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A LOCKING-FREE FEM IN ACTIVE VIBRATION CONTROL OF A TIMOSHENKO BEAM*

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Abstract. In this paper we analyze the numerical approximation of an active vibration control problem of a Timoshenko beam. In order to avoid locking, we focus on the finite element method used to compute the beam vibration, to minimize it. Optimal order error estimates are obtained for the control variable, which is the amplitude of secondary forces modeled as Dirac's delta distributions. These estimates are valid with constants that do not depend on the thickness of the beam. In order to assess the performance of the method, numerical tests are reported.

Key words. optimal control problems, finite element method, Timoshenko beam, error estimates, locking free method

AMS subject classifications. 49K20, 65N30, 65N15

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1. Introduction.

In recent years a large amount of work on the active control of the flexible structures vibrations has been done. The concrete engineering problems that motivate this study come from very different fields: aerospace, structures, meteorology, nanotechnology, etc. An overview can be found in the book of Fuller, Elliot, and Nelson [13]. A typical engineering problem in which this area is applied is to reduce the acoustic noise produced by the vibrating thin structures.

It is well-known that standard finite element methods applied to models of thin structures—like beams, rods, and plates—are subject to the so-called *locking* phenomenon (see the book of Chapelle and Bathe [7]). This means that they produce very unsatisfactory results when the thickness is small with respect to the other dimensions of the structure. From the point of view of the numerical analysis, this phenomenon usually reveals itself in that the a priori error estimates for these methods depend on the thickness of the structure in such a way that they degenerate when this parameter becomes small. To avoid locking, special methods based on reduced integration or mixed formulations have been devised and are typically used; among them we mention [9, 10], where methods for computing free vibration of plates were analyzed.

The first mathematical piece of work dealing with numerical locking and how to avoid it is the paper by Arnold [3], in which he proves that locking arises because of the shear term, and proposes and analyzes a locking-free method based on a mixed formulation. Recently, this proposed method has been used and analyzed when it is applied to the problem of free vibrations of a general curved rod (see [15]), which covers the Timoshenko beam case.

On the other hand, the problem of active vibration control (AVC) can be set in the framework of mathematical theory of optimal control as stated in the book of Lions [17]; by the way, this corresponds to a minimization problem governed by an elliptic

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partial differential equation. The numerical analysis of this kind of problem is an area of active research, beginning with the classical work of Falk [12] (see [2, 16, 19, 20], and references therein).

In this paper we study the numerical approximation of two problems of AVC applied to a Timoshenko beam. The goal of these problems is the reduction of the vibration in the two sensors considered: pointwise and distributed. The control variable is given by the real amplitudes of the secondary forces, which are modeled by Dirac's deltas. In this way, the problems consist of choosing the optimal real amplitudes of these secondary forces. Since our interest is the numerical analysis, we study a locking free finite element method for the Timoshenko equations in the frequency domain, then we study the approximations of the optimal control for two problems of active vibration control and, finally, optimal error estimates are obtained for the control variables.

There are many works on the analysis of control problems of Timoshenko equations; see, for example, Macchelli and Melchiorri [18] and Xu and Yung [23], and references therein. Despite this, to the best of the author's knowledge, this problem has not yet been analyzed from the numerical analysis point of view.

The outline of this paper is as follows. In section 2 we introduce the physical problems and pose them in the framework of optimal control theory. In section 3 we analyze the Timoshenko equations in the frequency domain (state equations). We prove existence and uniqueness of the solution for general terms in the Sobolev space H^{-1}. Using the results presented in [15], we obtain estimates that do not degenerate with the thickness of the beam. It includes, in addition, local $W^{2,\infty}$ a priori estimates for the state equations with source terms in L^2. In section 4 we introduce a locking-free finite element method to approximate the state equations. We prove L^2 and pointwise error estimates, with constants that do not depend on the thickness. In section 5 we state an optimal control problem to determine the optimal amplitudes of the actuator and show that it is well-posed. We approximate it by using the locking-free finite element method introduced in the previous section, and prove an optimal order error estimate for the approximate optimal control problem. In the last section, we report some numerical experiments which confirm our theoretical assertions. In all sections, C denotes a strictly positive constant, not necessarily the same at each occurrence, but always independent of the thickness t and of the mesh-size h.

2. Mathematical model. The optimal control problem. Let us consider an elastic beam of thickness $t \in (0,1]$, with reference configuration $\mathbf{I} \times (-t/2, t/2)$, where $\mathbf{I} := (0, L)$ with L the length of the beam. The deformation of the beam in the frequency domain is described by means of the Timoshenko model in terms of the rotations amplitude θ of its midplane and the transverse displacement amplitude w (see [22]). Assuming that the beam is clamped, its deformation is the solution of the following problem:

Find (w, θ) such that

$$\text{(2.1)} \qquad \left\{ \begin{array}{rcll} -\rho A \hat{\omega}^2 w(x) - kAG\left(\dfrac{d^2 w}{dx^2} - \dfrac{d\theta}{dx}\right) &=& f(x) & x \in \mathbf{I}, \\ -\rho I \hat{\omega}^2 \theta(x) - EI\dfrac{d^2\theta}{dx^2} - kAG\left(\dfrac{dw}{dx} - \theta\right) &=& g(x) & x \in \mathbf{I}, \\ w(0) = w(L) = \theta(0) = \theta(L) &=& 0, & \end{array} \right.$$

where $\hat{\omega}$ is the angular frequency, the coefficients ρ, E, and I, that will be assumed constants, represent the mass density, the Young modulus, and the inertia moment, respectively. The coefficient k is a correction factor usually taken as $5/6$; A and G represent the sectional area of the beam and elasticity modulus of shear. The term f represents a point or distributed force and g the bending moment.

These equations will be called *state equations* and play an important role in the control problems described below.

In our case, the term source f will be the sum of a pointwise or distributed external force, denoted by f_e, and a control force split as a linear combination of N Dirac's delta measures supported at given points, $y_1, y_2, \ldots, y_N \in \mathbf{I}$ with real amplitudes $u_1, \ldots, u_N$ to be determined, i.e.,

$$f = f_e + \sum_{i=1}^{N} u_i \delta_{y_i}, \qquad \text{with } u_i \in \mathbb{R}, \quad i = 1, \ldots, N. \tag{2.2}$$

Now, we introduce the control problems that concern us, for the mathematical optimal control framework we use the notation and the context introduced in [4]. For the sake of simplicity, we describe the two kinds of sensors, punctual and distributed, separately.

Punctual sensors. The problem of AVC consists of reducing the vibration in M given points, called *punctual sensors*. In order to state this problem mathematically, we make the following choices:

(A.1) the *state of the system* is given by the transversal displacement $w(x)$ of the beam;

(A.2) the *control variable* $\boldsymbol{u}$ is the vector of reals amplitudes of actuators,

$$\boldsymbol{u} = (u_1, \ldots, u_N) \in \mathbb{R}^N,$$

which define the source term f in the problem (2.1) by means of (2.2);

(A.3) the set of *admissible controls* is a convex, not empty, closed set $\mathrm{U}_{ad} \subset \mathbb{R}^N$;

(A.4) the *model of the system* that relates the control variable with the state is the Timoshenko problem, i.e., problem (2.1);

(A.5) the observation $\boldsymbol{z}$ is the set of displacement values at M sensors located at given points $p_1, \ldots, p_M \in \mathbf{I}$,

$$\boldsymbol{z}(\boldsymbol{u}) := (w(\boldsymbol{u}, p_1), \ldots, w(\boldsymbol{u}, p_M)) \in \mathbb{R}^M,$$

where, for $\boldsymbol{u} \in \mathbb{R}^N$, $w(\boldsymbol{u}, \cdot)$ denotes the solution of problem (2.1) with f given by (2.2);

(A.6) the *cost function* to be minimized depends on the observation and eventually on the cost of the control itself, namely,

$$J(\boldsymbol{u}) := \frac{1}{2} \|\boldsymbol{z}(\boldsymbol{u}) - \boldsymbol{z}_d\|_2^2 + \frac{\nu}{2} \|\boldsymbol{u}\|_2^2 , \tag{2.3}$$

where, $\nu \geq 0$ denotes a weighting factor that represents the cost of the control, $\|\cdot\|_2$ is the Euclidian norm in $\mathbb{R}^N$ or $\mathbb{R}^M$, and $\boldsymbol{z}_d$ denotes the desired state, which in our case will be $\boldsymbol{z}_d = \mathbf{0}$.

Thus, the *optimal control problem* will be the following:

Find $\boldsymbol{u}^{op} \in \mathrm{U}_{ad}$ such that

$$J(\boldsymbol{u}^{op}) = \inf_{\boldsymbol{u} \in \mathrm{U}_{ad}} J(\boldsymbol{u}). \tag{2.4}$$

Distributed sensor. The problem of AVC consists of reducing the vibration along the beam, or on a segment of it, namely (a_1, a_2), with $0 \leq a_1 < a_2 \leq L$.

To write this problem mathematically, we make the same choices as the problem above, but changing (A.5) and (A.6), in fact, we have that

(A.5) the observation $\boldsymbol{y}$ is the transverse displacement on (a_1, a_2), i.e., $\boldsymbol{y}(\boldsymbol{u}, x) = w(\boldsymbol{u}, x)|_{(a_1,a_2)}$.

(A.6) the cost function to be minimized in this problem is

$$\mathcal{J}(\boldsymbol{u}) := \frac{1}{2} \|\boldsymbol{y}(\boldsymbol{u}) - \boldsymbol{y}_d\|^2_{L_2(a_1,a_2)} + \frac{\nu}{2} \|\boldsymbol{u}\|_2^2, \tag{2.5}$$

where $\boldsymbol{y}_d$ denotes the desired state, which in our case is $\boldsymbol{y}_d = \mathbf{0}$.

The optimal control problem of distributed sensor is written as follows:

Find $\boldsymbol{u}^{op} \in \mathrm{U}_{ad}$ such that

$$\mathcal{J}(\boldsymbol{u}^{op}) = \inf_{\boldsymbol{u} \in \mathrm{U}_{ad}} \mathcal{J}(\boldsymbol{u}). \tag{2.6}$$

In both optimal control problems, any solution $\boldsymbol{u}^{op}$ of the minimization problem will be called an *optimal control*. Notice that this optimal control depends directly on the amplitude of the transverse displacement $w(x)$ of a Timoshenko beam. For this reason, the mathematical analysis of the problem (2.1) will be considered in what follows.

3. State equations. In this section we prove existence and uniqueness of solution of the state equations considering an adequate framework for the mathematical analysis of numerical locking that appears when standard finite element methods are applied to the beam equation (2.1). Additional regularity of the solution is also included, it will be used to study the optimal control problems in the following sections.

