Noname manuscript No.
(will be inserted by the editor)

Error estimates for FEM discretizations of the Navier–Stokes equations with Dirac measures

Felipe Lepe · Enrique Otárola · Daniel Quero

Received: date / Accepted: date

Abstract We analyze, on two dimensional polygonal domains, classical low–order inf-sup stable finite element approximations of the stationary Navier–Stokes equations with singular sources. We operate under the assumptions that the continuous and discrete solutions are sufficiently small. We perform an a priori error analysis on convex domains. On Lipschitz, but not necessarily convex, polygonal domains, we design an a posteriori error estimator and prove its global reliability. We also explore efficiency estimates. We illustrate the theory with numerical tests.

Keywords Navier–Stokes equations · Dirac measures · A priori error estimates · A posteriori error estimates · Adaptive finite elements

Mathematics Subject Classification (2010) 35Q35, 35Q30, 35R06, 65N15, 65N30, 65N50

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain with Lipschitz boundary $\partial\Omega$. In this work, we shall be interested in the study of a priori and a posteriori error

FL is partially supported by ANID-Chile through FONDECYT postdoctoral project 3190204 and FONDECYT project 11200529. EO is partially supported by ANID-Chile through FONDECYT project 11180193.

Felipe Lepe
GIMNAP-Departamento de Matemática, Universidad del Bío–Bío, Concepción, Chile.
E-mail: flepe@ubiobio.cl

Enrique Otrola
Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile.
E-mail: enrique.otarola@usm.cl

Daniel Quero
Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile.
E-mail: daniel.quero@alumnos.usm.cl

estimates for classical low–order inf-sup stable finite element discretizations of the stationary Navier–Stokes problem

$$-\nu\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla\pi=\boldsymbol{f}\delta_z \text{ in } \Omega, \quad \operatorname{div}\boldsymbol{u}=0 \text{ in } \Omega, \quad \boldsymbol{u}=\mathbf{0} \text{ on } \partial\Omega. \quad (1.1)$$

The data of problem (1.1) are the kinematic viscosity $\nu > 0$, the vector $\boldsymbol{f} \in \mathbb{R}^2$, and the interior point $z \in \Omega$; δ_z corresponds to the Dirac delta supported at z. The unknowns are $\boldsymbol{u}$ and π, the velocity and the pressure, respectively.

The stationary Navier–Stokes system models the motion of a stationary, incompressible, Newtonian fluid. In view of the fundamental importance of such a system in mathematical fluid mechanics, the analysis and design of solution techniques, at least in *energy–type* spaces, has received a tremendous attention; see [14, 16, 19, 20, 21] and references therein. However, in recent times, new models have emerged where the motion of a fluid is described by (1.1), or a variation of it, and due to the singular nature of the force $\boldsymbol{f}\delta_z$, the problem must be understood in a completely different setting for which rigorous approximation techniques are scarce. An instance where singular forces appear is in the modeling of the movement of active thin structures in a viscous fluid [15]. Another instance is in optimal control where the state is governed by fluid equations and the control variable corresponds to the amplitude of forces modeled as point sources [10] .

Recently, the authors of [7] have analyzed an optimal control problem for the stationary Navier–Stokes equations, where the control variable is measure valued. The authors provide, in two dimensions and under the assumption that Ω is of class C^2, a complete existence theory for the Navier–Stokes equations in $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ with $p \in [4/3, 2)$. Reference [7], however, is not concerned with approximation. Finite element approximations of (1.1) have been recently considered in [18] and [4], where a priori and a posteriori error estimates have been analyzed, respectively. The authors, however, operate under a complete different approach which is the one inherited by a suitable class of Muckenhoupt weights.

In the present paper, we continue with our research program and extend the linear a posteriori error analysis developed in [9] to the Navier–Stokes system (1.1). In contrast to [9], we also perform an a priori error analysis on *quasiuniform meshes*. We begin our studies by deriving, on the basis of a standard contraction argument, the existence and uniqueness of solutions in $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ for small data. We operate in two dimensions and under the assumptions that Ω is Lipschitz and $p \in (4/3 - \varepsilon, 2)$, where $\varepsilon = \varepsilon(\Omega) > 0$; see Remark 1 for a discussion. Under this framework, we develop an a priori error analysis on *quasiuniform meshes* for suitable low–order inf-sup stable finite element schemes. To perform such an analysis, the stability of the Stokes projection [12, 13] is essential. In view of the reduced regularity properties of solutions to (1.1), which are due to the singular nature of the forcing term, the derived a priori error estimates cannot be optimal in terms of approximation. This motivates the design and analysis of a posteriori error estimates for suitable finite element discretizations of (1.1) on families of conforming and *shape regular meshes*. Inspired by [3], we introduce a Ritz projection to control the energy norm of the error between the solution of (1.1) and its corresponding finite element approximation. This is the key result to obtain global reliability estimates. To provide local efficiency results, we invoke suitable bubble functions whose construction we owe to [5].

The rest of the paper is organized as follows. In section 2 we introduce some terminology used throughout this work. In section 3 we introduce the functional setting in which we will operate and a suitable weak formulation for problem (1.1). We derive existence and uniqueness results for small data; our main novelty here is that we only assume the domain to be Lipschitz. Section 4 presents basic ingredients of finite element methods and an a priori error analysis for classical low–order inf-stable finite elements. In section 5 we devise and analyze local error indicators and a posteriori error estimator. We derive, in section 5.4, the global reliability of the devised error estimator. We explore efficiency estimates in section 5.5. Finally, in section 6, we report a series of numerical tests that illustrate the performance of the devised a posteriori error estimator.

2 Notation and preliminaries

Let us fix notation and the setting in which we will operate. Throughout this work, Ω is an open and bounded polygonal domain of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$; the convexity of Ω will be imposed only to derive the a priori error analysis of section 4.5. If $\mathscr{X}$ and $\mathscr{Y}$ are normed vector spaces, we write $\mathscr{X} \hookrightarrow \mathscr{Y}$ to denote that $\mathscr{X}$ is continuously embedded in $\mathscr{Y}$. We denote by $\mathscr{X}'$ and $\|\cdot\|_{\mathscr{X}}$ the dual and the norm of $\mathscr{X}$, respectively.

For $1 < p < +\infty$, we denote by $p\prime$ its *conjugate*, which is a real number that satisfies $\frac{1}{p} + \frac{1}{p\prime} = 1$.

The relation $\mathtt{a} \lesssim \mathtt{b}$ indicates that $\mathtt{a} \leq C\mathtt{b}$, with a positive constant C which is independent of $\mathtt{a}$, $\mathtt{b}$, and the size of the elements in the mesh. The value of C might change at each occurrence.

3 The model problem

In this section we introduce a suitable weak formulation for problem (1.1) and show, under a smallness assumption on the data, existence and uniqueness of solutions.

3.1 Weak- formulation

Let $p \in (1, 2)$. We define the product spaces

$$\mathcal{X} := \mathbf{W}_0^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}, \qquad \mathcal{Y} := \mathbf{W}_0^{1,p\prime}(\Omega) \times L^{p\prime}(\Omega)/\mathbb{R}.$$

We also define the bilinear forms

$$a : \mathbf{W}_0^{1,p}(\Omega) \times \mathbf{W}_0^{1,p'}(\Omega) \to \mathbb{R}, \qquad a(\boldsymbol{w},\boldsymbol{v}) := \int_\Omega \nabla \boldsymbol{w} : \nabla \boldsymbol{v}, \tag{3.1}$$

$$b_+ : \mathbf{W}_0^{1,p}(\Omega) \times L^{p'}(\Omega) \to \mathbb{R}, \qquad b_+(\boldsymbol{w},q) := -\int_\Omega q \operatorname{div} \boldsymbol{w}, \tag{3.2}$$

$$b_- : \mathbf{W}_0^{1,p'}(\Omega) \times L^{p}(\Omega) \to \mathbb{R}, \qquad b_-(\boldsymbol{v},r) := -\int_\Omega r \operatorname{div} \boldsymbol{v}, \tag{3.3}$$

and the trilinear form

$$c : [\mathbf{W}_0^{1,p}(\Omega)]^2 \times \mathbf{W}_0^{1,p'}(\Omega) \to \mathbb{R}, \qquad c(\boldsymbol{u},\boldsymbol{w};\boldsymbol{v}) := -\int_\Omega \boldsymbol{u} \otimes \boldsymbol{w} : \nabla \boldsymbol{v}. \tag{3.4}$$

With definitions (3.1)–(3.4) at hand, we introduce the following weak formulation of problem (1.1): Find $(\boldsymbol{u},\pi) \in \mathcal{X}$ such that

$$\nu a(\boldsymbol{u},\boldsymbol{v}) + b_-(\boldsymbol{v},\pi) + c(\boldsymbol{u},\boldsymbol{u};\boldsymbol{v}) = \langle \boldsymbol{f}\delta_z, \boldsymbol{v}\rangle, \quad b_+(\boldsymbol{u},q) = 0 \quad \forall (\boldsymbol{v},q) \in \mathcal{Y}. \tag{3.5}$$

Here, $\langle \cdot,\cdot \rangle$ denotes the duality pairing between $\mathbf{W}^{-1,p}(\Omega) := \mathbf{W}_0^{1,p'}(\Omega)'$ and $\mathbf{W}_0^{1,p'}(\Omega)$.

