e^{\mathbf{A_P}\tau} y_0, \qquad \bar{u}(x,\tau) = -\mathbf{G}e^{\mathbf{A_P}\tau} y_0, \quad \forall \tau \geq 0 \tag{9}$$

where $\mathbf{G} = \mathbf{B}^*\mathbf{P}$ denotes the continuous gain operator, $\mathbf{A_P} = \mathbf{A} - \mathbf{BB}^*\mathbf{P}$ is the operator related to the closed loop dynamics and $\mathbf{P} = \mathbf{P}^* \in \mathcal{L}(Y)$ is the unique nonnegative operator that satisfies the following algebraic Riccati equation (ARE):

$$(\mathbf{A}^*\mathbf{P}x, y)_Y + (\mathbf{PA}x, y)_Y - (\mathbf{B}^*\mathbf{P}x, \mathbf{B}^*\mathbf{P}y)_U + (\mathbf{R}^*\mathbf{R}x, y)_Y = 0, \tag{10}$$

for all $(x, y) \in \mathcal{D}(\mathbf{A}) \times \mathcal{D}(\mathbf{A})$, where $(\cdot,\cdot)$ denotes the inner product over the corresponding space.

3. A locking-free numerical scheme

The following step is to construct a discretization of the optimal control problem (7)–(8). At this point an efficient solution of the Timoshenko model is fundamental. In this context, in [6] it is shown that standard finite element methods applied to the load problem associated with the static Timoshenko beam are subject to the locking phenomenon. This means that they produce unsatisfactory results for very thin beams; this effect is caused by the shear stress term. To avoid the numerical locking in the static case, Arnold in [6] introduces and analyzes a locking-free method based on a mixed formulation of the problem, and also proves that this mixed method is equivalent to using a scheme with reduced order for the integration of the shear term in the primal formulation.

We follow the structure presented in Chapter 4 of [15]. Then, we start by selecting a finite dimensional approximating subspace $\mathcal{V}_h \subset H_0^1(\mathbf{I})$ to be the piecewise linear finite element space. For this reason, we consider a family $\{\mathcal{T}_h\}$ of partitions of the interval $\mathbf{I}$:

$$\mathcal{T}_h : 0 = x_0 < x_1 < \cdots < x_n = L, \tag{11}$$

with mesh size

$$h := \max_{j=1,\ldots,n} \left(x_j - x_{j-1}\right).$$

The subspace $\mathcal{V}_h$ can be written as follows:

$$\mathcal{V}_h := \left\{ v \in H_0^1(\mathbf{I}) : v|_{[x_{j-1}-x_j]} \in \mathbb{P}_1, j = 1,\ldots,n \right\} \subset H_0^1(\mathbf{I}). \tag{12}$$

Let $\mathcal{V}_{h1}$ consist of the elements of $\mathcal{V}_h$ and be equipped with the $H^1(\mathbf{I})$ seminorm and let $\mathcal{V}_{h2}$ consist of the elements of $\mathcal{V}_h$, equipped with the $L^2(\mathbf{I})$ norm. We set $V_h = \mathcal{V}_{h1}^2 \times \mathcal{V}_{h2}^2$.

Moreover, to define a locking-free scheme for the approximation of the Timoshenko equation, we also consider the following discrete space (see [6]):

$$\mathcal{W}_h := \left\{ \frac{\mathrm{d}v}{\mathrm{d}x} + c,\, v \in \mathcal{V}_h,\, c \in \mathbb{R} \right\} \subset L^2(\mathbf{I}). \tag{13}$$

We denote by $\mathcal{P}_h$ the orthogonal projection from $L^2(\mathbf{I})^4$ onto V_h, i.e., $\mathcal{P}_h := \pi_h \mathbf{I}_4$, where $\mathbf{I}_4$ denotes the identity matrix in the square matrices of size 4, and π_h represents the orthogonal projection from $L^2(\mathbf{I})$ onto $\mathcal{V}_h$. It is standard that the subspace $\mathcal{V}_h$ satisfies the approximation property

$$\|\pi_h v - v\|_{H^l(\mathbf{I})} \leq C h^{s-l} \|v\|_{H^s(\mathbf{I})}, \quad v \in H^s(\mathbf{I}) \cap H_0^1(\mathbf{I}), \tag{14}$$

with $0 \leq l \leq s \leq 2$.

In order to write the Galerkin approximation of the operator $\mathbf{A}$ we need to define the following weighted $L^2(\mathbf{I}) \times L^2(\mathbf{I})$ inner product:

$$\langle (\eta, \varsigma), (v, \beta) \rangle_t = (\eta, v) + \frac{t^2}{12}(\varsigma, \beta). \tag{15}$$