3.1. A locking-free scheme. From now on, we assume square transversal section of the beam, with physical parameters $I = t^4/12$, $A = t^2$, and $G = E/2(1+\bar{\nu})$, where $\bar{\nu}$ denotes the Poisson ratio. By considering $f, g \in H^{-1}(\mathbf{I})$ and $v, \beta \in H^1_0(\mathbf{I})$, as test functions, using integration by parts, and considering the boundary conditions, we obtain the following variational formulation associated to the problem (2.1):

Find $(w_t, \theta_t) \in H^1_0(\mathbf{I})^2$ such that

$$\left\{\begin{array}{l} \dfrac{E}{12} \displaystyle\int_{\mathbf{I}} \frac{d\theta_t}{dx} \frac{d\beta}{dx} dx + \frac{\kappa}{t^2} \int_{\mathbf{I}} \left(\frac{dw_t}{dx} - \theta_t \right) \left(\frac{dv}{dx} - \beta \right) dx - \omega_t^2 \int_{\mathbf{I}} w_t v dx \\ -\omega_t^2 \dfrac{t^2}{12} \displaystyle\int_{\mathbf{I}} \theta_t \beta dx = \langle f, v \rangle + \frac{t^2}{12} \langle g, \beta \rangle \qquad \forall (v, \beta) \in H^1_0(\mathbf{I})^2, \end{array}\right. \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual parity between the spaces $H^{-1}(\mathbf{I})$ and $H^1_0(\mathbf{I})$, $\omega_t^2 := \rho \widehat{\omega}^2 / t^2$ is the rescaled angular frequency, and $\kappa = Ek/2(1+\bar{\nu})$. Note that, according to [7], the transversal and shear load have been adequately rescaled.

We denote by $a_{\omega t}$ the bilinear continuous form in $H^1_0(\mathbf{I})^2$ that appears in the left-hand side of (3.1):

$$\begin{aligned} a_{\omega t}\left((w_t, \theta_t), (v, \beta)\right) = \frac{E}{12} \int_{\mathbf{I}} \frac{d\theta_t}{dx} \frac{d\beta}{dx} dx + \frac{\kappa}{t^2} \int_{\mathbf{I}} \left(\frac{dw_t}{dx} - \theta_t \right) \left(\frac{dv}{dx} - \beta \right) dx \\ -\omega_t^2 \int_{\mathbf{I}} w_t v dx - \omega_t^2 \frac{t^2}{12} \int_{\mathbf{I}} \theta_t \beta dx \quad (w_t, \theta_t), (v, \beta) \in H^1_0(\mathbf{I})^2. \end{aligned} \tag{3.2}$$

This bilinear form is not positive definite, for this reason the *Lax–Milgram lemma* cannot be applied to obtain existence and uniqueness of solution of the variational problem (3.1). However, it is clear that problem (3.1) has a unique solution if ω_t^2 is not a eigenvalue of the following homogeneous problem:

Find $(w_t, \theta_t) \in H_0^1(\mathbf{I})^2$ such that

$$a_{\omega t}\left((w_t, \theta_t), (v, \beta)\right) = 0 \quad \forall (v, \beta) \in H_0^1(\mathbf{I})^2. \tag{3.3}$$

This eigenvalue problem has been recently analyzed in [15], as a particular case of a more general problem, and it is proved that the spectrum consists of a sequence of finite multiplicity real eigenvalues converging to infinite.

On the other hand, it is easy to prove than for a fixed t the bilinear form $a_{\omega t}$ satisfies the following Gårding inequality:

$$a_{\omega t}\left((w_t, \theta_t), (w_t, \theta_t)\right) + C_\omega \|(w_t, \theta_t)\|_{L^2(\mathbf{I})^2}^2 \geq \alpha \|(w_t, \theta_t)\|_{H^1(\mathbf{I})^2}^2 \tag{3.4}$$

for all pair $(w_t, \theta_t) \in H_0^1(\mathbf{I})^2$, with positive constants C_ω and α such that $C_\omega > \omega_t^2$ and $\alpha = \max(E/12, C_\omega, C/t^2)$. Thus, according to Theorem 6.5.15 in [14] (3.3) satisfies Fredholm's alternative; i.e., uniqueness of solution of the problem (3.1) implies existence of solution giving the following result.

THEOREM 3.1. *Let $\omega_t \in \mathbb{R}$ such that $\omega_t^2 \notin \mathcal{S}$, where $\mathcal{S}$ denotes the spectrum of problem (3.3), $t \in (0, 1]$, and $f, g \in H^{-1}(\mathbf{I})$. Then, problem (3.1) has existence and uniqueness of solution $(w_t, \theta_t) \in H_0^1(\mathbf{I})^2$, and, moreover, the following estimate holds:*

$$\|(w_t, \theta_t)\|_{H^1(\mathbf{I})^2} \leq C_t \left(\|f\|_{H^{-1}(\mathbf{I})} + t^2 \|g\|_{H^{-1}(\mathbf{I})}\right). \tag{3.5}$$

Proof. Note that, we only need to prove (3.5). Let us denote by $\langle \cdot, \cdot \rangle_t$ a weighting interior product in $L^2(\mathbf{I})^2$ defined by $\langle (u, v), (w, z) \rangle_t := (u, w)_{L^2(\mathbf{I})} + t^2/12 \cdot (v, z)_{L^2(\mathbf{I})}$.

Consider the operator $A_{\omega t}$ defined by

$$\langle A_{\omega t}(w_t, \theta_t), (v, \beta) \rangle_t = a_{\omega t}\left((w_t, \theta_t), (v, \beta)\right) \quad \forall (v, \beta) \in H_0^1(\mathbf{I})^2.$$

Clearly, the operator $A_{\omega t}$ is linear, continuous, and bijective, and then, using the open mapping theorem, $A_{\omega t}$ has a linear and continuous inverse, i.e.,

$$\|(w_t, \theta_t)\|_{H^1(\mathbf{I})^2} \leq C_t \|(f, g)\|_{H^{-1}(\mathbf{I})},$$

where, clearly, the constant C_t depends on α in (3.4). ∎

The analysis done so far is valid only for t fixed, i.e., it is not uniformly valid in t. In this context, in [3] it is shown that standard finite elements methods applied to the load problem associated to the static Timoshenko beam are subject to the locking phenomenon, this means that they produce unsatisfactory results for very thin beams; this effect is caused by the shear stress term. The same phenomenon occurs in our case, where a dynamic Timoshenko beam is considered, because the shear effect is also present in our model. In fact, according to [21], because the bilinear form $a_{\omega t}$ satisfies the Gårding inequality (3.4), if we consider standard finite element methods to solve (3.1), we obtain existence and uniqueness of the discrete solution (w_{th}, θ_{th}) only for $h < C/t^2$ and the following poor estimation holds:

$$\|(w_t, \theta_t) - (w_{th}, \theta_{th})\|_{L^2(\mathbf{I})^2} \leq \frac{C}{t^2} h^2 \|(w_t, \theta_t)\|_{H^2(\mathbf{I})^2}.$$

To avoid the numerical-locking in the static case, Arnold [3] introduces and analyzes a locking-free method based on a mixed formulation of the problem, and also proves that this mixed method is equivalent to using a reduced-order scheme for the integration of the shear term in the primal formulation. These ideas have been extended to the vibration modes of a Timoshenko curved rod with arbitrary geometry in [15].

We will use a mixed formulation to obtain a locking-free scheme that applies to our dynamic problem. In order to achieve this purpose, it is necessary to obtain a stability condition with a constant that does not degenerate when the thickness t goes to zero. For this reason, we introduce the operator

$$T_t : H^{-1}(\mathbf{I})^2 \longrightarrow H_0^1(\mathbf{I})^2,$$

defined by $T_t(f,g) := (w^t, \theta^t)$, where (w^t, θ^t) is solution of the following load problem associated with the Timoshenko equations in static case, written in mixed form:

Find $(w^t, \theta^t, \gamma^t) \in H_0^1(\mathbf{I})^2 \times L^2(\mathbf{I})$ such that

$$\left\{ \begin{aligned} \frac{E}{12}\int_{\mathbf{I}} \frac{d\theta^t}{dx}\frac{d\beta}{dx}dx + \int_{\mathbf{I}} \gamma^t \left(\frac{dv}{dx} - \beta\right) dx &= \langle f, v\rangle + \frac{t^2}{12}\langle g, \beta\rangle \quad \forall (v,\beta) \in H_0^1(\mathbf{I})^2, \\ \frac{t^2}{\kappa}\int_{\mathbf{I}} \gamma^t \eta dx &= \int_{\mathbf{I}} \left(\frac{dw^t}{dx} - \theta^t\right)\eta dx \qquad \forall \eta \in L^2(\mathbf{I}). \end{aligned} \right.$$

According to [15], as a particular case, this problem has a unique solution $(w^t, \theta^t, \gamma^t) \in H_0^1(\mathbf{I})^2 \times L^2(\mathbf{I})$ and there holds

$$\|w^t\|_{H^1(\mathbf{I})} + \|\theta^t\|_{H^1(\mathbf{I})} + \|\gamma^t\|_{L^2(\mathbf{I})} \le C\left(\|f\|_{H^{-1}(\mathbf{I})} + t^2\|g\|_{H^{-1}(\mathbf{I})}\right). \tag{3.6}$$

Now, using the operator T_t we can rewrite problem (3.1) in the following equivalent form:

Find $(w_t, \theta_t) \in H_0^1(\mathbf{I})$ such that

$$(w_t, \theta_t) - \omega_t^2 T_t(w_t, \theta_t) = T_t(f, g). \tag{3.7}$$

This new form to write our state equations is essential to obtain the desired condition for stability with a constant that does not degenerate when t goes to zero, which is obtained in the following theorem.

THEOREM 3.2. *Let $\omega_t \in \mathbb{R}$ such that $\omega_t^2 \notin \mathcal{S}$, and $f, g \in H^{-1}(\mathbf{I})$. Then, problem (3.1) has a unique solution $(w_t, \theta_t) \in H_0^1(\mathbf{I})^2$ that satisfies*

$$\|(w_t, \theta_t)\|_{H^1(\mathbf{I})^2} \le C\left(\|f\|_{H^{-1}(\mathbf{I})} + t^2\|g\|_{H^{-1}(\mathbf{I})}\right).$$

Moreover, if $f, g \in L^2(\mathbf{I})^2$, the solution belongs to $H^2(\mathbf{I})^2 \cap H_0^1(\mathbf{I})^2$, and there holds

$$\|(w_t, \theta_t)\|_{H^2(\mathbf{I})^2} \le C\left(\|f\|_{L^2(\mathbf{I})} + t^2\|g\|_{L^2(\mathbf{I})}\right).$$

Proof. Due to $\omega_t^2 \notin \mathcal{S}$, according to [15], $T_t^{-1}(I - \omega_t^2 T_t)$ is bounded and bijective, then from (3.7) we have the following estimation:

$$\|(w_t, \theta_t)\|_{H^1(\mathbf{I})^2} \le \frac{1}{\varsigma}\|T_t^{-1}(I - \omega_t^2 T_t)(w_t, \theta_t)\|_{H^{-1}(\mathbf{I})^2},$$

where ς denotes the positive constant in the open mapping theorem. Thus, the theorem follows from the fact that ς depends on the operator's norm $T_t^{-1}(I - \omega_t^2 T_t)$ and, as a consequence of (3.6), $\|T_t^{-1}\| \le C$.

Finally, the additional regularity is obtained in a similar way to Proposition 3 in [6]. ∎

3.2. Local $W^{2,\infty}$ a priori estimates. In this section we obtain L^∞ estimates for the second derivatives of the solution of problem (2.1), in the cases that f and g belong to $L^2(\mathbf{I})$. These estimates will be used to obtain a part of the main result of this paper in section 5: optimal error estimates for the optimal control problem with punctual sensors.