Since $p < 2$, we have that $\mathbf{W}_0^{1,p'}(\Omega) \hookrightarrow \mathbf{C}(\bar{\Omega})$ and thus that $\boldsymbol{f}\delta_z \in \mathbf{W}^{-1,p}(\Omega)$. On the other hand, a trivial application of Hölder's inequality reveals that, for $(\boldsymbol{w},r) \in \mathcal{X}$ and $(\boldsymbol{v},q) \in \mathcal{Y}$, the terms $a(\boldsymbol{w},\boldsymbol{v})$, $b_+(\boldsymbol{w},q)$, and $b_-(\boldsymbol{v},r)$ are bounded. The boundedness of the convective term is as follows:

$$\begin{aligned} |c(\boldsymbol{u},\boldsymbol{w};\boldsymbol{v})| &\leq \|\boldsymbol{u}\|_{\mathbf{L}^{2p}(\Omega)} \|\boldsymbol{w}\|_{\mathbf{L}^{2p}(\Omega)} \|\nabla \boldsymbol{v}\|_{\mathbf{L}^{p'}(\Omega)} \\ &\leq C_{2p\to p}^2 \|\nabla \boldsymbol{u}\|_{\mathbf{L}^{p}(\Omega)} \|\nabla \boldsymbol{w}\|_{\mathbf{L}^{p}(\Omega)} \|\nabla \boldsymbol{v}\|_{\mathbf{L}^{p'}(\Omega)}, \end{aligned} \tag{3.6}$$

where $C_{2p\to p}$ denotes the best constant in the embedding $\mathbf{W}_0^{1,p}(\Omega) \hookrightarrow \mathbf{L}^{2p}(\Omega)$, which holds because $p > 1$ [2, Theorem 4.12].

Since $\partial\Omega$ is Lipschitz, $\nabla\cdot : \mathbf{W}_0^{1,r}(\Omega) \to L^r(\Omega)/\mathbb{R}$, with $r \in (1,\infty)$, is surjective [1, Theorem 2.6]. We thus have the following inf–sup conditions:

$$\inf_{0\neq q \in L^{p'}(\Omega)/\mathbb{R}} \sup_{\boldsymbol{0}\neq \boldsymbol{w} \in \mathbf{W}_0^{1,p}(\Omega)} \frac{b_+(\boldsymbol{w},q)}{\|q\|_{L^{p'}(\Omega)} \|\nabla \boldsymbol{w}\|_{\mathbf{L}^{p}(\Omega)}} \geq \gamma_+. \tag{3.7}$$

$$\inf_{0\neq r \in L^{p}(\Omega)/\mathbb{R}} \sup_{\boldsymbol{0}\neq \boldsymbol{v} \in \mathbf{W}_0^{1,p'}(\Omega)} \frac{b_-(\boldsymbol{v},r)}{\|r\|_{L^{p}(\Omega)} \|\nabla \boldsymbol{v}\|_{\mathbf{L}^{p'}(\Omega)}} \geq \gamma_-. \tag{3.8}$$

Remark 1 (two dimensions) The boundedness of the convective term (3.6) is the sole reason why our analysis is restricted to two dimensions. Observe that, in three dimensions, we only have that $\mathbf{W}_0^{1,p}(\Omega) \hookrightarrow \mathbf{L}^{2p}(\Omega)$ for $p \geq 3/2$ [2, Theorem 4.12, **Case C**] while the solution $\boldsymbol{u}$ to problem (5.4) is sought in $\mathbf{W}_0^{1,p}(\Omega)$ with $p < 3/2$. Observe also that the related Stokes operator has a bounded inverse provided $p \in (3/2 - \varepsilon, 3/2 + \varepsilon)$, where $\varepsilon = \varepsilon(\Omega) > 0$ [17, Corollary 1.7].

3.2 Existence and uniqueness for small data

In this section we show, via a contraction argument, that provided the problem data is sufficiently small, we have existence and uniqueness of solutions. The contraction argument is rather standard and allows the domain to be merely Lipschitz; see, for instance, [21, Chapter 2].

To begin with our analysis, we define $\mathcal{S} : \mathcal{X} \to \mathcal{Y}'$, $\mathcal{NL} : \mathcal{X} \to \mathcal{Y}'$, and $\mathcal{F} \in \mathcal{Y}'$ by

$$\langle \mathcal{S}(\boldsymbol{u}, \pi), (\boldsymbol{v}, q)\rangle = a(\boldsymbol{u}, \boldsymbol{v}) + b_-(\boldsymbol{v}, \pi) + b_+(\boldsymbol{u}, q), \ \ \langle \mathcal{NL}(\boldsymbol{u}, \pi), (\boldsymbol{v}, q)\rangle = c(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{v}),$$

and $\langle \mathcal{F}, (\boldsymbol{v}, q)\rangle = \langle \boldsymbol{f}\delta_z, \boldsymbol{v}\rangle$, respectively. With $\mathcal{S}$, $\mathcal{NL}$, and $\mathcal{F}$ at hand, we can rewrite problem (1.1) as the following nonlinear functional equation in $\mathcal{Y}'$:

$$\mathcal{S}(\nu \boldsymbol{u}, \pi) + \mathcal{NL}(\boldsymbol{u}, \pi) = \mathcal{F}.$$

It is immediate that the *Stokes operator* $\mathcal{S}$ is bounded and linear. In addition, since Ω is Lipschitz, $\mathcal{S}$ has a bounded inverse provided $p \in (4/3-\varepsilon, 4+\varepsilon)$, where $\varepsilon = \varepsilon(\Omega)$ denotes a positive constant that depends on Ω [17, Corollary 1.7]. For $p \in (4/3-\varepsilon, 2)$, we can thus define the nonlinear mapping

$$\mathcal{T} : \mathcal{X} \to \mathcal{X}, \qquad (\nu \boldsymbol{u}, \pi) = \mathcal{T}(\boldsymbol{w}, r) := \mathcal{S}^{-1}(\mathcal{F} - \mathcal{NL}(\boldsymbol{w}, r)).$$

We shall denote by $\|\mathcal{S}^{-1}\|$ the $\mathcal{Y}' \to \mathcal{X}$ norm of $\mathcal{S}^{-1}$. Consequently, showing the existence of a solution for (1.1) is equivalent to finding a fixed point of $\mathcal{T}$.

We follow [18] and show existence and uniqueness for sufficiently small data. To accomplish this task, we define, for $K > 0$,

$$\mathcal{B}_K := \{\boldsymbol{w} \in \mathbf{W}_0^{1,p}(\Omega) : \operatorname{div} \boldsymbol{w} = 0, \ \|\nabla \boldsymbol{w}\|_{\mathbf{L}^p(\Omega)} \leq K\}$$

and $\mathcal{T}_1 : \mathbf{W}_0^{1,p}(\Omega) \to \mathbf{W}_0^{1,p}(\Omega)$ as $\boldsymbol{w} \mapsto \frac{1}{\nu}\mathrm{Pr}\mathcal{T}(\boldsymbol{w}, 0)$, where $\mathrm{Pr} : \mathcal{X} \to \mathbf{W}_0^{1,p}(\Omega)$ denotes the projection onto the velocity component.

Proposition 1 (contraction) *Let Ω be a Lipschitz domain. Assume that the forcing term $\boldsymbol{f}\delta_z$ is sufficiently small, or the viscosity ν is sufficiently large, so that*

$$\frac{C_{2p\to p}^2}{\nu^2}\|\mathcal{S}^{-1}\|^2\|\boldsymbol{f}\delta_z\|_{\mathbf{W}^{-1,p}(\Omega)} < \frac{1}{6}. \tag{3.9}$$

Set $K := \frac{\nu}{3}\frac{1}{C_{2p\to p}^2\|\mathcal{S}^{-1}\|}$. Hence, $\mathcal{T}_1$ maps $\mathcal{B}_K$ to itself and it is a contraction in it.