Then, we can write the Galerkin approximation of the operator $\mathbf{A}$ on V_h as follows:

$$\mathbf{A_h} = \begin{bmatrix} \mathbf{0} & \mathbf{\Pi_h} \\ -\mathcal{A}_h & -\alpha\mathcal{A}_h \end{bmatrix} : V_h \to V_h, \tag{16}$$

where $\mathbf{\Pi_h}$ represents a projection given by

$$\mathbf{\Pi_h} = \begin{bmatrix} \pi_h & 0 \\ 0 & \pi_h \end{bmatrix},$$

and $\mathcal{A}_h$ denotes the locking-free approximation of the operator $\mathcal{A}$, which is defined by means of the following bilinear form

$$\langle \mathcal{A}_h(w_{th}, \theta_{th}), (v_h, \beta_h) \rangle_t = \frac{E}{12\hat{\rho}} \int_{\mathbf{I}} \frac{\mathrm{d}\theta_{th}}{\mathrm{d}x} \frac{\mathrm{d}\beta_h}{\mathrm{d}x} \mathrm{d}x + \frac{\kappa}{t^2\hat{\rho}} \int_{\mathbf{I}} \pi_h^0 \left(\frac{\mathrm{d}w_{th}}{\mathrm{d}x} - \theta_{th} \right) \pi_h^0 \left(\frac{\mathrm{d}v_h}{\mathrm{d}x} - \beta \right) \mathrm{d}x, \tag{17}$$

for all $(w_{th}, \theta_{th}), (v_h, \beta_h) \in \mathcal{V}_h^2$, where π_h^0 denotes the projection from $L^2(\mathbf{I})$ onto $\mathcal{W}_h$.

Notice that the bilinear form $\mathcal{A}_h$ has been obtained by considering a beam with square transverse section with physical parameters $A = t^2$ and $I = t^4/12$, and $G = E/2(1+\nu)$, where ν denotes the Poisson ratio. Moreover, we have considered a rescaling in the density of the material, $\rho = \hat{\rho}t^2$. This rescaling is justified by the fact that the method to be used should remain stable as the thickness becomes small, and a way to obtain it is by considering a rescaling in the density (see the book of Chapelle and Bathe [5]).

Now, we can proceed to write the approximation of the operator $\mathbf{B}$, defined by the expression (5). In fact

$$\mathbf{B_h}u := \mathcal{P}_h\mathbf{B}u = \begin{bmatrix} 0 \\ 0 \\ \pi_h \mathcal{I}_c u(\tau) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathcal{I}_c u(\tau) \\ 0 \end{bmatrix} : \mathbb{R} \to V_h. \tag{18}$$

On the other hand, it is easy to verify that the adjoint operators of $\mathbf{A_h}$ and $\mathbf{B_h}$ in (16) and (18), respectively, are given by

$$\mathbf{A_h^*} = \begin{bmatrix} \mathbf{0} & -\mathbf{\Pi_h} \\ \mathcal{A}_h & \rho\mathcal{A}_h \end{bmatrix} : V_h \to V_h, \tag{19}$$

$$\mathbf{B_h^*}v_h = (v_{h3}, \chi_c)_{L^2(\mathbf{I})}, \qquad v_h = \begin{bmatrix} v_{h1} & v_{h2} & v_{h3} & v_{h4} \end{bmatrix}^T. \tag{20}$$

Now, we can formulate the approximated optimal control problem using the finite element scheme defined above. In fact, we are interested in obtaining a solution $(\bar{y}_h, \bar{u}_h)$ of the following problem:

$$\inf \mathcal{J}(y_h, u_h) = \frac{1}{2}\int_0^\infty \left\{ \|\mathbf{R}y_h(\cdot,\tau)\|_Z^2 + |u_h(\tau)|^2 \right\} \mathrm{d}\tau \tag{21}$$

$$\text{s.t.}: \quad \dot{y}_h = \mathbf{A_h}y_h + \mathbf{B_h}u_h \tag{22}$$

$$y_h(0) = \mathcal{P}_h y_0. \tag{23}$$

The approximated dynamics $\dot{y}_h = \mathbf{A_h}y_h + \mathbf{B_h}u_h$ are given, via (16) and (18), by

$$\begin{cases} \left((\ddot{w}_{th}, \ddot{\theta}_{th}), (v_h, \beta_h)\right) - \langle \mathcal{A}_h(w_{th}, \theta_{th}), (v_h, \beta_h)\rangle_t \\ -\alpha \left\langle \mathcal{A}_h(\dot{w}_{th}, \dot{\theta}_{th}), (v_h, \beta_h)\right\rangle_t = (\mathcal{I}_c u_h, v_h) \quad \forall (v_h, \beta_h) \in \mathcal{V}_h^2 \\ (w_{th}(0), v_h) = (f, v_h), \qquad (\dot{w}_{th}(0), v_h) = (\zeta, v_h) \quad \forall v_h \in \mathcal{V}_h, \\ (\theta_{th}(0), v_h) = (g, v_h), \qquad (\dot{\theta}_{th}(0), v_h) = (\eta, v_h) \quad \forall v_h \in \mathcal{V}_h, \end{cases} \tag{24}$$

with $y_h = [\ w_{th} \quad \theta_{th} \quad \dot{w}_{th} \quad \dot{\theta}_{th}\ ]$, and $(\cdot,\cdot)$ denoting the inner product in $L^2(\mathbf{I})$ or $L^2(\mathbf{I})^2$ as appropriate. Then, the optimal feedback control law for the approximated problem is

$$\bar{u}_h(\tau, \mathcal{P}_h y_0) = -\mathbf{G_h}\mathrm{e}^{\mathbf{A_{P_h}}\tau}\mathcal{P}_h y_0, \tag{25}$$