Considering the external source f as Dirac's delta concentrated in $y \in \mathbf{I}$ and $g = 0$, problem (2.1) is written as follows:

Find (w^y, θ^y) such that

$$(3.8) \qquad \begin{cases} -\rho A\hat{\omega}^2 w^y(x) - kAG\left(\dfrac{d^2w^y}{dx^2} - \dfrac{d\theta^y}{dx}\right) = \delta_y, & x \in \mathbf{I}, \\ -\rho I\hat{\omega}^2\theta^y(x) - EI\dfrac{d^2\theta^y}{dx^2} - kAG\left(\dfrac{dw^y}{dx} - \theta^y\right) = 0, & x \in \mathbf{I}, \\ w^y(0) = w^y(L) = \theta^y(0) = \theta^y(L) = 0, \end{cases}$$

where the first equation should be understood in the distributional sense.

Let (ϕ^y, φ^y) be the fundamental solution of the dynamic Timoshenko equations written in the frequency domain, by considering $f = \delta_y$ and $g = 0$. Such a solution is explicitly known in the work [1, section 3.1.2], but we only need to recall that it satisfies $\phi^y(x), \varphi^y(x) \in \mathcal{C}^0(\mathbf{I})$, $\phi^y(x), \varphi^y(x) \in \mathcal{C}^\infty(\mathbb{R}\backslash\{y\})$. Moreover, let $d > 0$ be such that $\mathbf{I}^d(y) := \{x \in \mathbb{R} : |x - y| < d\} \subset\subset \mathbf{I}$, then $\|\phi^y\|_{H^2(\mathbf{I}\backslash\mathbf{I}^d(y))}$ and $\|\varphi^y\|_{H^2(\mathbf{I}\backslash\mathbf{I}^d(y))}$ remains bounded with constants that only depend on d (see [1] for further details).

Now, solution of (3.8) can be split in the following way:

$$(w^y, \theta^y) = (\phi^y|_{\mathbf{I}}, \varphi^y|_{\mathbf{I}}) + (\zeta^y, \eta^y),$$

where (ζ^y, η^y) denotes the solution of

$$(3.9) \qquad \begin{cases} -\rho A\hat{\omega}^2 \zeta^y(x) - kAG\left(\dfrac{d^2\zeta^y}{dx^2} - \dfrac{d\eta^y}{dx}\right) = 0, & x \in \mathbf{I}, \\ -\rho I\hat{\omega}^2\eta^y(x) - EI\dfrac{d^2\eta^y}{dx^2} - kAG\left(\dfrac{d\zeta^y}{dx} - \eta^y\right) = 0, & x \in \mathbf{I}, \\ \zeta^y(0) = -\phi^y(0), \qquad \eta^y(0) = -\varphi^y(0), \\ \zeta^y(L) = -\phi^y(L), \quad \eta^y(L) = -\varphi^y(L). \end{cases}$$

It is easy to see that, by standard argument on nonhomogeneous Dirichlet problems and using Theorem 3.1, there exists a unique solution (ζ^y, η^y) of (3.9) and satisfies

$$(3.10) \qquad \|(\zeta^y, \eta^y)\|_{H^2(\mathbf{I})^2} \leq C.$$

Hence, we obtain

$$(3.11) \qquad \begin{aligned} \|w^y\|_{H^2(\mathbf{I}\backslash\mathbf{I}^d(y))} &\leq \|\phi^y\|_{H^2(\mathbf{I}\backslash\mathbf{I}^d(y))} + \|\zeta^y\|_{H^2(\mathbf{I})} \leq C, \\ \|\theta^y\|_{H^2(\mathbf{I}\backslash\mathbf{I}^d(y))} &\leq \|\varphi^y\|_{H^2(\mathbf{I}\backslash\mathbf{I}^d(y))} + \|\eta^y\|_{H^2(\mathbf{I})} \leq C. \end{aligned}$$

These estimates will be used in the proof of the following theorem.

THEOREM 3.3. *Let $\mathbf{I}_0$ and $\mathbf{I}_1$ be two open subsets of $\mathbf{I}$ such that $\mathbf{I}_0, \mathbf{I}_1 \subset\subset \mathbf{I}$. Let $d > 0$ such that $dist(\mathbf{I}_0, \mathbf{I}_1) > d$ and $dist(\mathbf{I}_1, \partial\mathbf{I}) > d$. Then, if f and $g \in L^2(\mathbf{I})$ satisfy $supp(f)$, $supp(g) \subset \mathbf{I}_0$, there exists a constant C, only depending on d, such that the solution (w_t, θ_t) of problem* (3.1) *satisfies*

$$(3.12) \qquad \|(w_t, \theta_t)\|_{W^{2,\infty}(\mathbf{I}_1)^2} \leq C\left(\|f\|_{L^2(\mathbf{I})} + t^2\|g\|_{L^2(\mathbf{I})}\right).$$

Proof. We adapt the proof of Lemma 3.4 in [4] to our case. Let us consider subsets of $\mathbf{I}$: $\mathbf{I}_2 := \{x \in \mathbf{I} : \operatorname{dist}(x, \mathbf{I}_1) < d/4\}$ and $\mathbf{I}_3 := \{x \in \mathbf{I} : \operatorname{dist}(x, \mathbf{I}_1) < d/2\}$, that satisfies $\operatorname{dist}(\mathbf{I}_0, \mathbf{I}_3) \geq d/2$ and $\operatorname{dist}(\mathbf{I}_3, \partial\mathbf{I}) \geq d/2$. Moreover, we denote by χ a cut-off real function of class $\mathcal{C}^\infty$ with support $\mathbf{I}_3$ such that $\chi|_{\mathbf{I}_2} = 1$ and $\|\chi\|_{W^{2,\infty}(\mathbb{R})}$ is bounded.

Given $z \in \mathbf{I}_1$, let (w^z, θ^z) be the solution of (3.8) with y replaced by z. By standard computation (see, for instance, [11]) we have

$$\begin{aligned} w_t(z) = \langle \delta_z, \chi w_t \rangle &= \left\langle -\rho A \hat{\omega}^2 w^z - kAG\left(\frac{d^2 w^z}{dx^2} - \frac{d\theta^z}{dx}\right), \chi w_t \right\rangle \\ &= \left\langle w^z, -\rho A\hat{\omega}^2 \chi w_t - kAG\frac{d^2\chi}{dx^2} w_t - 2kAG\frac{d\chi}{dx}\frac{dw_t}{dx} - kAG\chi\frac{d^2 w_t}{dx^2} \right\rangle \\ &\quad - \left\langle \theta^z, kAG\frac{d\chi}{dx} w_t + kAG\chi\frac{dw_t}{dx} \right\rangle. \end{aligned}$$

On the other hand, from the second equation of (3.8) we have

$$\begin{aligned} 0 &= \left\langle -\rho I\hat{\omega}^2 \theta^z(x) - EI\frac{d^2\theta^z}{dx^2} - kAG\left(\frac{dw^z}{dx} - \theta^z\right), \chi\theta_t \right\rangle \\ &= \left\langle \theta^z, -\rho I\hat{\omega}^2\chi\theta_t - EI\frac{d^2\chi}{dx^2}\theta_t - 2EI\frac{d\chi}{dx}\frac{d\theta_t}{dx} - EI\chi\frac{d^2\theta_t}{dx^2} + kAG\chi\theta_t \right\rangle \\ &\quad + \left\langle w^z, kAG\frac{d\chi}{dx}\theta_t + kAG\chi\frac{d\theta_t}{dx} \right\rangle. \end{aligned}$$

By adding the last two equations, considering that (w_t, θ_t) is the solution of problem (3.1) and noting the fact that $\chi f = 0$ and $\chi g = 0$, we obtain that

$$\begin{aligned} w_t(z) &= \int_{\mathbf{I}_3 \setminus \overline{\mathbf{I}_2}} w^z \left(kAG\frac{d\chi}{dx}\theta_t - kAG\frac{d^2\chi}{dx^2} w_t - 2kAG\frac{d^2\chi}{dx^2}\frac{dw_t}{dx} \right) dx \\ &\quad + \int_{\mathbf{I}_3 \setminus \overline{\mathbf{I}_2}} \theta^z \left(-EI\frac{d^2\chi}{dx^2}\theta_t - 2EI\frac{d\chi}{dx}\frac{d\theta_t}{dx} - kAG\frac{d\chi}{dx} w_t \right) dx, \end{aligned}$$

where the integral is written on $\mathbf{I}_3 \setminus \overline{\mathbf{I}_2}$, because $\operatorname{supp}(\chi')$, $\operatorname{supp}(\chi'') \subset \mathbf{I}_3 \setminus \overline{\mathbf{I}_2}$.

Since w^z is symmetric, because the operator $I - \omega_t^2 T_t$ is symmetric, $w^z(x) = w^x(z)$ for all $x, z \in \mathbf{I}$: $y \neq z$, we can differentiate the expression above to obtain

$$\begin{aligned} \frac{d}{dz^2} w_t(z) &= \int_{\mathbf{I}_3 \setminus \overline{\mathbf{I}_2}} \frac{d}{dx^2} w^z \left(kAG\frac{d\chi}{dx}\theta_t - kAG\frac{d^2\chi}{dx^2} w_t - 2kAG\frac{d^2\chi}{dx^2}\frac{dw_t}{dx} \right) dx \\ &\quad + \int_{\mathbf{I}_3 \setminus \overline{\mathbf{I}_2}} \frac{d}{dx^2}\theta^z \left(-EI\frac{d^2\chi}{dx^2}\theta_t - 2EI\frac{d\chi}{dx}\frac{d\theta_t}{dx} - kAG\frac{d\chi}{dx} w_t \right) dx. \end{aligned}$$

Then, using (3.11) and Theorem 3.2, we obtain the following result:

$$\begin{aligned} \left| \frac{d}{dz^2} w_t(z) \right| &\leq C \|\chi\|_{W^{2,\infty}(\mathbf{I}_3 \setminus \overline{\mathbf{I}_2})} \left(\|w^z\|_{H^2(\mathbf{I}_3 \setminus \overline{\mathbf{I}_2})} + \|\theta^z\|_{H^2(\mathbf{I}_3 \setminus \overline{\mathbf{I}_2})} \right) \\ &\quad \cdot \left(\|w_t\|_{H^1(\mathbf{I}_3 \setminus \overline{\mathbf{I}_2})} + \|\theta_t\|_{H^1(\mathbf{I}_3 \setminus \overline{\mathbf{I}_2})} \right) \\ &\leq C \left(\|f\|_{L^2(\mathbf{I})} + t^2 \|g\|_{L^2(\mathbf{I})} \right). \end{aligned}$$

For the estimation on θ we proceed in the same way. ∎

4. Numerical approximation of the state equations. In this section we will study the numerical approximation of the state equations in the case in which the source term f is a pointwise or distributed external force. We obtain pointwise error estimates which will be used in the next section to obtain uniform error estimates of the optimal control problem.