Proof We begin the proof by showing that $\mathcal{T}_1$ maps $\mathcal{B}_K$ into itself. Let $\boldsymbol{w} \in \mathcal{B}_K$ and $\boldsymbol{v} = \mathcal{T}_1(\boldsymbol{w})$. Then, $\operatorname{div} \boldsymbol{v} = 0$ and

$$\|\nabla \boldsymbol{v}\|_{\mathbf{L}^p(\Omega)} \leq \frac{\|\mathcal{S}^{-1}\|}{\nu}\|\boldsymbol{f}\delta_z\|_{\mathbf{W}^{-1,p}(\Omega)} + C_{2p\to p}^2\frac{\|\mathcal{S}^{-1}\|}{\nu}\|\nabla \boldsymbol{w}\|_{\mathbf{L}^p(\Omega)}^2 < \frac{K}{2} + \frac{K}{3} = \frac{5K}{6},$$

where we have used the definition of K and the smallness assumption (3.9). This immediately implies that $\boldsymbol{v} \in \mathcal{B}_K$. We now prove that $\mathcal{T}_1$ is a contraction. Let $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathcal{B}_K$ be such that $\boldsymbol{v}_i = \mathcal{T}_1(\boldsymbol{w}_i)$, with $i = 1, 2$. Then

$$
\begin{aligned}
\|\nabla(\boldsymbol{v}_1 - \boldsymbol{v}_2)\|_{\mathbf{L}^p(\Omega)} &\leq C_{2p\to p}^2 \frac{\|\mathcal{S}^{-1}\|}{\nu} \left(\|\nabla \boldsymbol{w}_1\|_{\mathbf{L}^p(\Omega)} + \|\nabla \boldsymbol{w}_2\|_{\mathbf{L}^p(\Omega)}\right) \\
\cdot \|\nabla(\boldsymbol{w}_1 - \boldsymbol{w}_2)\|_{\mathbf{L}^p(\Omega)} &\leq C_{2p\to p}^2 \frac{\|\mathcal{S}^{-1}\|}{\nu} \frac{2\nu}{3C_{2p\to p}^2\|\mathcal{S}^{-1}\|} \|\nabla(\boldsymbol{w}_1 - \boldsymbol{w}_2)\|_{\mathbf{L}^p(\Omega)} \\
&= \frac{2}{3}\|\nabla(\boldsymbol{w}_1 - \boldsymbol{w}_2)\|_{\mathbf{L}^p(\Omega)}. \quad (3.10)
\end{aligned}
$$

This shows that $\mathcal{T}_1$ is a contraction and concludes the proof.

We present the following existence and uniqueness result for small data.

Proposition 2 (existence and uniqueness) *Let Ω be Lipschitz. Assume that the forcing term $\boldsymbol{f}\delta_z$ is sufficiently small, or the viscocity ν is sufficiently large, so that* (3.9) *holds. If $p \in (4/3 - \varepsilon, 2)$, where $\varepsilon = \varepsilon(\Omega) > 0$ denotes a constant that depends on Ω, then there exists a unique solution of* (3.5) *which satisfies the estimates*

$$
\|\nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} \leq \frac{3}{2}\frac{\|\mathcal{S}^{-1}\|}{\nu}\|\boldsymbol{f}\delta_z\|_{\mathbf{W}^{-1,p}(\Omega)} \quad (3.11)
$$

$$
\|\pi\|_{L^p(\Omega)} \lesssim \|\nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|\nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}^2 + \|\boldsymbol{f}\delta_z\|_{\mathbf{W}^{-1,p}(\Omega)}. \quad (3.12)
$$

The hidden constant is independent of the solution $(\boldsymbol{u}, \pi)$ and $\boldsymbol{f}\delta_z$.

Proof Existence and uniqueness of the velocity field follow from Proposition 1. Similar arguments to those elaborated in the proof of (3.10) allow us to obtain (3.11). Since $\nabla\cdot : \mathbf{W}_0^{1,p}(\Omega) \to L^p(\Omega)/\mathbb{R}$ is surjective, the existence of a unique pressure follows from de Rahm's theorem [8, Theorem B.73]. To obtain (3.12) we invoke (3.8):

$$
\|\pi\|_{L^p(\Omega)} \lesssim \sup_{\mathbf{0}\neq\boldsymbol{v}\in\mathbf{W}_0^{1,p'}(\Omega)} \frac{b_-(\boldsymbol{v},\pi)}{\|\nabla \boldsymbol{v}\|_{\mathbf{L}^{p'}(\Omega)}} \quad \forall \pi \in L^p(\Omega)/\mathbb{R}.
$$

From this estimate, the first equation of problem (3.5), and the estimate (3.6), for the convective term, we obtain the desired pressure estimate.

We conclude the section with the following inf-sup condition [6]: If $p \in (4/3 - \varepsilon, 2)$, where $\varepsilon = \varepsilon(\Omega) > 0$ denotes a constant that depends on Ω, then

$$
\begin{aligned}
&\inf_{\mathbf{0}\neq\boldsymbol{w}\in\mathbf{W}_0^{1,p}(\Omega)} \sup_{\mathbf{0}\neq\boldsymbol{v}\in\mathbf{W}_0^{1,p'}(\Omega)} \frac{a(\boldsymbol{w},\boldsymbol{v})}{\|\nabla\boldsymbol{w}\|_{\mathbf{L}^p(\Omega)}\|\nabla\boldsymbol{v}\|_{\mathbf{L}^{p'}(\Omega)}} = \\
&\qquad \inf_{\mathbf{0}\neq\boldsymbol{v}\in\mathbf{W}_0^{1,p'}(\Omega)} \sup_{\mathbf{0}\neq\boldsymbol{w}\in\mathbf{W}_0^{1,p}(\Omega)} \frac{a(\boldsymbol{w},\boldsymbol{v})}{\|\nabla\boldsymbol{w}\|_{\mathbf{L}^p(\Omega)}\|\nabla\boldsymbol{v}\|_{\mathbf{L}^{p'}(\Omega)}} > 0. \quad (3.13)
\end{aligned}
$$

4 Finite element approximation

We now introduce the discrete setting in which we will operate. We first introduce some terminology and a few basic ingredients and assumptions that will be common to all our methods.

4.1 Triangulation and finite element spaces

Let $\mathscr{T} = \{T\}$ be a conforming partition, or mesh, of $\overline{\Omega}$ into closed simplices T with size $h_T = \text{diam}(T)$. Define $h_{\mathscr{T}} := \max_{T\in\mathscr{T}} h_T$. We denote by $\mathbb{T}$ the collection of conforming and shape regular meshes $\mathscr{T}$ that are refinements of an initial mesh $\mathscr{T}_0$ [8,14]. We define $\mathscr{S}$ as the set of internal two dimensional interelement boundaries S of $\mathscr{T}$. For $S \in \mathscr{S}$, we indicate by h_S the diameter of S. For $T \in \mathscr{T}$, let $\mathscr{S}_T$ denote the subset of $\mathscr{S}$ which contains the sides in $\mathscr{S}$ which are sides of T. We denote by $\mathcal{N}_S$ the subset of $\mathscr{T}$ that contains the two elements that have S as a side. For $T \in \mathscr{T}$, we define the *stars* or *patches* associated with an element T as

$$\mathcal{N}_T := \{T' \in \mathscr{T} : T \cap T' \neq \emptyset\}, \qquad \mathcal{N}_T^* := \{T' \in \mathscr{T} : \mathscr{S}_T \cap \mathscr{S}_{T'} \neq \emptyset\}.$$

In an abuse of notation, in what follows, by $\mathcal{N}_T$ and $\mathcal{N}_T^*$ we will indistinctively denote either these sets or the union of the triangles that comprise them.

For a discrete tensor valued function $\mathbf{W}_{\mathscr{T}}$, we denote by $[\![\mathbf{W}_{\mathscr{T}} \cdot \boldsymbol{n}]\!]$ the jump or interelement residual, which is defined, on the internal side $S \in \mathscr{S}$ shared by the distinct elements $T^+, T^- \in \mathcal{N}_S$, by $[\![\mathbf{W}_{\mathscr{T}} \cdot \boldsymbol{n}]\!] = \mathbf{W}_{\mathscr{T}}|_{T^+} \cdot \boldsymbol{n}^+ + \mathbf{W}_{\mathscr{T}}|_{T^-} \cdot \boldsymbol{n}^-$. Here, $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$ are unit normals on S pointing towards T^+ and T^-, respectively.