where $\mathbf{G_h} = \mathbf{B_h^*}\mathbf{P_h}$ denotes the approximated gain operator and $\mathbf{P_h}$ denotes the unique nonnegative, self-adjoint solution of the following algebraic Riccati equation (ARE$_h$):

$$(\mathbf{A_h^*}\mathbf{P_h}\phi_h, \varphi_h)_Y + (\phi_h, \mathbf{A_h^*}\mathbf{P_h}\varphi_h)_Y - (\mathbf{B_h^*}\mathbf{P_h}\phi_h, \mathbf{B_h^*}\mathbf{P_h}\varphi_h)_U + (\mathbf{R}^*\mathbf{R}\phi_h, \varphi_h)_Y = 0 \quad \forall (\phi_h, \varphi_h) \in V_h \times V_h. \tag{26}$$

We proceed to obtain the principal result of this paper, namely, an optimal convergence rate for the linear quadratic regulator control problem developed above, and moreover, we are interested in proving that the convergence rate does not depend on the thickness of the beam, in order to avoid the so-called *numerical locking*. To achieve this, we follow the abstract framework of optimal control theory for partial differential equations, as stated in Chapter 4 of [15]. We begin with an auxiliary lemma that proves the assumptions stated in Theorem 4.1.4.1 in [15], that in this case turns to be:

(A.1) $\mathbf{A_h}$ is the infinitesimal generator of a uniformly analytic semigroup on V_h.

(A.2)

$$\|\mathbf{A}^{-1}\mathcal{P}_h - \mathbf{A_h}^{-1}\mathcal{P}_h\|_{\mathcal{L}(H_0^1(\mathbf{I})^2 \times L^2(\mathbf{I})^2)} \le Ch.$$

(A.3)

$$|\mathbf{B}^*\psi_h| \le C\|\psi_h\|_{H_0^1(\mathbf{I})^2 \times L^2(\mathbf{I})^2}, \quad \forall \psi_h \in V_h.$$

(A.4)

$$|\mathbf{B}^*(\mathcal{P}_h - I)\psi| \le Ch\|\psi\|_{D(\mathbf{A}^*)}, \quad \forall \psi \in D(\mathbf{A}^*).$$

(A.5)

$$|\mathbf{B}^*\mathcal{P}_h\psi| \le C\|\psi\|_{H_0^1(\mathbf{I})^2 \times L^2(\mathbf{I})^2}, \quad \forall \psi \in L^2(\mathbf{I})^4.$$

Note that we have considered, in the same form as in [12], a variation of the second original assumption, which has the objective of recovering the same convergence rates when the initial conditions are replaced by their projections onto V_h. Also notice that (A.5) in [15] is omitted because $\mathbf{B_h} = \mathcal{P}_h\mathbf{B}$.

Lemma 3.1. *For the optimal control problem (7)–(8),* (A.1)–(A.5) *hold.*

Proof. The proof of these assumptions is a direct consequence of Lemma 2 in [12]. However, we need to verify that the estimations (A.2)–(A.5) do not depend on the thickness of the beam.

First, we begin with the assumption (A.2). In fact, notice that

$$\mathbf{A}^{-1}=\begin{bmatrix}-\alpha I & -\mathcal{A}^{-1}\\ I & \mathbf{0}\end{bmatrix},\qquad \mathbf{A_h}^{-1}=\begin{bmatrix}-\rho\mathbf{\Pi_h} & -\mathcal{A}_h^{-1}\\ \mathbf{\Pi_h} & \mathbf{0}\end{bmatrix}.$$

Using the definition of the projection $\mathcal{P}_h$, and considering the inner product in $H_0^1(\mathbf{I})^2\times L^2(\mathbf{I})^2$,

$$\left(\begin{bmatrix}x_1\\x_2\end{bmatrix},\begin{bmatrix}y_1\\y_2\end{bmatrix}\right)_{H_0^1(\mathbf{I})^2\times L^2(\mathbf{I})^2}=(x_1,y_1)_{H^1(\mathbf{I})^2}+(x_2,y_2)_{L^2(\mathbf{I})^2},$$

we obtain the following estimate:

$$\begin{aligned}\|(\mathbf{A}^{-1}\mathcal{P}_h-\mathbf{A_h}^{-1}\mathcal{P}_h)\psi\|_{H_0^1(\mathbf{I})^2\times L^2(\mathbf{I})^2}&\le\|\mathcal{A}^{-1}\mathbf{\Pi_h}(\psi_3\psi_4)^T-\mathcal{A}_h^{-1}\mathbf{\Pi_h}(\psi_3\psi_4)^T\|_{H^1(\mathbf{I})^2}\\&\le Ch\|\psi\|_{L^2(\mathbf{I})^4}.\end{aligned}$$

The last inequality is obtained using the error estimate given in Theorem 3.1 in [19], as a particular case, and it does not depend on the thickness of the beam.

Finally, following the proof of Lemma 2 in [12], it is clear that the estimations (A.2)–(A.5), which are related only to the operator $\mathbf{B}$, do not depend on the thickness parameter. □

Now we can obtain the optimal convergence rates for our optimal control problem. These results are consequences of the assumptions (A.1)–(A.5) and the theoretical framework developed in [15].