Following [3], we consider a family $\{\mathcal{T}_h\}$ of partitions of the interval $\mathbf{I}$:

$$\mathcal{T}_h : 0 = s_0 < s_1 < \cdots s_n = L,$$

with mesh-size

$$h := \max_{j=1,\dots,n} (s_j - s_{j-1}).$$

We define the following finite element spaces:

$$\mathcal{V}_h := \left\{ v \in H_0^1(\mathbf{I}) : v|_{[s_{j-1},s_j]} \in \mathbb{P}_1, j = 1,\dots,n \right\} \subset H_0^1(\mathbf{I}),$$

and

$$\mathcal{W}_h := \left\{ \frac{dv}{dx} + c : \ v \in \mathcal{V}_h, c \in \mathbb{R} \right\} \subset L^2(\mathbf{I}).$$

Thus, we can write the discrete version of variational problem (3.1) as follows:

Find $(w_{th}, \theta_{th}) \in \mathcal{V}_h^2$ such that

$$a_{\omega th}\left((w_{th}, \theta_{th}), (v_h, \beta_h)\right) = \langle f, v_h \rangle + \frac{t^2}{12}\langle g, \beta_h \rangle \qquad \forall (v_h, \beta_h) \in \mathcal{V}_h^2, \tag{4.1}$$

where the bilinear form $a_{\omega th}$ is given by

$$\begin{aligned} a_{\omega th}\left((w_{th}, \theta_{th}), (v_h, \beta_h)\right) &= \frac{E}{12}\int_{\mathbf{I}} \frac{d\theta_{th}}{dx}\frac{d\beta_h}{dx}dx - \omega_t^2 \int_{\mathbf{I}} w_{th} v_h dx - \omega_t^2 \frac{t^2}{12}\int_{\mathbf{I}} \theta_{th}\beta_h dx \\ &+ \frac{\kappa}{t^2}\int_{\mathbf{I}} \pi_h \left(\frac{dw_{th}}{dx} - \theta_{th}\right) \pi_h \left(\frac{dv_h}{dx} - \beta_h\right) dx \end{aligned} \tag{4.2}$$

for all $(w_{th}, \theta_{th}), (v_h, \beta_h) \in \mathcal{V}_h^2$, where π_h denotes the L^2-projector onto $\mathcal{W}_h$.

As in the continuous case, in order to study existence and uniqueness of the solution of the problem above, we introduce the discrete operator

$$T_{th} : H^{-1}(\mathbf{I})^2 \longrightarrow \mathcal{V}_h^2,$$

defined by $T_{th}(f, g) := (w_h^t, \theta_h^t)$, where (w_h^t, θ_h^t) denotes the solution of the discrete version of the mixed problem, i.e.,

Find $(w_h^t, \theta_h^t, \gamma_h^t) \in \mathcal{V}_h^2 \times \mathcal{W}_h$ such that

$$\left\{ \begin{aligned} \frac{E}{12}\int_{\mathbf{I}} \frac{d\theta_h^t}{dx}\frac{d\beta_h}{dx}dx + \int_{\mathbf{I}} \gamma_h^t \left(\frac{dv_h}{dx} - \beta_h\right) dx &= \langle f, v_h \rangle + \frac{t^2}{12}\langle g, \beta_h \rangle, \\ \frac{t^2}{\kappa}\int_{\mathbf{I}} \gamma_h^t \eta_h dx &= \int_{\mathbf{I}} \left(\frac{dw_h^t}{dx} - \theta_h^t\right) \eta_h dx \end{aligned} \right.$$

for all $(v_h, \beta_h) \in \mathcal{V}_h^2$ and for all $\eta_h \in \mathcal{W}_h$, respectively.

As a consequence of Theorem 3.1 in [15], the operator T_{th} is well-defined and continuous; in fact, the solution of the mixed problem above satisfies

$$\|w_h^t\|_{H^1(\mathbf{I})} + \|\theta_h^t\|_{H^1(\mathbf{I})} + \|\gamma_h^t\|_{L^2(\mathbf{I})} \leq C\left(\|f\|_{H^{-1}(\mathbf{I})} + t^2\|g\|_{H^{-1}(\mathbf{I})}\right).$$

Now we use the operator T_{th} to study existence and uniqueness of our discrete problem (4.1). First, we introduce the following technical result.

LEMMA 4.1. *For t and h small enough it holds that if μ_t is not an eigenvalue of the operator T_t, then neither is an eigenvalue of the operator T_{th}.*

Proof. According to Lemma 2.6 in [15], for t and h small enough we can guarantee the separation of isolated parts of the spectrum of T_t. Then, this lemma is a direct consequence of the spectral approximation of the discrete operator T_{th} (see Theorem 3.5 in [15]). □

THEOREM 4.2. *Given $f, g \in H^{-1}(\mathbf{I})$ and $\omega_t^2 \in \mathbb{R}$ such that $\omega_t^2 \notin \mathcal{S}$, there exists $h_0 > 0$ such that, for all $h < h_0$, problem (4.1) has a unique solution (w_{th}, θ_{th}). Moreover, if $f, g \in L^2(\mathbf{I})$, the following estimates hold:*

$$\|(w_t, \theta_t) - (w_{th}, \theta_{th})\|_{H^1(\mathbf{I})^2} \leq Ch\left(\|f\|_{L^2(\mathbf{I})} + t^2\|g\|_{L^2(\mathbf{I})}\right), \tag{4.3}$$

$$\|(w_t, \theta_t) - (w_{th}, \theta_{th})\|_{L^2(\mathbf{I})^2} \leq Ch^2\left(\|f\|_{L^2(\mathbf{I})} + t^2\|g\|_{L^2(\mathbf{I})}\right). \tag{4.4}$$

On the other hand, if $f = \delta_x$ and $g = \delta_y$, where $x, y \in \boldsymbol{I}$ are grid-point, then

$$\|(w_t, \theta_t) - (w_{th}, \theta_{th})\|_{H^1(\mathbf{I})^2} \leq Ch, \tag{4.5}$$

$$\|(w_t, \theta_t) - (w_{th}, \theta_{th})\|_{L^2(\mathbf{I})^2} \leq Ch^2. \tag{4.6}$$

Proof. Analogously to the continuous case, the discrete problem (4.1) can be written using the operator T_{th} in the following manner:

$$(w_{th}, \theta_{th}) - \omega_t^2 T_{th}(w_{th}, \theta_{th}) = T_{th}(f, g).$$

Using Lemma 4.1, it is easy to see that ω_t^2 either belongs to the spectrum of the operator T_{th} and, as a consequence of this, we conclude existence and uniqueness for problem (4.1) for all $\omega_t \in \mathbb{R}$ such that $\omega_t^2 \notin \mathcal{S}$. Moreover, by means of the same arguments shown in the proof of Theorem 3.2, we obtain the following discrete stability condition:

$$\|(w_{th}, \theta_{th})\|_{H^1(\mathbf{I})^2} \leq C\left(\|f\|_{H^{-1}(\mathbf{I})} + t^2\|g\|_{H^{-1}(\mathbf{I})}\right). \tag{4.7}$$

On the other hand, as in [3], it is easy to show that problem (4.1) is equivalent to the following mixed problem:

Find $(w_{th}, \theta_{th}, \gamma_{th}) \in \mathcal{V}_h^2 \times \mathcal{W}_h$ such that

$$\begin{cases} a\left((w_{th}, \theta_{th}), (v_h, \beta_h)\right) + \displaystyle\int_{\mathbf{I}} \gamma_{th}\left(\frac{dv_h}{dx} - \beta_h\right)dx = \langle f, v_h\rangle + \frac{t^2}{12}\langle g, \beta_h\rangle, \\ \displaystyle\int_{\mathbf{I}}\left(\frac{dw_{th}}{dx} - \theta_{th}\right)\eta_h dx - \frac{t^2}{\kappa}\int_{\mathbf{I}} \gamma_{th}\eta_h dx = 0 \end{cases}$$

for all $(v_h, \beta_h) \in \mathcal{V}_h^2$ and for all $\eta_h \in \mathcal{W}_h$, respectively. Here, the bilinear form is defined by

$$a\left((w_{th}, \theta_{th}), (v_h, \beta_h)\right) = \frac{E}{12}\int_{\mathbf{I}} \frac{d\theta_{th}}{dx}\frac{d\beta_h}{dx}dx - \omega_t^2\int_{\mathbf{I}} w_{th}v_h dx - \omega_t^2\frac{t^2}{12}\int_{\mathbf{I}} \theta_{th}\beta_h dx.$$

Note that, this problem corresponds to the discretization of the continuous equation (3.1) written as a mixed problem. Then, (4.3) is a consequence of the second part of Proposition II.2.11 from [5], considering the stability condition given in (4.7).

The estimation (4.4) is obtained by adapting the duality argument used in the proof of Theorem 2 in [6] to our case.

Finally, note that, where x and y are grid-point, the solution of problem (4.1) locally belongs to H^2, then (4.5) and (4.6) follow by similar arguments. ∎

The following lemmas will be used to obtain an error estimate for $|w-w_{th}|$. First, we introduce the following standard projections of w and θ over $\mathcal{V}_h$ (see [8]):

$$((\mathcal{P}w_t - w_t)', v')_{L^2(\mathbf{I})} = 0 \quad \forall v \in \mathcal{V}_h, \tag{4.8}$$

$$((\mathcal{P}\theta_t - \theta_t)', \beta')_{L^2(\mathbf{I})} = 0 \quad \forall \beta \in \mathcal{V}_h. \tag{4.9}$$

Here and therein v' stands for dv/dx.

LEMMA 4.3. *Given $f, g \in L^2(\mathbf{I})$ and $\omega_t^2 \notin \mathcal{S}$. Let (w_t, θ_t) and (w_{th}, θ_{th}) be the solution of problems* (3.1) *and* (4.1)*, respectively. Then, there exists $h_0 > 0$ such that, for all $h < h_0$, the following estimation holds:*

$$\|\mathcal{P}w_t - w_{th}\|_\infty \leq Ch^2 \left(\|f\|_{L^2(\mathbf{I})} + t^2\|g\|_{L^2(\mathbf{I})}\right).$$

Proof. By taking $(v, \beta) = (\mathcal{P}w_t - w_{th}, 0)$ in (3.1) and, subtracting from (4.1) with $(v_h, \beta_h) = (\mathcal{P}w_t - w_{th}, 0)$, we obtain the error equation

$$a_{wt}\left((w_t, \theta_t), (\mathcal{P}w_t - w_{th}, 0)\right) - a_{wth}\left((w_{th}, \theta_{th}), (\mathcal{P}w_t - w_{th}, 0)\right) = 0. \tag{4.10}$$

In order to simplify the notation we will use the following expressions:

$$\begin{aligned} \varsigma &= \theta_t - \theta_{th}, \\ \sigma &= w_t - w_{th}, \\ \bar{\sigma} &= \mathcal{P}w_t - w_{th}. \end{aligned}$$

From (4.10) and the definition of a_{wt} and a_{wth}, we obtain

$$\begin{aligned} \frac{k}{t^2}(\sigma', \bar{\sigma}') &= \frac{k}{t^2}(\theta_t - \pi_h\theta_{th}, \bar{\sigma}') + \omega_t^2(\sigma, \bar{\sigma}), \\ &= \frac{k}{t^2}(\varsigma, \bar{\sigma}') + \frac{k}{t^2}(\theta_{th} - \pi_h\theta_{th}, \bar{\sigma}') + \omega_t^2(\sigma, \bar{\sigma}). \end{aligned}$$

Now, by using the definition of π_h and the Cauchy–Schwarz inequality in $L^2(\mathbf{I})$ and $\mathbb{R}^2$, respectively, we obtain

$$\begin{aligned} k(\sigma', \bar{\sigma}') &\leq k\|\varsigma\|_{L^2(\mathbf{I})}\|\bar{\sigma}'\|_{L^2(\mathbf{I})} + \omega_t^2 t^2\|\sigma\|_{L^2(\mathbf{I})}\|\bar{\sigma}\|_{L^2(\mathbf{I})} \\ &\leq C\|(\varsigma, \sigma)\|_{L^2(\mathbf{I})^2}\|\bar{\sigma}\|_{H^1(\mathbf{I})}. \end{aligned}$$