4.2 Finite element spaces

Given a mesh $\mathscr{T} \in \mathbb{T}$, we denote by $\mathbf{V}(\mathscr{T})$ and $\mathcal{P}(\mathscr{T})$ the finite element spaces that approximate the velocity field and the pressure, respectively, constructed over $\mathscr{T}$. We assume that, for every $p \in (1,\infty)$, $\mathbf{V}(\mathscr{T}) \subset \mathbf{W}_0^{1,\infty}(\Omega) \subset \mathbf{W}_0^{1,p}(\Omega)$ and $\mathcal{P}(\mathscr{T}) \subset L^\infty(\Omega)/\mathbb{R} \subset L^p(\Omega)/\mathbb{R}$. Moreover, we require that $\mathbf{V}(\mathscr{T})$ and $\mathcal{P}(\mathscr{T})$ satisfy the following compatibility conditions [8, Proposition 4.13]: there exists $\beta > 0$ such that, for all $\mathscr{T} \in \mathbb{T}$,

$$\begin{aligned}
\inf_{q_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})} \sup_{\boldsymbol{v}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T})} \frac{b_+(\boldsymbol{v}_{\mathscr{T}}, q_{\mathscr{T}})}{\|\nabla \boldsymbol{v}_{\mathscr{T}}\|_{\mathbf{L}^p(\Omega)} \|q_{\mathscr{T}}\|_{L^{p'}(\Omega)}} \geq \beta, \\
\inf_{q_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})} \sup_{\boldsymbol{v}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T})} \frac{b_-(\boldsymbol{v}_{\mathscr{T}}, q_{\mathscr{T}})}{\|\nabla \boldsymbol{v}_{\mathscr{T}}\|_{\mathbf{L}^{p'}(\Omega)} \|q_{\mathscr{T}}\|_{L^{p}(\Omega)}} \geq \beta.
\end{aligned} \tag{4.1}$$

The following particular elections satisfy the aforementioned assumptions; see [8, Lemma 4.20] and [8, Lemma 4.23].

(a) The mini–element [8, Section 4.2.4]: Here,

$$\mathbf{V}(\mathscr{T}) = \{\boldsymbol{v}_{\mathscr{T}} \in \boldsymbol{C}(\overline{\Omega}) : \boldsymbol{v}_{\mathscr{T}}|_T \in [\mathbb{W}(T)]^2 \ \forall T \in \mathscr{T}\} \cap \mathbf{W}_0^{1,p}(\Omega), \tag{4.2}$$

$$\mathcal{P}(\mathscr{T}) = \{q_{\mathscr{T}} \in C(\overline{\Omega}) : q_{\mathscr{T}}|_T \in \mathbb{P}_1(T) \ \forall T \in \mathscr{T}\} \cap L^p(\Omega)/\mathbb{R}, \tag{4.3}$$

where $\mathbb{W}(T) = \mathbb{P}_1(T) \oplus \mathbb{B}(T)$ and $\mathbb{B}(T)$ denotes the space spanned by local bubble functions.

(b) The lowest order Taylor–Hood element [8, Section 4.2.5]: In this case,

$$\mathbf{V}(\mathscr{T}) = \{\boldsymbol{v}_{\mathscr{T}} \in \boldsymbol{C}(\overline{\Omega}) : \boldsymbol{v}_{\mathscr{T}}|_T \in [\mathbb{P}_2(T)]^2 \ \forall T \in \mathscr{T}\} \cap \mathbf{W}_0^{1,p}(\Omega), \tag{4.4}$$

$$\mathcal{P}(\mathscr{T}) = \{q_{\mathscr{T}} \in C(\overline{\Omega}) : q_{\mathscr{T}}|_T \in \mathbb{P}_1(T) \ \forall T \in \mathscr{T}\} \cap L^p(\Omega)/\mathbb{R}. \tag{4.5}$$

Finally, we introduce the discrete divergence–free finite element space

$$\mathbf{X}(\mathscr{T}) := \{\boldsymbol{w}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T}) : b_+(\boldsymbol{w}_{\mathscr{T}}, q_{\mathscr{T}}) = 0 \, \forall q_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})\}. \tag{4.6}$$

4.3 Finite element formulation

We define the following finite element approximation of problem (3.5): Find $(\boldsymbol{u}_{\mathscr{T}}, \pi_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ such that

$$\begin{aligned} \nu a(\boldsymbol{u}_{\mathscr{T}}, \boldsymbol{v}_{\mathscr{T}}) + b_-(\boldsymbol{v}_{\mathscr{T}}, \pi_{\mathscr{T}}) + c(\boldsymbol{u}_{\mathscr{T}}, \boldsymbol{u}_{\mathscr{T}}; \boldsymbol{v}_{\mathscr{T}}) &= \langle \boldsymbol{f}\delta_z, \boldsymbol{v}_{\mathscr{T}} \rangle, \\ b_+(\boldsymbol{u}_{\mathscr{T}}, q_{\mathscr{T}}) &= 0, \end{aligned} \tag{4.7}$$

for all $\boldsymbol{v}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T})$ and $q_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})$, respectively.

4.4 The Stokes projection

Assume that $\mathbb{T} = \{\mathscr{T}_h\}$ is a collection of conforming and quasiuniform meshes of $\overline{\Omega}$ parametrized by their mesh size $h_{\mathscr{T}} > 0$. The *Stokes projection* of a velocity–pressure pair $(\boldsymbol{\varphi}, \psi) \in \mathbf{W}_0^{1,1}(\Omega) \times L^1(\Omega)/\mathbb{R}$, with zero divergence velocity, is defined as the solution to the following problem: Find $(\boldsymbol{\varphi}_{\mathscr{T}}, \psi_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ such that

$$a(\boldsymbol{\varphi}_{\mathscr{T}}, \boldsymbol{v}_{\mathscr{T}}) + b_-(\boldsymbol{v}_{\mathscr{T}}, \psi_{\mathscr{T}}) = a(\boldsymbol{\varphi}, \boldsymbol{v}_{\mathscr{T}}) + b_-(\boldsymbol{v}_{\mathscr{T}}, \psi), \quad b_+(\boldsymbol{\varphi}_{\mathscr{T}}, q_{\mathscr{T}}) = 0, \tag{4.8}$$

for all $\boldsymbol{v}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T})$ and $q_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})$, respectively.

We present the following stability estimate for the finite element Stokes projection $(\boldsymbol{\varphi}_{\mathscr{T}}, \psi_{\mathscr{T}})$ of the velocity–pressure pair $(\boldsymbol{\varphi}, \psi)$ [12,13].

Proposition 3 (stability of Stokes projection) *Let $\Omega \subset \mathbb{R}^2$ be a convex polygon. Let $s \in (1, 2)$ and $(\boldsymbol{\varphi}, \psi) \in \mathbf{W}_0^{1,s}(\Omega) \times L^s(\Omega)/\mathbb{R}$ with $\boldsymbol{\varphi}$ solenoidal. If $\mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ is given by* (4.2)–(4.3) *or* (4.4)–(4.5)*, then the Stokes projection $(\boldsymbol{\varphi}_{\mathscr{T}}, \psi_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$, defined as the solution to* (4.8)*, satisfies*

$$\|\nabla \boldsymbol{\varphi}_{\mathscr{T}}\|_{\mathbf{L}^s(\Omega)} + \|\psi_{\mathscr{T}}\|_{L^s(\Omega)} \lesssim \|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^s(\Omega)} + \|\psi\|_{L^s(\Omega)}.$$

Proof The proof follows from combining the maximum–norm stability of the Stokes projection derived in [13, Theorems 8.2 and 8.4], the basic stability estimate $\|\nabla \boldsymbol{\varphi}_{\mathscr{T}}\|_{\mathbf{L}^2(\Omega)} + \|\psi_{\mathscr{T}}\|_{L^2(\Omega)} \lesssim \|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)} + \|\psi\|_{L^2(\Omega)}$, and the duality argument elaborated in [12, Remark 4].