Theorem 3.2. *There exists $h_0>0$ such that for all $h<h_0$, (ARE_h) in (26) admits a unique, nonnegative, self-adjoint solution P_h. Moreover, there exist $\omega_0>0$ and $C>0$ independent of h, τ and t, such that for any $\epsilon>0$, $\tau>0$, the following convergence rate is obtained:*

$$\|\mathbf{G}-\mathbf{G_h}\mathcal{P}_h\|_{\mathcal{L}(H_0^1(\mathbf{I})^2\times L^2(\mathbf{I})^2)}\le Ch. \tag{27}$$

Proof. First, notice that, as a consequence of the compact injection $H_0^1(\mathbf{I})\hookrightarrow L^2(\mathbf{I})$, we have that the operators $\mathbf{B}^*\mathbf{A}^{-*}$ and $\mathbf{A}^{-1}\mathbf{KR}$ are compact. For the last operator, $\mathbf{K}\in\mathcal{L}(Y,Z)$ denotes the operator involved in the detectability condition stated in Chapter 4 of [15], which exists as a consequence of the finite cost condition. Then, according to Lemmas 1 and 2, and using the Theorem 4.1.4.1 from [15], we have the existence of $h_0>0$ such that for all $h<h_0$, $(\mathrm{ARE})_h$ admits a unique, nonnegative and self-adjoint solution.

On the other hand, because $\mathbf{B_h}=\mathcal{P}_h\mathbf{B}$, the assumptions (A.7)–(A.9) stated in Chapter 4 in [15] hold; then (27) is obtained by using the estimation (ii) in Theorem 4.6.2.1 and Corollary 4.6.2.5 in [15] with $\gamma=0$, $s=1$ and $r_0=r_1=s$. However, it is important to note that the proof of Theorem 4.6.2.1 in [15] follows from Theorem 4.5.4.1 in [15], in which the assumptions (A.1)–(A.5) proved in Lemma 3.1 are strongly used. This result allow us to guarantee that the constant involved in the estimation (27) does not depend on the thickness parameter. □

4. Computational implementation and numerical experiments

In this section we perform a computational implementation of the above mentioned problem in order to exhibit the optimal convergence rates obtained theoretically. Since the mesh size parameter is directly proportional to numbers of nodes (also with d.o.f.), in this section we adopt the following notation: discrete variables and operators are identified by a superscript ^{N} rather than a subscript $_h$.

We consider a uniform partition of the interval $\mathbf{I}$, $\mathcal{T}_h$ as in (11), and the finite dimensional space of piecewise linear and continuous functions over $\mathbf{I}$ that vanishes in $x=0$ and $x=L$, i.e. $\mathcal{V}_h$, defined in (12). We seek a solution of (24) assuming a Galerkin approximation of the form

$$w^N(x,t)=\sum_{j=1}^N w_j(t)\varphi_j(x),\qquad \theta^N(x,t)=\sum_{j=1}^N\theta_j(t)\varphi_j(x),$$

where $\{\varphi_j\}_{j=1}^N$ denotes a basis of the discrete space $\mathcal{V}_h$ and $N=\dim(\mathcal{V}_h)$.

Now, replacing this expression in (24) we obtain a second-order system of differential equations of the form

$$M^N\ddot{c}(t)+D^N\dot{c}(t)+K^Nc(t)=B^Nu(t)$$

for $c(t)=[w_1(t),\dots,w_N(t),\theta_1(t),\dots,\theta_N(t)]$, where the matrices M^N, D^N, and K^N denote the mass, damping and stiffness matrices respectively while B^N stands for a load vector. It is important to stress that the stiffness matrix K^N and, therefore, the damping operator D^N have been constructed using the reduced integration procedure previously described in order to avoid the numerical locking.

The initial conditions for this second-order problem are obtained taking the Galerkin approximation of the initial conditions of the continuous problem, in fact,

$$(w^N(0), \varphi_j)_{L^2(\mathbf{I})} = (f, \varphi_j)_{L^2(\mathbf{I})},$$
$$(\theta^N(0), \varphi_j)_{L^2(\mathbf{I})} = (g, \varphi_j)_{L^2(\mathbf{I})},$$
$$(\dot{w}^N(0), \varphi_j)_{L^2(\mathbf{I})} = (\zeta, \varphi_j)_{L^2(\mathbf{I})},$$
$$(\dot{\theta}^N(0), \varphi_j)_{L^2(\mathbf{I})} = (\eta, \varphi_j)_{L^2(\mathbf{I})}.$$

Defining the vector state in the same way as we have done in the abstract problem, i.e., $y^N = [c(t), \dot{c}(t)]^T$, we formally obtain a classical first-order state space representation form for the system dynamics:

$$\dot{y}^N = \mathbf{A}^N y^N + \mathbf{B}^N u \qquad y^N(0) = y_0^N$$

where

$$\mathbf{A}^N = \begin{bmatrix} 0 & I \\ -(M^N)^{-1}K^N & -\alpha(M^N)^{-1}K^N \end{bmatrix} \quad \text{and} \quad \mathbf{B}^N = \begin{bmatrix} 0 \\ -(M^N)^{-1}B^N \end{bmatrix}.$$