From the definition of the projectors, we can use that $(\sigma', \bar{\sigma}') = (\bar{\sigma}', \bar{\sigma}')$, and the Poincare inequality to obtain

$$\|\bar{\sigma}\|_{H^1(\mathbf{I})}^2 \leq C\|(\varsigma, \sigma)\|_{L^2(\mathbf{I})^2}\|\bar{\sigma}\|_{H^1(\mathbf{I})}. \tag{4.11}$$

Finally, the desired estimation follows from the continuous inclusion $H^1(\mathbf{I}) \hookrightarrow L^\infty(\mathbf{I})$ and Theorem 4.2. ∎

LEMMA 4.4. *Given $f, g \in L^2(\mathbf{I})$ such that $supp(f), supp(g) \subset\subset \mathbf{I}$ and $\omega_t^2 \notin \mathcal{S}$, let us denote by (w_t, θ_t) the solution of problem* (3.1). *Given $x \in \mathbf{I} \backslash (supp(f) \cup supp(g))$, let $d > 0$ such that $dist(x, supp(f) \cup supp(g)) \geq d$ and $dist(supp(f) \cup supp(g), \partial\mathbf{I}) \geq d$. Then, there exists C, depending on d, and $h_0 > 0$ such that, for all $h < h_0$, the following pointwise error estimate holds:*

$$|(w_t(x), \theta_t(x)) - (\mathcal{P}w_t(x), \mathcal{P}\theta_t(x))| \leq Ch^2 \left(\|f\|_{L^2(\mathbf{I})} + \|g\|_{L^2(\mathbf{I})} \right).$$

Proof. By considering $\mathbf{I}_1 = \{x \in \mathbf{I} : \operatorname{dist}(x, \operatorname{supp}(f) \cup \operatorname{supp}(g)) \geq d\}$, because of Theorem 3.3, we have $(w_t, \theta_t) \in W^{2,\infty}(\mathbf{I}_1)^2$, then, recalling Lemma 4.3 in [8], we have

$$\|(w_t, \theta_t) - (\mathcal{P}w_t, \mathcal{P}\theta_t)\|_{L^\infty(\mathbf{I}_1)^2} \leq Ch^2 \|(w_t, \theta_t)\|_{W^{2,\infty}(\mathbf{I}_1)^2},$$

then the desired result follows using the estimation in Theorem 3.3. ∎

THEOREM 4.5. *Given $f, g \in L^2(\mathbf{I})$, such that $supp(f) \subset\subset \mathbf{I}$ and $supp(g) \subset\subset \mathbf{I}$, and $\omega_t^2 \notin \mathcal{S}$, let us denote by (w_t, θ_t) and (w_{th}, θ_{th}) the solution of problems* (3.1) *and* (4.1), *respectively. Given $x \in \mathbf{I} \backslash (supp(f) \cup supp(g))$, let $d > 0$ such that $dist(x, supp(f) \cup supp(g)) \geq d$. Then, there exist C, only depending on d, and $h_0 > 0$ such that, for all $h < h_0$, the following estimate holds:*

$$|w_t(x) - w_{th}(x)| \leq Ch^2 \left(\|f\|_{L^2(\mathbf{I})} + t^2 \|g\|_{L^2(\mathbf{I})} \right).$$

Proof. The proof is a direct consequence of the previous lemmas. ∎

Finally, for the case where the source term is Dirac's delta, we use standard results on classical Sobolev inequalities.

THEOREM 4.6. *Given $f = \delta_x$ with $x \in \mathbf{I}$, $g = 0$, and $\omega_t^2 \notin \mathcal{S}$, let us denote by (w_t, θ_t) and (w_{th}, θ_{th}) the solutions of problems* (3.1) *and* (4.1), *respectively. Then, there exist $h_0 > 0$ such that, for all $h < h_0$, the following estimate holds:*

$$\|(w_t, \theta_t) - (w_{th}, \theta_{th})\|_{L^\infty(\mathbf{I})^2} \leq Ch.$$

Proof. This result is a consequence of the continuous inclusion $H^1(\mathbf{I}) \hookrightarrow L^\infty(\mathbf{I})$ and the estimate (4.5) in Theorem 4.2. ∎

5. Optimal amplitudes of actuators. Numerical methods. In this section we obtain locking-free error estimates for the optimal control problems proposed in section 1: *punctual sensors* and *distributed sensors*. In order to obtain such estimates, we proceed as in the numerical framework used in [4, Chapter 5], and we use the results obtained in the previous sections.

Due to the linearity of problem (2.1) and keeping in mind that the source term f is written in terms of the control variable $\mathbf{u} := (u_1, \ldots, u_N) \in \mathbb{R}^N$,

$$f = f_e + \sum_{i=1}^{N} u_i \delta_{y_i}, \qquad \text{with } u_i \in \mathbb{R},\ i = 1, \ldots, N. \tag{5.1}$$

The unique solution (w_t, θ_t) can be written in terms of the control variable as

$$(w_t, \theta_t) = (w_{t0}, \theta_{t0}) + \sum_{i=1}^{N} u_i \left(w_{ti}, \theta_{ti} \right),$$

where (w_{t0}, θ_{t0}) is the solution of problem (2.1) without control, considering only the external force f_e, i.e., $\mathbf{u} = \mathbf{0}$ in (5.1), and (w_{ti}, θ_{ti}) is the solution of problem (2.1) with $f = \delta_{y_i}$ and $g = 0$, i.e., when the system is only excited by the ith actuator with unit amplitude, excluding the effect of the external force f_e.

To prove existence and uniqueness of the optimal control problems (2.4) and (2.6), we consider *transfer functions* that correspond to mappings $\boldsymbol{z}(\boldsymbol{u})$ and $\boldsymbol{y}(\boldsymbol{u})$ that establish the relation between control and observation in the respective control problem.

Distributed sensors:

$$\begin{array}{cccc} \mathbb{R}^N & \longrightarrow & \mathcal{C}^0(\mathbf{I}), & \\ \boldsymbol{u} & \longmapsto & \boldsymbol{y}(\boldsymbol{u}) & = w_t(\boldsymbol{u}, x)|_{(a_1,a_2)}. \end{array}$$

Punctual sensors:

$$\begin{array}{cccc} \mathbb{R}^N & \longrightarrow & \mathbb{R}^M, & \\ \boldsymbol{u} & \longmapsto & \boldsymbol{z}(\boldsymbol{u}) & = (w_t(\boldsymbol{u}, p_1), \ldots, w_t(\boldsymbol{u}, p_M)). \end{array}$$

Since the transfer functions $\boldsymbol{z}(\boldsymbol{u})$ and $\boldsymbol{y}(\boldsymbol{u})$ are affine, it is clear that both first terms in the cost functions (2.3) and (2.5), respectively, are quadratic. Moreover, the second terms are strictly convex when $\nu > 0$. Therefore, it is clear that the cost functions are strictly convex if $\nu > 0$, or well if $\nu \geq 0$ and the observations are one to one. Thus, we may conclude that both optimal control problems, distributed sensors, and punctual sensors have unique solutions under these considerations.

Note that, in the problem of punctual sensors, in the case that the number of sensors is greater than or equal to the number the actuators (i.e., $M \geq N$), the observations to each single actuator are linearly independent and, therefore, transfer function is one to one, or the observations are one to one. In the distributed case, the same follows from the existence and uniqueness of the solution of (2.1).

In order to analyze both optimal control problems simultaneously, we will consider the following, more general, cost functional:

$$\mathfrak{J}(\boldsymbol{u}) := \frac{1}{2}\left\|\boldsymbol{p}(\boldsymbol{u}) - \boldsymbol{p}_d\right\|^2_{\mathcal{H}} + \frac{\nu}{2}\left\|\boldsymbol{u}\right\|^2_2, \tag{5.2}$$

where, $\mathcal{H} = \mathbb{R}^M$ and $\boldsymbol{p}(\boldsymbol{u}) = \boldsymbol{z}(\boldsymbol{u})$ in the case of the punctual sensor and, $\mathcal{H} = L^2(a_1, a_2)$ and $\boldsymbol{p}(\boldsymbol{u}) = \boldsymbol{y}(\boldsymbol{u})$ in the case of the distributed sensor. Thus, considering $\boldsymbol{p}_d = \mathbf{0}$, the optimal control problems is the following:

Find $\boldsymbol{u}^{op} \in \mathrm{U}_{ad}$ such that

$$\mathfrak{J}(\boldsymbol{u}^{op}) = \inf_{\boldsymbol{u} \in \mathrm{U}_{ad}} \mathfrak{J}(\boldsymbol{u}). \tag{5.3}$$

The global observation $\boldsymbol{p}$ is written in terms of the control variable $\boldsymbol{u} \in \mathbb{R}^n$ and the observations $\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N$ in the following way:

$$\boldsymbol{p}(\boldsymbol{u}) = \boldsymbol{p}_0 + \sum_{i=1}^{N} u_i \boldsymbol{p}_i,$$

where, if $\boldsymbol{e}_i$ represents the ith element of the canonic basis of $\mathbb{R}^N$, we set

Distributed sensors:

$$\boldsymbol{p}_0 := w_{t0}(\mathbf{0}, x)|_{(a_1,a_2)}, \tag{5.4}$$

$$\boldsymbol{p}_i := w_{ti}(\boldsymbol{e}_i, x)|_{(a_1,a_2)}, \quad i = 1, \ldots, N. \tag{5.5}$$

Punctual sensors:

$$\boldsymbol{p}_0 := (w_{t0}(\boldsymbol{0}, p_1), \ldots, w_{t0}(\boldsymbol{0}, p_M)), \tag{5.6}$$

$$\boldsymbol{p}_i := (w_{ti}(\boldsymbol{e}_i, p_1), \ldots, w_{ti}(\boldsymbol{e}_i, p_M)), \quad i = 1, \ldots, N. \tag{5.7}$$

Thus, the cost function (5.2) becomes

$$\begin{aligned}\mathfrak{J}(\boldsymbol{u}) &= \frac{1}{2}\left\|\boldsymbol{p}_0 + \sum_{i=1}^{N} u_i \boldsymbol{p}_i\right\|_{\mathcal{H}}^2 + \frac{\nu}{2}\|\boldsymbol{u}\|_2^2 \\ &= \frac{1}{2}\left\{(\boldsymbol{p}_0, \boldsymbol{p}_0)_{\mathcal{H}} + 2\sum_{i=1}^{N} u_i(\boldsymbol{p}_i, \boldsymbol{p}_0)_{\mathcal{H}} + \sum_{i=1}^{N}\sum_{j=1}^{N} u_i u_j \left((\boldsymbol{p}_i, \boldsymbol{p}_j)_{\mathcal{H}} + \nu\delta_{ij}\right)\right\}.\end{aligned}$$

Then, by introducing the matrix $\boldsymbol{P} \in \mathbb{R}^{N\times N}$ and the vector $\boldsymbol{b} \in \mathbb{R}^N$, defined by

$$\begin{aligned}(\boldsymbol{P})_{ij} &:= (\boldsymbol{p}_j, \boldsymbol{p}_i)_{\mathcal{H}}, \quad i, j = 1, \ldots, N, \\ (\boldsymbol{b})_i &:= (\boldsymbol{p}_0, \boldsymbol{p}_i)_{\mathcal{H}}, \quad i = 1, \ldots, N,\end{aligned}$$

the optimal control problem (5.3) is equivalent to the following quadratic programming problem:

Find $\boldsymbol{u}^{op}$ such that

$$\mathfrak{J}(\boldsymbol{u}^{op}) = \inf_{\boldsymbol{u}\in U_{ad}} \frac{1}{2}\left\{\left((\boldsymbol{P} + \nu\mathbf{I})\,\boldsymbol{u}, \boldsymbol{u}\right) + 2\,(\boldsymbol{u}, \boldsymbol{b}) + \|\boldsymbol{p}_0\|_2^2\right\},$$

where $(\cdot,\cdot)$ denotes the standard inner product in $\mathbb{R}^N$.