As a consequence of Proposition 3, $\mathcal{S}_{\mathscr{T}}$, the discrete version of $\mathcal{S}$, is a bounded linear operator whose inverse $S_{\mathscr{T}}^{-1}$ is bounded uniformly, in $\mathcal{X}$, with respect to $h_{\mathscr{T}}$. We will make use of this fact to perform an a priori error analysis for problem (4.7).

4.5 A priori error estimates

In this section, and this section only, we will assume that Ω is convex. We also assume that $\mathbb{T} = \{\mathscr{T}_h\}$ is a family of quasiuniform triangulations of $\overline{\Omega}$. We begin with the following existence and uniqueness result for small data.

Proposition 4 (well–posedness) *Assume that $\boldsymbol{f}\delta_z$ is sufficiently small or the viscocity ν is sufficiently large so that* (3.9) *with $\mathcal{S}^{-1}$ replaced by $\mathcal{S}_{\mathscr{T}}^{-1}$ holds. Let $\mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ be given by* (4.2)–(4.3) *or* (4.4)–(4.5). *If $p \in (4/3 - \varepsilon, 2)$, where $\varepsilon = \varepsilon(\Omega) > 0$, then there exists a unique pair $(\boldsymbol{u}_{\mathscr{T}}, \pi_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ that solves* (4.7) *and satisfies an estimate similar to that of Proposition 2.*

Proof Repeat the arguments developed in the proof of Propositions 1 and 2 upon replacing $\mathcal{S}^{-1}$ by $\mathcal{S}_{\mathscr{T}}^{-1}$; the latter being uniformly bounded with respect to $h_{\mathscr{T}}$.

In what follows we obtain a priori error estimates for finite element discretizations of problem (3.5). As a first step, we derive a general approximation result that states that approximation of a function in $\mathbf{W}_0^{1,p}(\Omega)$ from $\mathbf{X}(\mathscr{T})$ is as good as from $\mathbf{V}(\mathscr{T})$.

Lemma 1 (approximation in $\mathbf{X}(\mathscr{T})$) *Let $p \in (1, \infty)$ and $\boldsymbol{v} \in \mathbf{W}_0^{1,p}(\Omega)$ be such that $b_+(\boldsymbol{v}, q_{\mathscr{T}}) = 0$ for all $q_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})$. Then, there exist $\bar{\boldsymbol{v}}_{\mathscr{T}} \in \mathbf{X}(\mathscr{T})$ such that*

$$\|\nabla(\boldsymbol{v} - \bar{\boldsymbol{v}}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)} \lesssim \inf_{\boldsymbol{v}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T})} \|\nabla(\boldsymbol{v} - \boldsymbol{v}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)}, \tag{4.9}$$

where the hidden constant is independent of $\boldsymbol{v}$ and $h_{\mathscr{T}}$.

Proof Let $\boldsymbol{v}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T})$ arbitrary. In view of the continuous (3.7) and discrete (4.1) inf-sup conditions, we can apply the Fortin criterion [8, Lemma 4.19] to conclude that there exists $\boldsymbol{w}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T})$ such that $b_+(\boldsymbol{w}_{\mathscr{T}}, q_{\mathscr{T}}) = b_+(\boldsymbol{v} - \boldsymbol{v}_{\mathscr{T}}, q_{\mathscr{T}})$, for all $q_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})$, together with the bound

$$\|\nabla \boldsymbol{w}_{\mathscr{T}}\|_{\mathbf{L}^p(\Omega)} \lesssim \|\nabla(\boldsymbol{v} - \boldsymbol{v}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)}. \tag{4.10}$$

Define $\bar{\boldsymbol{v}}_{\mathscr{T}} := \boldsymbol{v}_{\mathscr{T}} + \boldsymbol{w}_{\mathscr{T}}$. We can thus immediately obtain that

$$b_+(\bar{\boldsymbol{v}}_{\mathscr{T}}, q_{\mathscr{T}}) = b_+(\boldsymbol{v}_{\mathscr{T}}, q_{\mathscr{T}}) + b_+(\boldsymbol{w}_{\mathscr{T}}, q_{\mathscr{T}}) = b_+(\boldsymbol{v}, q_{\mathscr{T}}) = 0 \quad \forall q_{\mathscr{T}} \in \mathcal{P}(\mathscr{T}),$$

where we used that $b_+(\boldsymbol{v}, q_{\mathscr{T}}) = 0$ for all $q_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})$. Consequently, $\bar{\boldsymbol{v}}_{\mathscr{T}} \in \mathbf{X}(\mathscr{T})$. Finally, using the triangle inequality and estimate (4.10), we arrive at

$$\|\nabla(\boldsymbol{v}-\bar{\boldsymbol{v}}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)} \leq \|\nabla \boldsymbol{w}_{\mathscr{T}}\|_{\mathbf{L}^p(\Omega)} + \|\nabla(\boldsymbol{v}-\boldsymbol{v}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)} \lesssim \|\nabla(\boldsymbol{v}-\boldsymbol{v}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)}.$$

The desired estimate (4.9) thus follows from the arbitrariness of $\boldsymbol{v}_{\mathscr{T}}$.

We now present the main result of this section.

Theorem 1 (a priori error estimate) *Assume that $\boldsymbol{f}\delta_z$ is sufficiently small or ν is sufficiently large so that* (3.5) *and* (4.7) *have a unique solution, with sufficiently small norms. Let $\mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ denote the mini–element or the lowest Taylor–Hood finite element. If $p \in (4/3-\varepsilon, 2)$, where $\varepsilon = \varepsilon(\Omega) > 0$, then we have the a priori error estimate*

$$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)} \lesssim \inf_{\boldsymbol{w}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T})} \|\nabla(\boldsymbol{u}-\boldsymbol{w}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)} + \inf_{r_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})} \|\pi - r_{\mathscr{T}}\|_{L^p(\Omega)},$$

where the hidden constant may depend on $\boldsymbol{f}\delta_z$, ν and $\boldsymbol{u}$, but is independent of $h_{\mathscr{T}}$.

Proof Denote by $(\mathcal{S}_{\mathscr{T}}\boldsymbol{u}, \mathcal{S}_{\mathscr{T}}\pi)$ the Stokes projection of $(\boldsymbol{u}, \pi)$, which is defined as the solution to (4.8). We write $\boldsymbol{u} - \boldsymbol{u}_{\mathscr{T}} = (\boldsymbol{u} - \mathcal{S}_{\mathscr{T}}\boldsymbol{u}) + (\mathcal{S}_{\mathscr{T}}\boldsymbol{u} - \boldsymbol{u}_{\mathscr{T}})$ and proceed on the basis of three steps.

Step 1. Let $(\boldsymbol{w}_{\mathscr{T}}, r_{\mathscr{T}}) \in \mathbf{X}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ be arbitrary. Since problem (4.8) is linear, we obtain, for all $(\boldsymbol{v}_{\mathscr{T}}, q_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$, that

$$\begin{aligned} &a(\mathcal{S}_{\mathscr{T}}\boldsymbol{u} - \boldsymbol{w}_{\mathscr{T}}, \boldsymbol{v}_{\mathscr{T}}) + b_-(\boldsymbol{v}_{\mathscr{T}}, \mathcal{S}_{\mathscr{T}}\pi - r_{\mathscr{T}}) = a(\boldsymbol{u} - \boldsymbol{w}_{\mathscr{T}}, \boldsymbol{v}_{\mathscr{T}}) + b_-(\boldsymbol{v}_{\mathscr{T}}, \pi - r_{\mathscr{T}}), \\ &b_+(\mathcal{S}_{\mathscr{T}}\boldsymbol{u} - \boldsymbol{w}_{\mathscr{T}}, q_{\mathscr{T}}) = 0 = b_+(\boldsymbol{u} - \boldsymbol{w}_{\mathscr{T}}, q_{\mathscr{T}}). \end{aligned}$$

We recall that $\mathbf{X}(\mathscr{T})$ is defined in (4.6). Notice that, since $(\mathcal{S}_{\mathscr{T}}\boldsymbol{u}, \mathcal{S}_{\mathscr{T}}\pi)$ is the Stokes projection of $(\boldsymbol{u}, \pi)$ and $\boldsymbol{w}_{\mathscr{T}} \in \mathbf{X}(\mathscr{T})$, we immediately conclude that $b_+(\mathcal{S}_{\mathscr{T}}\boldsymbol{u} - \boldsymbol{w}_{\mathscr{T}}, q_{\mathscr{T}}) = 0$ for every $q_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})$.