By means of the same ansatz we obtain a finite dimensional version of the cost functional:

$$\begin{aligned} \mathcal{J}(y^N, u) &= \frac{1}{2}\int_0^\infty \left\{ \|\mathbf{R}^N y^N\|_Z^2 + |u(\tau)|^2 \right\} \mathrm{d}\tau \\ &= \frac{1}{2}\int_0^\infty \left\{ (y^N)^T(\mathbf{R}^N)^T \mathbf{Q}^N \mathbf{R}^N y^N + u(\tau)^2 \right\} \mathrm{d}\tau, \end{aligned} \tag{28}$$

where $\mathbf{R}^N y^N = [w_1(\tau) \cdots w_N(\tau), \dot{w}_1(\tau) \cdots \dot{w}_N(\tau)]^T$ and

$$\mathbf{Q}^N = \begin{bmatrix} Q_1^N & 0 \\ 0 & Q_2^N \end{bmatrix}, \tag{29}$$

with

$$Q_{1,ij}^N = (\phi_i, \phi_j)_{L^2(\mathbf{I})} + (\phi_i', \phi_j')_{L^2(\mathbf{I})}, \qquad Q_{2,ij}^N = (\phi_i, \phi_j)_{L^2(\mathbf{I})}. \tag{30}$$

These finite dimensional operators are inputs in the following algebraic Riccati equation for $\mathbf{P}^N$:

$$\mathbf{A}^N\mathbf{P}^N + \mathbf{P}^N\mathbf{A}^N - \mathbf{P}^N\mathbf{B}^N(\mathbf{B}^N)^T\mathbf{P}^N + (\mathbf{R}^N)^T\mathbf{Q}^N\mathbf{R}^N = 0,$$

and the discrete control law is given by $u_h = -(\mathbf{B}^N)^T\mathbf{P}^N y^N$. Our analysis is focused on the convergence properties of the feedback gain operator $\mathbf{G}^N = (\mathbf{B}^N)^T\mathbf{P}^N$. Proceeding in a similar manner to in [13], we write the action of the full gain operator $\mathbf{G}^N$ as the action of four separate kernels or functional gains, one for each state of the abstract control problem. Formally,

$$u_h(t) = -\mathbf{G}^N y(t) = -(f_1^N, w^N)_{H^1(\mathbf{I})} - (f_2^N, \theta^N)_{H^1(\mathbf{I})} - (f_3^N, \dot{w}^N)_{L^2(\mathbf{I})} - (f_4^N, \dot{\theta}^N)_{L^2(\mathbf{I})} \tag{31}$$

$$= -\begin{bmatrix} f_1^N & f_2^N & f_3^N & f_4^N \end{bmatrix} \begin{bmatrix} Q_1^N & 0 & 0 & 0 \\ 0 & Q_1^N & 0 & 0 \\ 0 & 0 & Q_2^N & 0 \\ 0 & 0 & 0 & Q_2^N \end{bmatrix} y^N(\tau), \tag{32}$$

which leads to an explicit characterization of the functional gains,

$$\begin{bmatrix} f_1^N & f_2^N & f_3^N & f_4^N \end{bmatrix} = \mathbf{G}^N \begin{bmatrix} Q_1^N & 0 & 0 & 0 \\ 0 & Q_1^N & 0 & 0 \\ 0 & 0 & Q_2^N & 0 \\ 0 & 0 & 0 & Q_2^N \end{bmatrix}^{-1}. \tag{33}$$

Under such an approach, analyzing the convergence of the abstract gain operator turns out to be equivalent to the problem of analyzing the convergence of the functional gains over the state space.

We present now numerical experiments performed in MATLAB, which are consistent with the above developed theoretical framework. After discretizing the system we compute the suboptimal feedback by means of the LQR command in MATLAB, and finally we recover the functional gains. For dynamic simulations we advance in time with a Runge–Kutta fourth-order solver.