Notice that, observation $\boldsymbol{p}$ involves the solution of a system of partial differential equations, given by the state equations, which has to be approximated by the finite element method described and analyzed in the last section. This leads to an approximate functional that will be considered.

Let w_{th} be the approximation of the transversal displacement solution of the discrete problem (4.1), with a load source f defined by (5.1) for a given vector $\boldsymbol{u} \in \mathbb{R}^N$.

Let us introduce, for $i = 0, \ldots, N$, the approximate observation $\boldsymbol{p}_{hi}$ which is obtained by replacing w_{ti} by w_{thi}, in (5.4)–(5.7). Then, the global observation $\boldsymbol{p}_h$ in problem (5.3) can be written in the following way:

$$\boldsymbol{p}_h(\boldsymbol{u}) := \boldsymbol{p}_{h0} + \sum_{i=1}^{N} u_i \boldsymbol{p}_{hi}.$$

Now, let $\boldsymbol{P}_h \in \mathbb{R}^{N\times N}$ and $\boldsymbol{b}_h$ be defined by

$$\begin{aligned}(\boldsymbol{P}_h)_{ij} &:= (\boldsymbol{p}_{hj}, \boldsymbol{p}_{hi})_{\mathcal{H}}, \quad i, j = 1, \ldots, N, \\ (\boldsymbol{b}_h)_i &:= (\boldsymbol{p}_{h0}, \boldsymbol{p}_{hi})_{\mathcal{H}}, \quad i = 1, \ldots, N.\end{aligned}$$

The approximate cost function can be written as

$$\begin{aligned}\mathfrak{J}_h(\boldsymbol{u}) &= \frac{1}{2}\|\boldsymbol{p}_h(\boldsymbol{u})\|_{\mathcal{H}}^2 + \frac{\nu}{2}\|\boldsymbol{u}\|_2^2 \\ &= \frac{1}{2}\left\{\left((\boldsymbol{P}_h + \nu\mathbf{I})\,\boldsymbol{u}, \boldsymbol{u}\right) + 2\,(\boldsymbol{u}, \boldsymbol{b}_h) + \|\boldsymbol{p}_{h0}\|_2^2\right\}.\end{aligned} \tag{5.8}$$

These definitions lead us to the following discrete optimal control problem:

Find $\boldsymbol{u}_h^{op}$ such that

$$\mathfrak{J}_h(\boldsymbol{u}_h^{op}) = \inf_{\boldsymbol{u}\in \mathrm{U}_{ad}} \frac{1}{2}\left\{ ((\boldsymbol{P}_h + \nu \mathbf{I})\,\boldsymbol{u}, \boldsymbol{u}) + 2\,(\boldsymbol{u}, \boldsymbol{b}_h) + \|\boldsymbol{p}_{h0}\|_2^2 \right\}. \tag{5.9}$$

The argument $\boldsymbol{u}_h^{op}$, where the minimum is attained, is expected to be a good approximation of the optimal control $\boldsymbol{u}^{op}$. Adapting the proof of Lemma 5.1 in [4], we obtain estimates for $\|\boldsymbol{P} - \boldsymbol{P}_h\|_2$ and $\|\boldsymbol{b} - \boldsymbol{b}_h\|_2$. Notice that, any norm can be used because the spaces have finite dimension. The estimates are shown separately, for each control problem, in the next lemma.

LEMMA 5.1. *There exists $h_0 > 0$, such that for all $h < h_0$ the following estimates hold:*

Punctual sensors:

$$\|\boldsymbol{P} - \boldsymbol{P}_h\|_2 \leq Ch, \tag{5.10}$$
$$\|\boldsymbol{b} - \boldsymbol{b}_h\|_2 \leq Ch. \tag{5.11}$$

Distributed sensors:

$$\|\boldsymbol{P} - \boldsymbol{P}_h\|_2 \leq Ch^2, \tag{5.12}$$
$$\|\boldsymbol{b} - \boldsymbol{b}_h\|_2 \leq Ch^2 \tag{5.13}$$

Proof. We will prove Lemma 5.1 in our general setting. Let us denote by $\Delta \boldsymbol{p}_i$ the approximation errors between the observation $\boldsymbol{p}_i$ and the approximate observation $\boldsymbol{p}_{hi}$, $i = 0, \ldots, N$, i.e., $\Delta \boldsymbol{p}_i = \boldsymbol{p}_{hi} - \boldsymbol{p}_i$. Using this notation, the definition of the matrices $\boldsymbol{P}$ and $\boldsymbol{P}_h$, and the Cauchy–Schwarz inequality in the space $\mathcal{H}$, we obtain

$$\begin{aligned}|(\boldsymbol{P})_{ij} - (\boldsymbol{P}_h)_{ij}| &= \left|(\boldsymbol{p}_j, \Delta\boldsymbol{p}_i)_{\mathcal{H}} + (\Delta\boldsymbol{p}_j, \boldsymbol{p}_i)_{\mathcal{H}} + (\Delta\boldsymbol{p}_j, \Delta\boldsymbol{p}_i)_{\mathcal{H}}\right| \\ &\leq \|\boldsymbol{p}_j\|_{\mathcal{H}}\|\Delta\boldsymbol{p}_i\|_{\mathcal{H}} + \|\boldsymbol{p}_j\|_{\mathcal{H}}\|\Delta\boldsymbol{p}_i\|_{\mathcal{H}} + \|\Delta\boldsymbol{p}_j\|_{\mathcal{H}}\|\Delta\boldsymbol{p}_i\|_{\mathcal{H}},\end{aligned}$$

analogously,

$$|(\boldsymbol{b})_i - (\boldsymbol{b}_h)_i| \leq \|\boldsymbol{p}_0\|_{\mathcal{H}}\|\Delta\boldsymbol{p}_i\|_{\mathcal{H}} + \|\boldsymbol{p}_i\|_{\mathcal{H}}\|\Delta\boldsymbol{p}_0\|_{\mathcal{H}} + \|\Delta\boldsymbol{p}_0\|_{\mathcal{H}}\|\Delta\boldsymbol{p}_i\|_{\mathcal{H}}.$$

Thus, estimates (5.10) and (5.11) follow from the definition of the observations, Theorem 4.6 or Theorem 4.5 (depending on the external force) for $i = 0$, and Theorem 4.6 for $i = 1, \ldots, N$. In the same way, estimates (5.12) and (5.13) follow from Theorem 4.2. ∎

As a consequence of this theorem, we have existence and uniqueness of the solution of the discrete optimal control problem for h small enough.

COROLLARY 5.2. *There exists $h_0 > 0$ such that for all $h < h_0$, problem* (5.3) *has a unique solution in the case of $\nu > 0$ or in the case of $\nu \geq 0$ and $\boldsymbol{p}(\boldsymbol{u})$ is one to one.*

Proof. From the previous theorem, for h small enough, $\boldsymbol{P}_h$ is positive definite, because it converges to $\boldsymbol{P}$, which is positive definite. ∎

Finally, in the next theorem we show the estimates for the approximate error of the optimal control; the proof is based on Theorem 5.4 in [4].

THEOREM 5.3. *Let us assume that $\nu > 0$ or $\nu \geq 0$ and $\boldsymbol{p}(\boldsymbol{u})$ is one to one. If $\mathbf{0} \in \mathrm{U}_{ad}$, then there exists a positive constant C independent of t and h_0 such that, for all $h \in (0, h_0]$,*

$$\|\boldsymbol{u}^{op} - \boldsymbol{u}_h^{op}\| \leq Ch$$

for the punctual sensor problem, and

$$\|\boldsymbol{u}^{op} - \boldsymbol{u}_h^{op}\| \leq Ch^2$$

for the distributed sensor problem.

Proof. These results are the consequence of Lemmas 5.1 and 5.3 in [4], adapting the notations. ∎

6. Numerical results. In this section we report the results of some numerical tests computed with a MATLAB code that implements the locking-free finite element scheme described above. We used a reduced-order scheme for the integration of the shear term in the primal formulation, such as the scheme proposed in [3], which is equivalent to the mixed formulation.

We consider representative examples of each control scheme. In order to quantify the effect of the control we use the following attenuation measure in both cases:

$$\text{Attenuation (dB)} = -10\log\left(\frac{\mathbf{J}(\boldsymbol{u})}{\mathbf{J}(\mathbf{0})}\right),$$

where $\mathbf{J}$ denotes the corresponding functional of each problem.

Punctual sensors. In order to describe this numerical control problem we consider the following choices:

- The domain of the beam is $\mathbf{I} := (0, 1)$.
- The physical parameters are:
  - elastic moduli: $E = 2.1 \times 10^{11}$Pa,
  - Poisson coefficient: $\bar{\nu} = 0.3$,
  - correction factor: $k = 5/6$,
  - density: $\rho = 7.8 \times 10^3 \text{Kg/m}^3$,
  - $\hat{\omega} = 22s^{-1}$.
- The external force is Dirac's delta localized in $x = 0.15$.
- There is one sensor in $x = 0.4$.
- There is one actuator in $x = 0.75$.
- The admissible control set is $\mathrm{U}_{ad} = \mathbb{R}$ and the weighting factor is $\nu = 0$.

Note that, as $U_{ad} = \mathbb{R}$ and $\nu = 0$, the classical Euler inequality

$$((\boldsymbol{P}_h + \nu\mathbf{I})\boldsymbol{u}_h^{op} + \boldsymbol{b}_h, v - \boldsymbol{u}_h^{op}) \geq 0 \quad \forall v \in U_{ad},$$

associated with the problem (5.9), is reduced to the following linear systems of equations:

$$\boldsymbol{P}_h \boldsymbol{u}_h^{op} = -\boldsymbol{b}_h.$$

In Table 6.1, we report the optimal control computed for different values of the thickness t and successively refined meshes. It also includes the computed order of convergence and the corresponding extrapolated optimal control, obtained by means of a least squares fitting of the model

$$\boldsymbol{u}_h^{op} \approx \boldsymbol{u}_{ex} + Ch^t.$$

Moreover, in Table 6.1, it can be seen that the order of convergence remains uniformly optimal with respect to t; this confirms that the method is locking-free.

The error, for $t = 0.05$ and $t = 0.005$, is shown in Figure 6.1, where it can be clearly seen that the order of convergence is essentially $\mathcal{O}(h)$, as predicted by theoretical results.