Let us now define $(\boldsymbol{\varphi}, \psi) \in \mathbf{W}_0^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ as the solution to

$$\begin{aligned} &a(\boldsymbol{\varphi}, \boldsymbol{v}) + b_-(\boldsymbol{v}, \psi) = a(\boldsymbol{u} - \boldsymbol{w}_{\mathscr{T}}, \boldsymbol{v}) + b_-(\boldsymbol{v}, \pi - r_{\mathscr{T}}) \quad \forall \boldsymbol{v} \in \mathbf{W}_0^{1,p'}(\Omega), \\ &b_+(\boldsymbol{\varphi}, q) = b_+(\boldsymbol{u} - \boldsymbol{w}_{\mathscr{T}}, q) \quad \forall q \in L^{p'}(\Omega)/\mathbb{R}. \end{aligned}$$

Since $p \in (4/3-\varepsilon, 2)$, this problem is well–posed [17, Corollary 1.7]. In addition, we have the estimate

$$\|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^p(\Omega)} + \|\psi\|_{L^p(\Omega)} \lesssim \|\nabla(\boldsymbol{u} - \boldsymbol{w}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)} + \|\pi - r_{\mathscr{T}}\|_{L^p(\Omega)}.$$

We now utilize the triangle inequality, the fact that $(\mathcal{S}_{\mathscr{T}}\boldsymbol{u} - \boldsymbol{w}_{\mathscr{T}}, \mathcal{S}_{\mathscr{T}}\pi - r_{\mathscr{T}})$ corresponds to the Stokes projection of $(\boldsymbol{\varphi}, \psi)$, and the previous estimate to arrive at

$$\|\nabla(\boldsymbol{u}-\mathcal{S}_{\mathscr{T}}\boldsymbol{u})\|_{\mathbf{L}^p(\Omega)} + \|\pi - \mathcal{S}_{\mathscr{T}}\pi\|_{L^p(\Omega)} \lesssim \|\nabla(\boldsymbol{u}-\boldsymbol{w}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)} + \|\pi - r_{\mathscr{T}}\|_{L^p(\Omega)}.$$

Since $(\boldsymbol{w}_{\mathscr{T}}, r_{\mathscr{T}})$ is arbitrary, we conclude that

$$\|\nabla(\boldsymbol{u} - \mathcal{S}_{\mathscr{T}}\boldsymbol{u})\|_{\mathbf{L}^p(\Omega)} + \|\pi - \mathcal{S}_{\mathscr{T}}\pi\|_{L^p(\Omega)} \lesssim \inf_{\boldsymbol{w}_{\mathscr{T}} \in \mathbf{X}(\mathscr{T})} \|\nabla(\boldsymbol{u} - \boldsymbol{w}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)} + \inf_{r_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})} \|\pi - r_{\mathscr{T}}\|_{L^p(\Omega)}. \quad (4.11)$$

To replace $\mathbf{X}(\mathscr{T})$ by $\mathbf{V}(\mathscr{T})$ on the right–hand side of (4.11) we invoke the result of Lemma 1.

Step 2. Define $\boldsymbol{e}_{\mathscr{T}} := \mathcal{S}_{\mathscr{T}}\boldsymbol{u} - \boldsymbol{u}_{\mathscr{T}}$ and $\xi_{\mathscr{T}} := \mathcal{S}_{\mathscr{T}}\pi - \pi_{\mathscr{T}}$. From (4.8), we infer that

$$\nu a(\boldsymbol{e}_{\mathscr{T}}, \boldsymbol{v}_{\mathscr{T}}) + b_-(\boldsymbol{v}_{\mathscr{T}}, \xi_{\mathscr{T}}) = -c(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{v}_{\mathscr{T}}) + c(\boldsymbol{u}_{\mathscr{T}}, \boldsymbol{u}_{\mathscr{T}}; \boldsymbol{v}_{\mathscr{T}}) \quad \forall \boldsymbol{v}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T}),$$
$$b_+(\boldsymbol{e}_{\mathscr{T}}, q_{\mathscr{T}}) = 0 \quad \forall q_{\mathscr{T}} \in \mathcal{P}(\mathscr{T}).$$

Invoke the discrete stability of the Stokes projection of Proposition 3 to obtain

$$\|\nabla \boldsymbol{e}_{\mathscr{T}}\|_{\mathbf{L}^p(\Omega)} + \|\xi_{\mathscr{T}}\|_{L^p(\Omega)} \lesssim \left(\|\nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|\nabla \boldsymbol{u}_{\mathscr{T}}\|_{\mathbf{L}^p(\Omega)}\right) \|\nabla(\boldsymbol{u} - \boldsymbol{u}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)}.$$

Step 3. Combining the estimates obtained in Steps 1 and 2, we arrive at

$$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)} \lesssim \inf_{\boldsymbol{w}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T})} \|\nabla(\boldsymbol{u} - \boldsymbol{w}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)} + \inf_{r_{\mathscr{T}} \in \mathcal{P}(\mathscr{T})} \|\pi - r_{\mathscr{T}}\|_{L^p(\Omega)} + \left(\|\nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|\nabla \boldsymbol{u}_{\mathscr{T}}\|_{\mathbf{L}^p(\Omega)}\right) \|\nabla(\boldsymbol{u} - \boldsymbol{u}_{\mathscr{T}})\|_{\mathbf{L}^p(\Omega)}.$$

The assumption that $\boldsymbol{u}$ and $\boldsymbol{u}_{\mathscr{T}}$ are sufficiently small allow us to absorb the last term on the right hand side of this inequality into the left and conclude.

4.6 Interpolation error estimates

Let $\mathscr{T} \in \mathbb{T}$ and $v \in \mathrm{W}_0^{1,p\prime}(\Omega)$ with $p\prime > 2$. Let $\mathcal{I}_{\mathscr{T}} v$ be the *Lagrange* interpolation operator onto continuous piecewise polynomials of degree $k \in \{1, 2\}$ over $\mathscr{T}$ that vanish on $\partial\Omega$. We will consider $k = 1$ for approximation based on mini–element and $k = 2$ for Taylor–Hood approximation. For $\boldsymbol{v} \in \mathbf{W}_0^{1,p\prime}(\Omega)$, with $p\prime > 2$, we set $\mathcal{I}_{\mathscr{T}}\boldsymbol{v}$ to be the Lagrange interpolation operator applied componentwise and present the following interpolation error estimates.

Lemma 2 (interpolation error estimates) *Let $T \in \mathscr{T}$. If $\boldsymbol{v} \in \mathbf{W}^{1,p\prime}(T)$, with $p\prime > 2$, then*

$$\|\boldsymbol{v} - \mathcal{I}_{\mathscr{T}}\boldsymbol{v}\|_{\mathbf{L}^{p\prime}(T)} \lesssim h_T \|\nabla \boldsymbol{v}\|_{\mathbf{L}^{p\prime}(T)}. \quad (4.12)$$

Let $T \in \mathscr{T}$ and $S \subset \mathscr{S}_T$. If $\boldsymbol{v} \in \mathbf{W}^{1,p\prime}(\mathcal{N}_S)$, with $p\prime > 2$, then

$$\|\boldsymbol{v} - \mathcal{I}_{\mathscr{T}}\boldsymbol{v}\|_{\mathbf{L}^{p\prime}(S)} \lesssim h_T^{1-1/p\prime} \|\nabla \boldsymbol{v}\|_{\mathbf{L}^{p\prime}(\mathcal{N}_S)}. \quad (4.13)$$

Proof See [9, Lemma 1].

5 A posteriori error estimates

In this section we propose and analyze an a posteriori error estimator for finite element approximations of the Navier–Stokes problem (3.5). We prove the global reliability of the devised estimator and also explore local efficiency estimates; the later being based on the existence of a suitable bubble function whose construction we owe to [5]. To perform a reliability analysis we follow [3] and invoke, as an instrumental ingredient, a Ritz projection $(\mathbf{\Phi}, \psi)$ of the residuals. The study of the aforementioned Ritz projection is the content of the following section.

To simplify the presentation, we assume that $\nu = 1$.