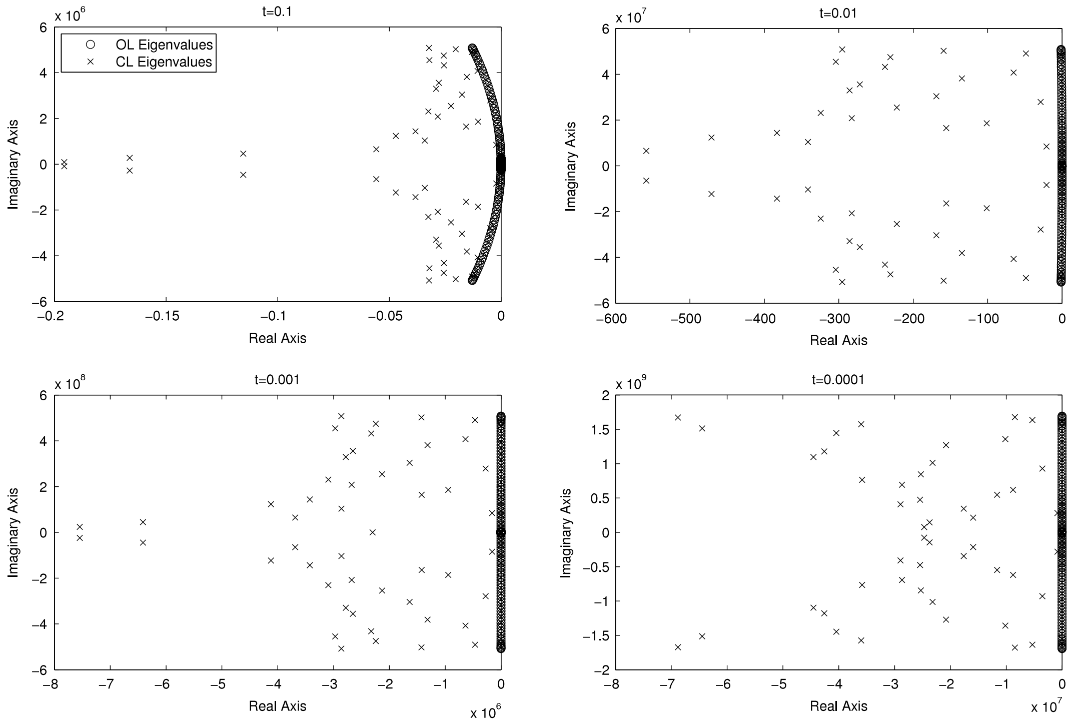

Fig. 1. Open and closed loop pole locations for different values of the thickness with $N = 100$.

In the absence of an exact solution the error estimate of the gain operator is computed relative to the solution obtained with 500 nodes, i.e.,

$$\text{error}^N = \|\mathbf{G}^N - \mathbf{G}^{500}\|. \tag{34}$$

In order to describe the numerical control problem we consider the following choices:

- The domain of the beam is $\mathbf{I} := (0, 1)$.
- The physical parameters are:
  - the elastic moduli: $E = 2.1 \times 10^1 1$ Pa,
  - the Poisson coefficient: $\nu = 0.3$,
  - the correction factor: $k = 5/6$,
  - the density: $\rho = 7.8 \times 10^3$ Kg/m^3.
- There is a distributed actuator over $\mathbf{I}_c := (0.4, 0.6)$.
- The thickness is analyzed from 1e−1 to a minor value of 1e−4.

4.1. Stabilization

Fig. 1 shows closed and open loop poles for different values of t. Open loop pole location is consistent with what should be expected for slender structure models with a strong damping term (as in [20]). Closed loop poles, independently of the thickness, exhibit an improvement on the location that will consequently lead to a faster stabilization of the vertical vibrations.

Figs. 2 and 3 show the stabilization in time at fixed positions, given an initial condition. We show the stabilization of the vertical displacement and velocity, as required in our minimization problem. Despite the dynamical behavior varying as the thickness is changed, in both cases it is possible to compute a control law that regulates the states of the system.

4.2. Convergence

So far we have presented a design procedure for computing stabilizing control laws for a Timoshenko beam by means of a low-order finite element method that involves a reduced integration calculation of the shear term. In such a context, our main result establishes an optimal convergence rate for the feedback operator, resulting from the optimal control problem setting. The convergence rate is obtained from a functional gain characterization of the feedback, and the proposed scheme ensures robustness of the order with respect to the thickness.