TABLE 6.1
Optimal controls for a punctual sensor problem (scaled by a factor 10^{11}).

t	$h = 1/80$	$h = 1/100$	$h = 1/120$	$h = 1/140$	ext.	order
0.8	1.6288181	1.6252264	1.6228266	1.6211095	1.6106959	0.99
0.1	2.1995987	2.2139872	2.2236505	2.2305880	2.2735737	0.97
0.05	2.2237015	2.2389812	2.2492472	2.2566195	2.3029090	0.96
0.01	2.2305049	2.2460195	2.2564446	2.2639318	2.3109361	0.96
0.005	2.2269057	2.2422274	2.2525226	2.2599163	2.3063355	0.96
0.0008	2.0188211	2.0238280	2.0271929	2.0296103	2.0447812	0.96

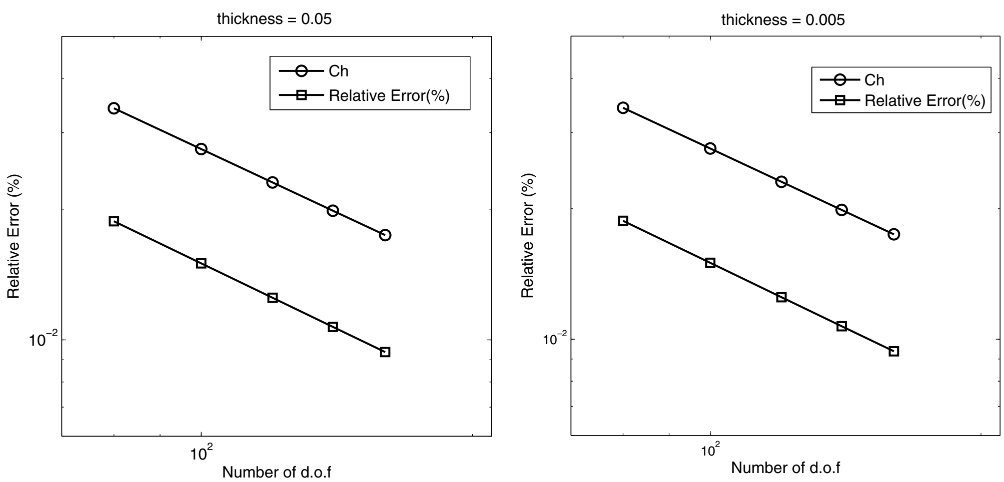

FIG. 6.1. *Relative error (%) versus number of d.o.f. (log-log scale).*

Now, we consider the following choice of parameters: $h = 1/100$, $t = 0.001$, and the first resonance frequency for this configuration, i.e., $\omega = 33s^{-1}$. The attenuation obtained is

$$\text{Attenuation (dB)} = -10\log\left(\frac{J(\boldsymbol{u})}{J(\mathbf{0})}\right) = 481.10.$$

The absolute value of the displacements with and without control can be seen in Figure 6.2, which clearly shows the reduction of the vibration at the sensor.

Finally, in Figure 6.3, we report the functionals with and without control as a function of the frequency, where $[0, 200]$ is the chosen frequency range. We can appreciate a vibration reduction in all frequencies of interest, in fact, the vibration is reduced by a factor of order 10^9.

Now, we consider a second numerical example, where the objective is to minimize the vibration in two points of the beam using two actuators. We use the same choices as the previous example, except the following:

- There are two sensors in $x = 0.4$ and $x = 0.7$.
- There are two actuators in $x = 0.1$ and $x = 0.5$.
- $\omega = 153s^{-1}$.

In Figure 6.4, we report the absolute value of the displacement with and without control for $t = 0.001$ and $h = 1/100$. In this figure we can see that the displacement

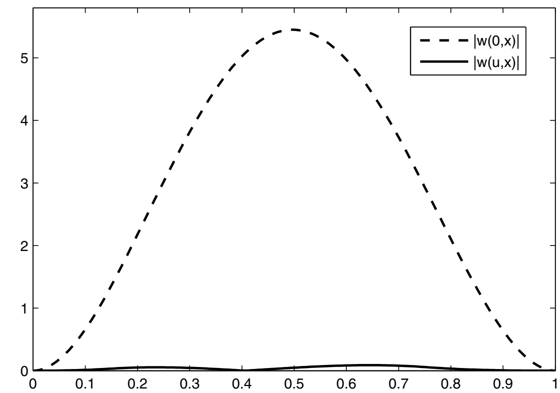

FIG. 6.2. *Reduction of vibration in the used sensor: Absolute values of the displacements with and without control as functions of a spatial coordinate.*

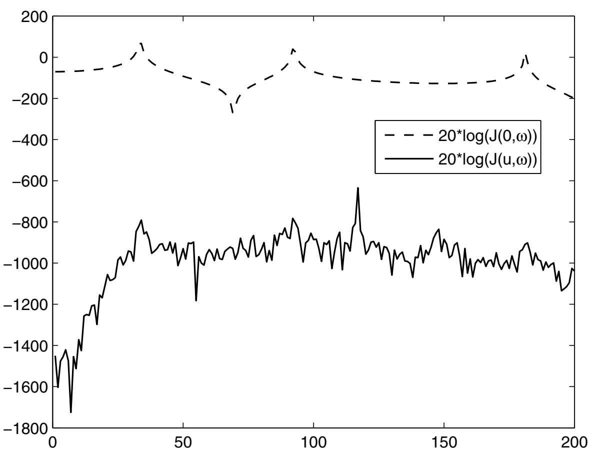

FIG. 6.3. *Cost functionals with and without control for punctual sensor control problem.*

in the used sensors are practically zero, in fact,

$$\begin{aligned} w(u,0.4) &= \quad 9.2646 \times 10^{-13}, \\ w(u,0.7) &= -2.77712 \times 10^{-12}. \end{aligned}$$

The attenuation obtained in this case is

$$\text{Attenuation (dB)} = -10 \log \left(\frac{J(\boldsymbol{u})}{J(\mathbf{0})} \right) = 443.14.$$

Notice that the aim of the control design in this case is to reduce the vibration at the sensors used, i.e., the vertical displacement on the points $x = 0.4$ and $x = 0.7$. However, this kind of control can increase significantly the displacement elsewhere on

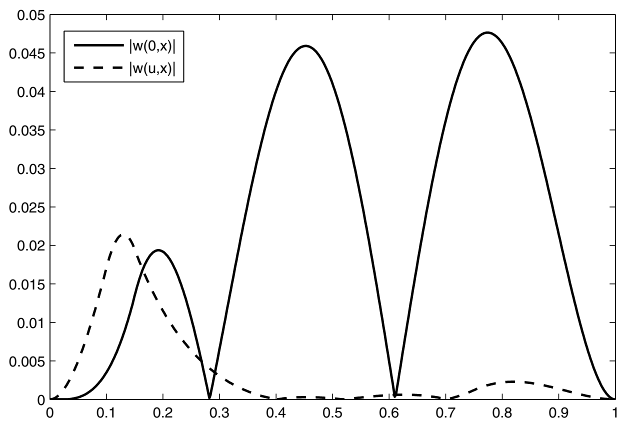

FIG. 6.4. *Reduction of vibration in the used sensors: Absolute value of the displacements with and without control as functions of a spatial coordinate.*

TABLE 6.2
Optimal controls for a distributed sensor problem (scaled by a factor 10^{10}*).*

t	$h = 1/80$	$h = 1/100$	$h = 1/120$	$h = 1/140$	ext	order
0.8	−9.7823054	−9.7817365	−9.7814275	−9.7812412	−9.7807252	2.00
0.1	−8.6262707	−8.6259863	−8.6258316	−8.6257382	−8.6254778	1.99
0.05	−8.4882370	−8.4881011	−8.4880270	−8.4879822	−8.4878566	1.98
0.01	−8.4649421	−8.4649080	−8.4648892	−8.4648777	−8.4648443	1.92
0.005	−8.5598335	−8.5599232	−8.5599717	−8.5600010	−8.5600815	2.01
0.0008	−6.5088359	−6.3125978	−6.2058951	−6.1415276	−5.9634914	2.00

the beam since the cost functional only includes the above mentioned observations. In this example, this behavior can be appreciated, approximately, on the interval $[0, 0.15]$. The effect of local control on structures can be appreciated in other control designs (see [13, section 6.2]).

Distributed sensors. The problem of AVC arises naturally from engineering problems; several of them have been reviewed in the book by Fuller, Elliot, and Nelson [13]. In this book they review a control scheme based on Fourier transform, applied to the Euler–Bernoulli beam model. While the control scheme proposed in this paper is different from that and, moreover, the structure is modeled with different equations, for Timoshenko's model for small values of the beam thickness we can expect similar results, because the objective of the control is the same and the shear effect included in Timoshenko's equations can be ignored. To formulate this control problem it is enough to consider one sensor along the beam ($a_1 = 0$ and a_2=1) and the same choices as the first numerical example (see [13, Chapter 6]).

Similar to the punctual sensor test in Table 6.2, we report the optimal control computed for different values of t and successively refined meshes. In this table, we can see how the order of convergence remains uniformly optimal in t. In the same way as above, the extrapolated value is considered as an accurate value of optimal control and it is used to compute the relative error. These errors are shown in Figure 6.5, for $t = 0.05$ and $t = 0.005$, where it can be clearly seen that the order of convergence is

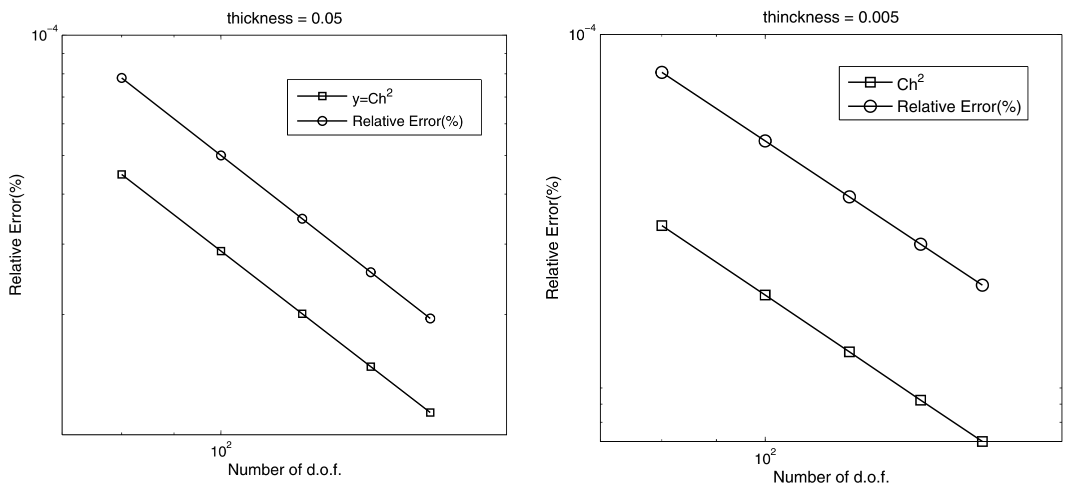

FIG. 6.5. *Relative error (%) versus number of d.o.f. (log-log scale).*

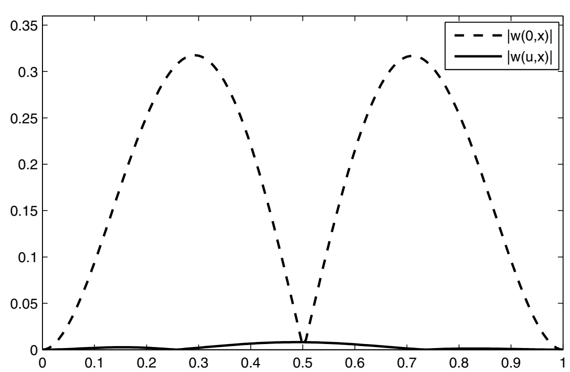

FIG. 6.6. *Reduction of vibration in the used sensor: Displacement with and without control as a function of a spatial coordinate.*

essentially $\mathcal{O}(h^2)$, as predicted by theoretical results.