5.1 Ritz projection

Define $\mathbf{e}_{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_{\mathscr{T}}$ and $e_\pi = \pi - \pi_{\mathscr{T}}$, the velocity and pressure errors, respectively. The Ritz projection of the residuals is defined as the solution to the following problem: Find $(\mathbf{\Phi}, \psi) \in \mathcal{X}$ such that

$$\begin{aligned} a(\mathbf{\Phi}, \boldsymbol{v}) &= a(\mathbf{e}_{\boldsymbol{u}}, \boldsymbol{v}) + b_-(\boldsymbol{v}, e_\pi) + c(\boldsymbol{u}, \mathbf{e}_{\boldsymbol{u}}; \boldsymbol{v}) + c(\mathbf{e}_{\boldsymbol{u}}, \boldsymbol{u}_{\mathscr{T}}; \boldsymbol{v}), \\ (\psi, q)_{L^2(\Omega)} &= b_+(\mathbf{e}_{\boldsymbol{u}}, q) \end{aligned} \tag{5.1}$$

for all $(\boldsymbol{v}, q) \in \mathcal{Y}$. To analyze problem (5.1), we introduce the linear functional

$$\mathcal{A} : \mathbf{W}_0^{1,p'}(\Omega) \to \mathbb{R}, \quad \boldsymbol{v} \mapsto a(\mathbf{e}_{\boldsymbol{u}}, \boldsymbol{v}) + b_-(\boldsymbol{v}, e_\pi) + c(\boldsymbol{u}, \mathbf{e}_{\boldsymbol{u}}; \boldsymbol{v}) + c(\mathbf{e}_{\boldsymbol{u}}, \boldsymbol{u}_{\mathscr{T}}; \boldsymbol{v}).$$

Notice that

$$\begin{aligned} |\mathcal{A}(\boldsymbol{v})| \leq{}& \|\nabla \mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\Omega)} \|\nabla \boldsymbol{v}\|_{\mathbf{L}^{p'}(\Omega)} + \|e_\pi\|_{L^p(\Omega)} \|\operatorname{div} \boldsymbol{v}\|_{\mathbf{L}^{p'}(\Omega)} \\ &+ (\|\boldsymbol{u}\|_{\mathbf{L}^{2p}(\Omega)} \|\nabla \boldsymbol{v}\|_{\mathbf{L}^{p'}(\Omega)} + \|\boldsymbol{u}_{\mathscr{T}}\|_{\mathbf{L}^{2p}(\Omega)} \|\nabla \boldsymbol{v}\|_{\mathbf{L}^{p'}(\Omega)}) \|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^{2p}(\Omega)}, \end{aligned} \tag{5.2}$$

which, in view of the embedding $\mathbf{W}_0^{1,p}(\Omega) \hookrightarrow \mathbf{L}^{2p}(\Omega)$, implies that $\mathcal{A} \in \mathbf{W}^{-1,p}(\Omega)$.

We present the following stability result for the Ritz projection.

Lemma 3 (Ritz projection) *Let Ω be Lipschitz. If $p \in (4/3 - \varepsilon, 2)$, with $\varepsilon = \varepsilon(\Omega) > 0$, then, problem (5.1) has a unique solution in $\mathcal{X}$. In addition, we have*

$$\begin{aligned} \|\nabla \mathbf{\Phi}\|_{\mathbf{L}^p(\Omega)} + \|\psi\|_{L^p(\Omega)} \lesssim{}& \|\nabla \mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\Omega)} + \|e_\pi\|_{L^p(\Omega)} \\ &+ \|\nabla \mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\Omega)} (\|\nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|\nabla \boldsymbol{u}_{\mathscr{T}}\|_{\mathbf{L}^p(\Omega)}), \end{aligned} \tag{5.3}$$

where the hidden constant is independent of $(\mathbf{\Phi}, \psi)$, $(\boldsymbol{u}, \pi)$, and $(\boldsymbol{u}_{\mathscr{T}}, \pi_{\mathscr{T}})$.

Proof Since $\mathcal{A} \in \mathbf{W}^{-1,p}(\Omega)$, the inf–sup condition (3.13) immediately yields the existence and uniqueness of $\boldsymbol{\Phi} \in \mathbf{W}_0^{1,p}(\Omega)$ together with the estimate

$$\begin{aligned} \|\nabla \boldsymbol{\Phi}\|_{\mathbf{L}^p(\Omega)} \lesssim \|\mathcal{A}\|_{\mathbf{W}^{-1,p}(\Omega)} \lesssim{}& \|\nabla \mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\Omega)} + \|e_\pi\|_{L^p(\Omega)} \\ &+ \|\nabla \mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\Omega)} \left(\|\nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|\nabla \boldsymbol{u}_{\mathscr{T}}\|_{\mathbf{L}^p(\Omega)} \right). \end{aligned}$$

We now focus on the existence of ψ. Since $\mathbf{e}_{\boldsymbol{u}} \in \mathbf{W}_0^{1,p}(\Omega)$, $b_+(\mathbf{e}_{\boldsymbol{u}},\cdot)$ defines a linear and bounded functional in $L^{p'}(\Omega)/\mathbb{R}$, an inf–sup condition for $(\cdot,\cdot)_{L^2(\Omega)}$ on $L^p(\Omega)/\mathbb{R} \times L^{p'}(\Omega)/\mathbb{R}$ yields the existence of a unique $\psi \in L^p(\Omega)/\mathbb{R}$ such that $(\psi,q)_{L^2(\Omega)} = b_+(\mathbf{e}_{\boldsymbol{u}},q)$ for all $q \in L^{p'}(\Omega)/\mathbb{R}$. Moreover, we have that $\|\psi\|_{L^p(\Omega)} \lesssim \|\operatorname{div}\mathbf{e}_{\boldsymbol{u}}\|_{L^p(\Omega)}$.

A combination of the derived estimates yields the desired bound (5.3). This concludes the proof.

5.2 Upper bound for the error

In this section, we prove that the energy norm of the error can be bounded in terms of the energy norm of the Ritz projection. To accomplish this task, we begin by observing that the pair $(\mathbf{e}_{\boldsymbol{u}}, e_\pi)$ can be seen as the solution to the following Stokes problem: Find $(\mathbf{e}_{\boldsymbol{u}}, e_\pi) \in \mathcal{X}$ such that

$$a(\mathbf{e}_{\boldsymbol{u}},\boldsymbol{v}) + b_-(\boldsymbol{v},e_\pi) = \mathcal{G}(\boldsymbol{v}), \quad b_+(\mathbf{e}_{\boldsymbol{u}},q) = (\psi,q)_{L^2(\Omega)} \quad \forall (\boldsymbol{v},q) \in \mathcal{Y}, \tag{5.4}$$

where $\mathcal{G} : \mathbf{W}_0^{1,p'}(\Omega) \to \mathbb{R}$ is defined by $\boldsymbol{v} \mapsto a(\boldsymbol{\Phi},\boldsymbol{v}) - c(\boldsymbol{u},\mathbf{e}_{\boldsymbol{u}};\boldsymbol{v}) - c(\mathbf{e}_{\boldsymbol{u}},\boldsymbol{u}_{\mathscr{T}};\boldsymbol{v})$. It is clear that $\mathcal{G} \in \mathbf{W}^{-1,p}(\Omega)$. In addition, we have

$$\begin{aligned}\|\mathcal{G}\|_{\mathbf{W}^{-1,p}(\Omega)} &\leq \|\nabla\boldsymbol{\Phi}\|_{\mathbf{L}^p(\Omega)} \\ &\quad + C_{2p\to p}^2 \|\nabla\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\Omega)} (\|\nabla\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|\nabla\boldsymbol{u}_{\mathscr{T}}\|_{\mathbf{L}^p(\Omega)}).\end{aligned} \tag{5.5}$$

To derive the following result, we assume that the forcing term $\boldsymbol{f}\delta_z$ is sufficiently small so that

$$1 - \|\mathcal{S}^{-1}\| C_{2p\to p}^2 \left(\|\nabla\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|\nabla\boldsymbol{u}_{\mathscr{T}}\|_{\mathbf{L}^p(\Omega)}\right) \geq \eta > 0. \tag{5.6}$$

Lemma 4 (upper bound for the error) *Assume that the forcing term $\boldsymbol{f}\delta_z$ is sufficiently small so that* (5.6) *holds. If $p \in (4/3-\varepsilon, 2)$, where $\varepsilon = \varepsilon(\Omega) > 0$ denotes a constant that depends on Ω, then*

$$\|\nabla\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\Omega)} + \|e_\pi\|_{L^p(\Omega)} \lesssim \|\nabla\boldsymbol{\Phi}\|_{\mathbf{L}^p(\Omega)} + \|\psi\|_{L^p(\Omega)}, \tag{5.7}$$

where the hidden constant is independent of $(\boldsymbol{u},\pi)$, $(\boldsymbol{u}_{\mathscr{T}},\pi_{\mathscr{T}})$, and $(\boldsymbol{\Phi},\psi)$.