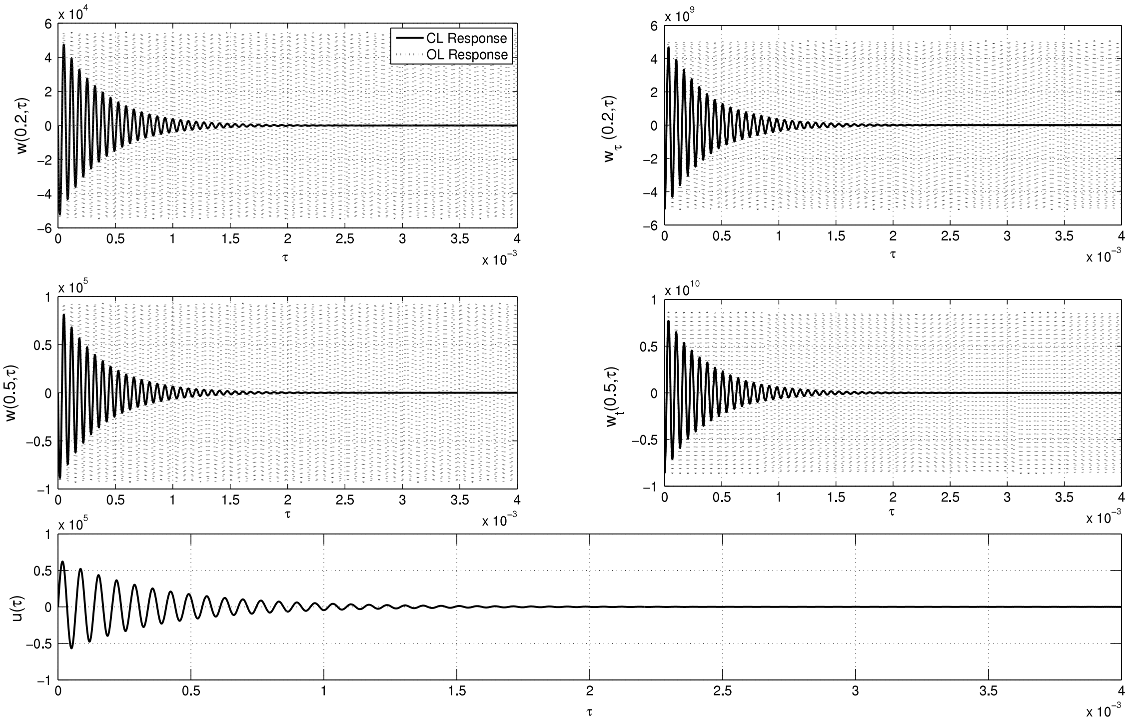

Fig. 2. Dynamical response of the uncontrolled and controlled state variables at $x = 0.2$ (top) and the control signal (bottom). $t = 0.01$.

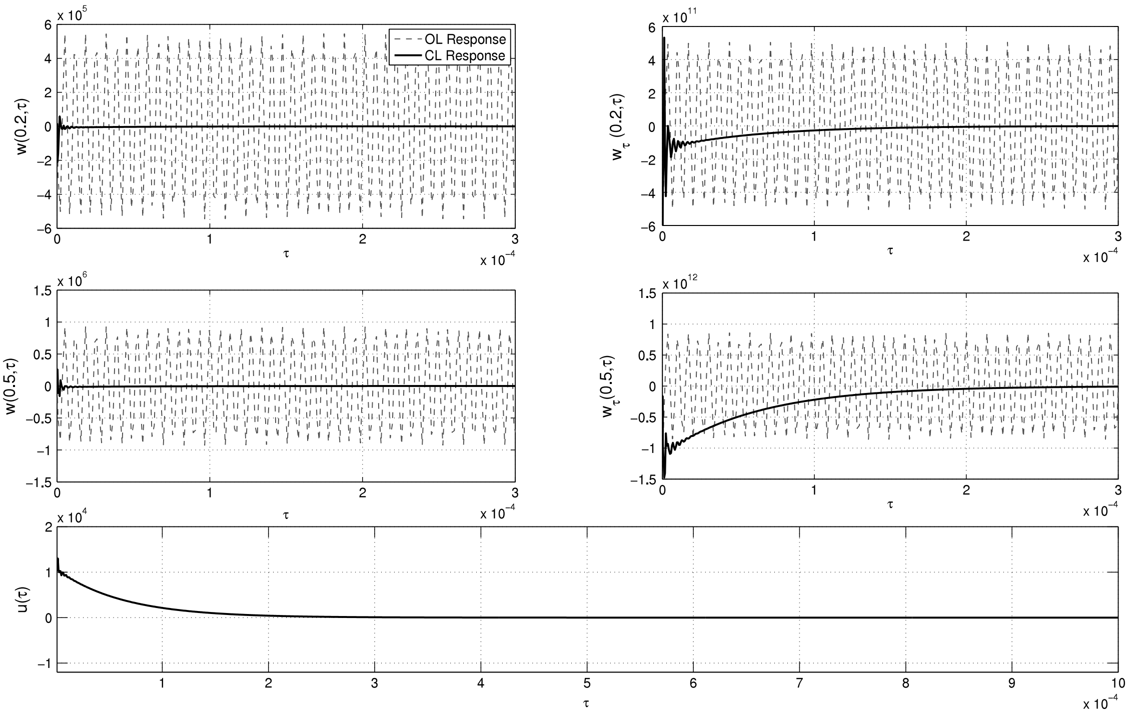

Fig. 3. Dynamical response of the uncontrolled and controlled state variables at $x = 0.5$ (top) and the control signal (bottom). $t = 0.0001$.

We assess the performance of the method with several numerical tests, which consist of the calculation of the functional gains for different numbers of nodes (N), and varying the thickness value t from 0.1 down to 10^{-4}. As can be observed in Figs. 4 and 5, for fixed values of the thickness ($t = 0.01$ and $t = 10^{-4}$), basic convergence of the kernels with respect to the numbers of nodes is verified, and therefore consistency with the locking-free property of the procedure.

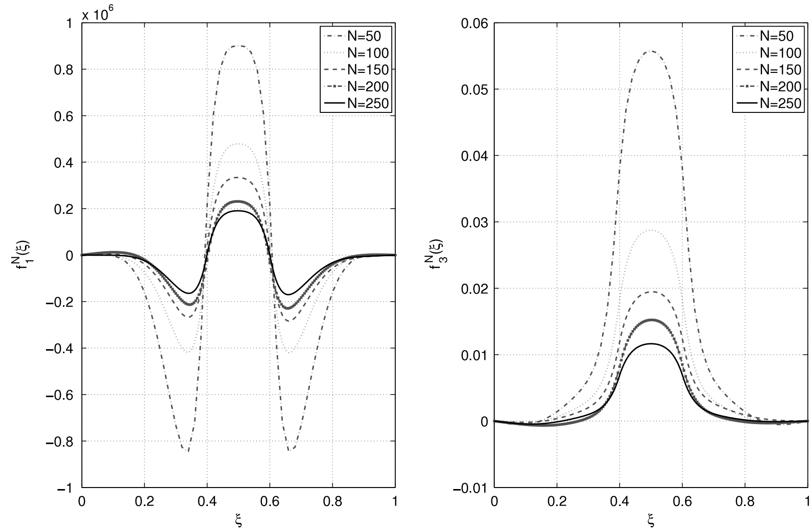

Fig. 4. Spatial convergence of $f_1^N(x)$ (left) and $f_3^N(x)$ (right), with $t = 0.01$.