Figure 6.6 shows the vibration reduction along the beam, which is the target of the control. For the following figures, we used $h = 1/100$, $t = 0.001$, and the external frequency as the second resonance frequency for this configuration: $\omega = 93s^{-1}$. The attenuation is given by

$$\text{Attenuation (dB)} = -10 \log \left(\frac{J(\boldsymbol{u})}{J(\mathbf{0})} \right) = 79.38.$$

Figure 6.7 shows the functionals $\mathcal{J}$ with and without control. It should be noted that this is qualitatively the same as what Fuller, Elliot, and Nelson have shown in their book [13, p. 271]. This allows us to validate our engineering scheme in terms of this mathematical setting.

Note that a large reduction in the cost function can be achieved in the resonance frequency of the beam. Nevertheless, between these frequencies our control scheme

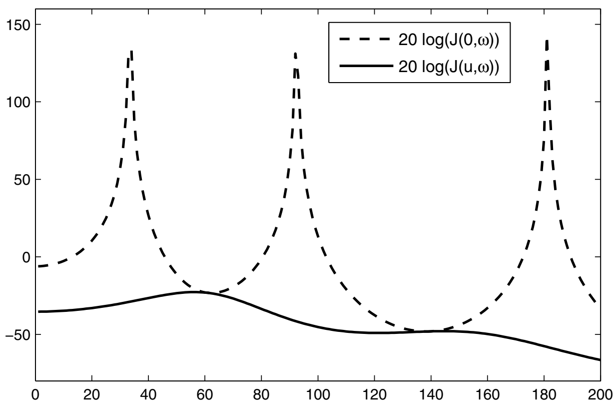

FIG. 6.7. *Cost functionals with and without control for distributed sensor control problem.*

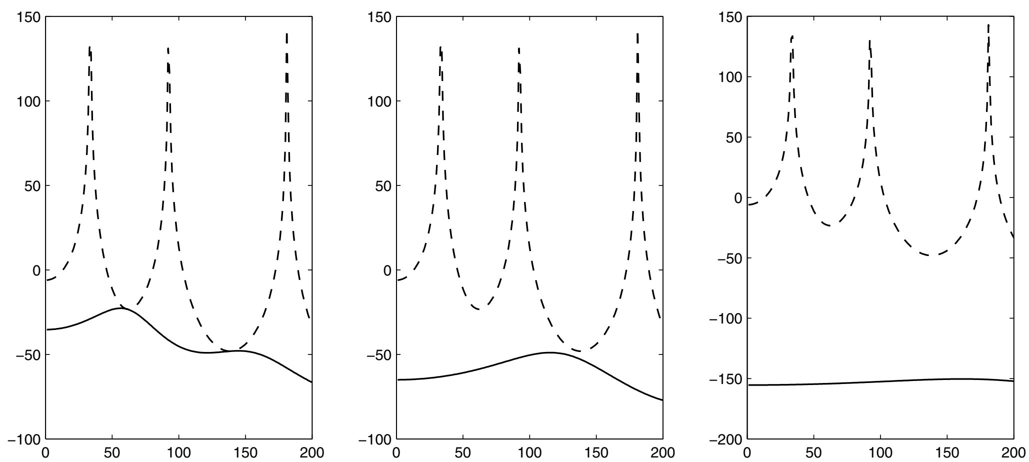

FIG. 6.8. *Cost functionals with and without control for different control designs: one, two, and three actuators, respectively. The segmented line represents the cost functional without control and the continuous line represents the cost functional with control.*

behavior is inefficient, due to the existence of frequencies where no reduction of vibration is achieved. For this reason, we implement two new control designs: Both designs consider the same choices as the numerical example *distributed sensor*; the first design added a second actuator in $x = 0.5$, whereas the second design incorporates two actuators in $x = 0.5$ and $x = 0.1$.

Figure 6.8 shows the functionals with and without control for the distributed control scheme and for the new designs proposed earlier. This figure shows the performance of the number of actuators with regard to the objective of control: If we consider two or three actuators, the reduction of vibration can be achieved in all interest frequencies and, clearly, the reduction achieved improves if we consider a higher

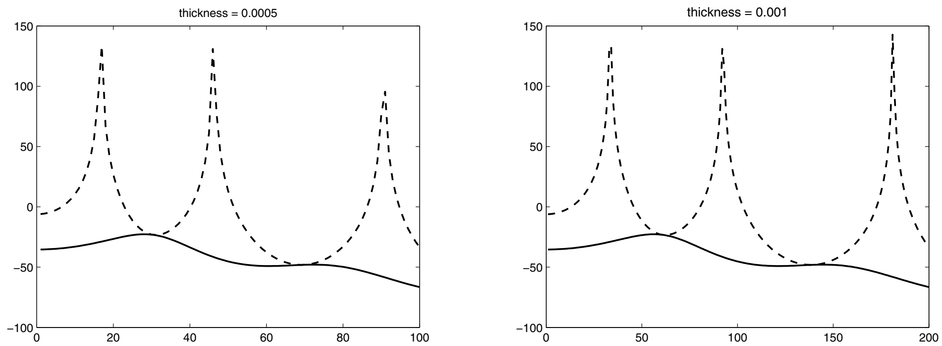

FIG. 6.9. *Cost functionals with and without control for different values of the beam thickness.*

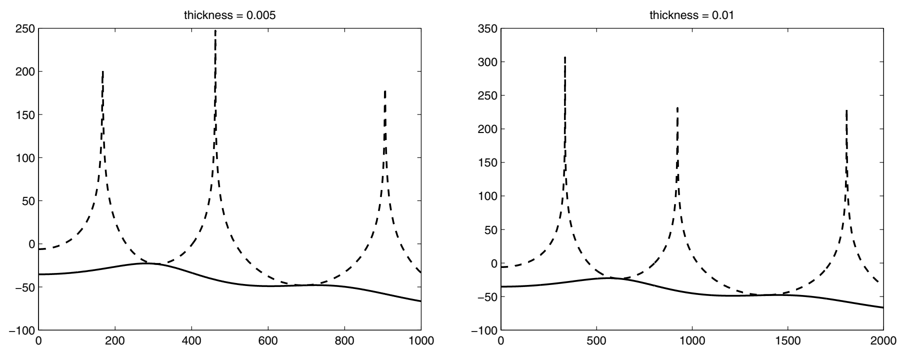

FIG. 6.10. *Cost functionals with and without control for different values of the beam thickness.*

number of actuators.

Figures 6.9 and 6.10 show influence of the thickness of the beam on the reduction of vibration. These figures show the cost functional with and without control for different values of the thickness beam: $t = 0.0005$, $t = 0.001$, $t = 0.005$, and $t = 0.01$. We can appreciate the stability and robustness of this control design with regard to the thickness parameter.

Finally, as a conclusion of this section, we note that in order to minimize the vibration in one point, the punctual control scheme is clearly more effective (see Figure 6.3). However, this scheme has the disadvantage that the associated cost functional only considers point observations. On the other hand, the functional associated to the distributed scheme allows us to reduce the vibration in all the beam, or in a sector of the beam, without considering punctual displacement.

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REFERENCES

[1] H. ANTES, M. SCHANZ, AND S. ALVERMANN, *Dynamic analyses of plane frames by integral equations for bars and Timoshenko beams*, J. Sound Vibration, 276 (2004), pp. 807–836.

[2] N. ARADA, E. CASAS, AND F. TRÖLTZSCH, *Error estimates for the numerical approximation of a semilinear elliptic control problem*, Comput. Optim. Appl., 23 (2002), pp. 201–229.

[3] D. N. Arnold, *Discretization by finite element of a model parameter dependent problem*, Numer. Math., 37 (1981), pp. 405–421.

[4] A. Bermúdez, P. Gamallo, and R. Rodríguez, *Finite element methods in local active control of sound*, SIAM J. Control Optim., 43 (2004), pp. 437–465.

[5] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[6] D. Chapelle, *A locking-free approximation of curved rods by straight beam elements*, Numer. Math., 77 (1997), pp. 299–322.

[7] D. Chapelle and K. J. Bathe, *The Finite Element Analysis of Shells—Fundamentals*, Comput. Fluid Solid Mech., Springer–Verlag, Berlin, 2003.

[8] J. Douglas, T. Dupont, and L. Wahlbim, *Optimal L_∞ error estimates for Galerkin approximations to solutions of two-point boundary value problems*, Math. Comp., 29 (1975), pp. 475–483.

[9] R. G. Durán, L. Hervella-Nieto, E. Liberman, R. Rodríguez, and J. Solomin, *Approximation of the vibration modes of a plate by Reissner–Mindlin equations,* Math. Comp., 68 (1999), pp. 1447–1463.

[10] R. G. Durán, E. Hernández, L. Hervella-Nieto, E. Liberman, and R. Rodríguez, *Error estimates for low-order isoparametric quadrilateral finite elements for plates*, SIAM J. Numer. Anal., 41 (2003), pp. 1751–1772.

[11] L. C. Evans, *Partial Differential Equations*, AMS, Providence, RI, 1998.

[12] R. S. Falk, *Approximation of a class of optimal control problems with order of convergence estimates*, J. Math. Anal. Appl., 44 (1973), pp. 28–47.

[13] C. R. Fuller, S. J. Elliot, and P. A. Nelson, *Active Control of Vibration*, Academic Press, London, 1997.

[14] W. Hackbusch, *Elliptic Partial Differential Equations: Theory and Numerical Treatment*, Springer-Verlag, Berlin, 1992.

[15] E. Hernández, E. Otárola, R. Rodríguez, and F. Sanhueza, *Approximation of the vibration modes of a Timoshenko curved rod of arbitrary geometry*, IMA J. Numer. Anal., 29 (2009), pp. 180–207.

[16] M. Hinze, *A variational discretization concept in control constrained optimization: The linear-quadratic case*, Comput. Optim. Appl., 30 (2005), pp. 45–61.

[17] J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, New York-Berlin, 1971.

[18] A. Macchelli and C. Melchiorri, *Modeling and control of the Timoshenko beam. The distributed port Hamiltonian approach*, SIAM J. Control Optim., 43 (2004), pp. 743–767.

[19] R. Rannacher and B. Vexler, *A priori error estimates for the finite element discretization of elliptic parameter identification problems with pointwise measurements*, SIAM J. Control Optim., 44 (2005), pp. 1844–1863.

[20] A. Rösch and B. Vexler, *Optimal control of the Stokes equations: A priori error analysis for finite element discretization with postprocessing*, SIAM J. Numer. Anal., 44 (2006), pp. 1903–1920.

[21] A. H. Schatz, *An observation concerning Ritz–Galerkin methods with indefinite bilinear forms*, Math. Comp., 28 (1974), pp. 959–962.

[22] S. P. Timoshenko, *On the correction for shear of the differential equation for transverse vibrations of prismatic bars*, Philos. Mag. Ser., 6 (1921), pp. 744–746.

[23] G. Q. Xu and S. P. Yung, *Stabilization of Timoshenko beam by means of pointwise controls*, ESAIM Control Optim. Calc. Var., 9 (2003), pp. 579–600.