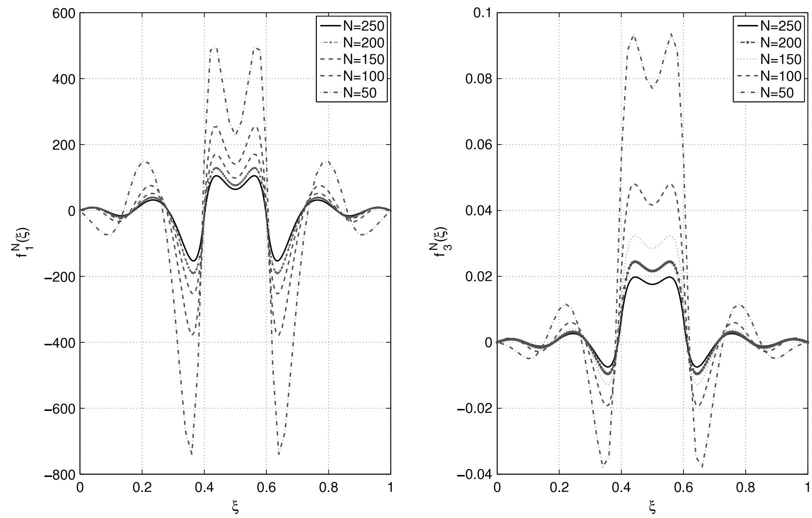

Fig. 5. Spatial convergence of $f_1^N(x)$ (left) and $f_3^N(x)$ (right), with $t = 0.0001$.

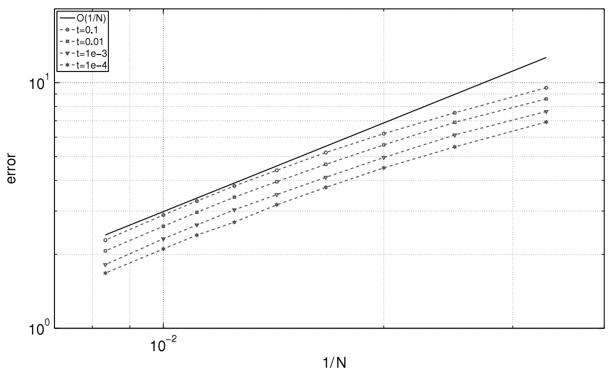

Fig. 6. Log–log plot of the error $\|f_1^N - f_1^{500}\|$ versus $1/N$ for different thickness values.

Furthermore, we perform calculations for the convergence rate of the feedback operator. Figs. 6 and 7 exhibit a comparison between the theoretical predicted order h and the numerical results. We show the convergence rate for f_1 and

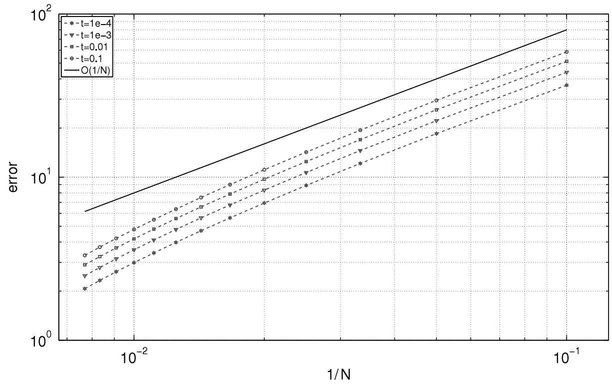

Fig. 7. Log–log plot of the error $\|f_3^N - f_3^{500}\|$ versus $1/N$ for different thickness values.

f_3 in norms H^1 and L^2 respectively. It can be observed that for f_1 the theoretical order h is verified and robustly preserved as the value of t is decreased. Such behavior represents the locking-free feature of our design strategy. The convergence rate of the full feedback operator depends on the convergence of all the functional gains separately; this can essentially be split into two categories, depending on the functional space where they belong, namely H^1 for f_1 and f_2, L^2 for f_3 and f_4. Therefore, we also include Fig. 7, showing the convergence rate of f_3, which turns out to be faster than expected. Certainly this result does not represent any contradiction with the theory, and is consistent, as it also preserves the locking-free property of the finite element approximation. The theoretical results are consequently validated.

5. Concluding remarks

We have developed a locking-free method for the computation of the optimal gain arising from the LQR control problem of a Timoshenko beam. The proposed method consists in an approximation of the system dynamics by means of a numerical procedure which leads to thickness-independent convergence properties. The solution of the LQR problem implies the solution of an algebraic Riccati equation which receives the dynamics approximated thus as an input and strongly depends on it, and consequently, generates an optimal gain operator that inherits the independence of the thickness. Computational simulations validated this approach.

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