Proof Since Ω is Lipschitz, $p \in (4/3-\varepsilon, 2)$, and $\mathcal{G} \in \mathbf{W}^{-1,p}(\Omega)$, we are in position to apply the results of [17, Corollary 1.7] and thus use (5.5) to conclude that

$$\begin{aligned}\|\nabla\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\Omega)} + \|e_\pi\|_{L^p(\Omega)} &\leq \|\mathcal{S}^{-1}\| \big[\|\nabla\boldsymbol{\Phi}\|_{\mathbf{L}^p(\Omega)} + \|\psi\|_{L^p(\Omega)} \\ &\quad + C_{2p\to p}^2 \|\nabla\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\Omega)} \left(\|\nabla\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|\nabla\boldsymbol{u}_{\mathscr{T}}\|_{\mathbf{L}^p(\Omega)}\right) \big].\end{aligned}$$

The desired estimate thus follows from (5.6). This concludes the proof.

5.3 A posteriori error estimators

In this section we introduce a posteriori error estimators for finite element discretizations of (4.7). The estimators depend on which scheme is considered. To be precise, let $T \in \mathscr{T}$. If $z \in T$ is such that

(i) z is not a vertex of T or a midpoint of a side of T, when Taylor–Hood approximation is considered, or
(ii) z is not a vertex of T, when the approximation based on the mini–element is considered, then we define the element error indicator

$$\eta_{p,T} := \Big(h_T^p \|\Delta \boldsymbol{u}_{\mathscr{T}} - (\boldsymbol{u}_{\mathscr{T}} \cdot \nabla)\boldsymbol{u}_{\mathscr{T}} - \operatorname{div} \boldsymbol{u}_{\mathscr{T}} \boldsymbol{u}_{\mathscr{T}} - \nabla \pi_{\mathscr{T}} \|_{\mathbf{L}^p(T)}^p$$
$$+ h_T \| [\![(\nabla \boldsymbol{u}_{\mathscr{T}} - \mathbb{I}\pi_{\mathscr{T}}) \cdot \boldsymbol{n}]\!] \|_{\mathbf{L}^p(\partial T \setminus \partial \Omega)}^p + \| \operatorname{div} \boldsymbol{u}_{\mathscr{T}} \|_{L^p(T)}^p + h_T^{2-p} |\boldsymbol{f}|^p \Big)^{\frac{1}{p}}. \quad (5.8)$$

If $z \in T$ and (i) or (ii) do not hold, then

$$\eta_{p,T} := \Big(h_T^p \|\Delta \boldsymbol{u}_{\mathscr{T}} - (\boldsymbol{u}_{\mathscr{T}} \cdot \nabla)\boldsymbol{u}_{\mathscr{T}} - \operatorname{div} \boldsymbol{u}_{\mathscr{T}} \boldsymbol{u}_{\mathscr{T}} - \nabla \pi_{\mathscr{T}} \|_{\mathbf{L}^p(T)}^p$$
$$+ h_T \| [\![(\nabla \boldsymbol{u}_{\mathscr{T}} - \mathbb{I}\pi_{\mathscr{T}}) \cdot \boldsymbol{n}]\!] \|_{\mathbf{L}^p(\partial T \setminus \partial \Omega)}^p + \| \operatorname{div} \boldsymbol{u}_{\mathscr{T}} \|_{L^p(T)}^p \Big)^{\frac{1}{p}}. \quad (5.9)$$

If $z \notin T$, then the indicator $\eta_{p,T}$ is defined as in (5.9). In (5.8) and (5.9), $\mathbb{I}$ denotes the identity matrix in $\mathbb{R}^{2\times 2}$ and $(\boldsymbol{u}_{\mathscr{T}}, \pi_{\mathscr{T}})$ corresponds to the solution to the discrete problem (4.7). We recall that we consider our elements T to be closed sets. The a posteriori error estimators are thus defined by

$$\eta_p := \left(\sum_{T \in \mathscr{T}} \eta_{p,T}^p \right)^{\frac{1}{p}}. \quad (5.10)$$

5.4 A posteriori error estimates: global reliability

In what follows we obtain a global reliability property for the a posteriori error estimators (5.10).

Theorem 2 (global reliability) *Let $(\boldsymbol{u}, \pi) \in \mathcal{X}$ be the solution of (3.5) and $(\boldsymbol{u}_{\mathscr{T}}, \pi_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ its finite element approximation given as the solution of (4.7). Assume that the forcing term $\boldsymbol{f}\delta_z$ is sufficiently small so that (3.9) holds. If $p \in (4/3 - \varepsilon, 2)$, where $\varepsilon = \varepsilon(\Omega) > 0$, then*

$$\|\nabla \mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\Omega)} + \|e_\pi\|_{L^p(\Omega)} \lesssim \eta_p, \quad (5.11)$$

where the hidden constant is independent of $(\boldsymbol{u}, \pi)$, $(\boldsymbol{u}_{\mathscr{T}}, \pi_{\mathscr{T}})$, the size of the elements in $\mathscr{T}$, and $\#\mathscr{T}$.

Proof We proceed in three steps.

Step 1. Let $(\boldsymbol{v},q) \in \mathcal{Y}$. Since $(\boldsymbol{\Phi},\psi) \in \mathcal{X}$ and $(\boldsymbol{u},\pi) \in \mathcal{X}$ solve (5.1) and (3.5), respectively, an elementwise integration by parts yields the identity

$$a(\boldsymbol{\Phi},\boldsymbol{v}) = \langle \boldsymbol{f}\delta_z, \boldsymbol{v}\rangle + \sum_{S\in\mathscr{S}} \int_S [\![(\nabla \boldsymbol{u}_{\mathscr{T}} - \mathbb{I}\pi_{\mathscr{T}}) \cdot \boldsymbol{n}]\!] \cdot \boldsymbol{v} \\ + \sum_{T\in\mathscr{T}} \int_T (\Delta \boldsymbol{u}_{\mathscr{T}} - (\boldsymbol{u}_{\mathscr{T}} \cdot \nabla)\boldsymbol{u}_{\mathscr{T}} - \operatorname{div} \boldsymbol{u}_{\mathscr{T}} \boldsymbol{u}_{\mathscr{T}} - \nabla \pi_{\mathscr{T}}) \cdot \boldsymbol{v}. \quad (5.12)$$

On the other hand, the first equation of problem (4.7) can be rewritten as

$$\langle \boldsymbol{f}\delta_z, \boldsymbol{v}_{\mathscr{T}}\rangle - a(\boldsymbol{u}_{\mathscr{T}}, \boldsymbol{v}_{\mathscr{T}}) - b_-(\boldsymbol{v}_{\mathscr{T}}, \pi_{\mathscr{T}}) - c(\boldsymbol{u}_{\mathscr{T}}, \boldsymbol{u}_{\mathscr{T}}; \boldsymbol{v}_{\mathscr{T}}) = 0 \quad \forall \boldsymbol{v}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T}).$$

Set $\boldsymbol{v}_{\mathscr{T}} = \mathcal{I}_{\mathscr{T}}\boldsymbol{v}$, the Lagrange interpolation operator applied componentwise. Notice that, since $p\prime > 2$, functions in $\mathbf{W}_0^{1,p\prime}(\Omega)$ are continuous and thus the Lagrange interpolation operator is well–defined on $\mathbf{W}_0^{1,p\prime}(\Omega)$. Subtract the obtained identity from (5.12), and invoke, again, an elementwise integrating by parts formula to arrive at

$$a(\boldsymbol{\Phi},\boldsymbol{v}) = \langle \boldsymbol{f}\delta_z, \boldsymbol{v} - \mathcal{I}_{\mathscr{T}}\boldsymbol{v}\rangle + \sum_{S\in\mathscr{S}} \int_S [\![(\nabla \boldsymbol{u}_{\mathscr{T}} - \mathbb{I}\pi_{\mathscr{T}}) \cdot \boldsymbol{n}]\!] \cdot (\boldsymbol{v} - \mathcal{I}_{\mathscr{T}}\boldsymbol{v}) \\ + \sum_{T\in\mathscr{T}} \int_T (\Delta \boldsymbol{u}_{\mathscr{T}} - (\boldsymbol{u}_{\mathscr{T}} \cdot \nabla)\boldsymbol{u}_{\mathscr{T}} - \operatorname{div} \boldsymbol{u}_{\mathscr{T}} \boldsymbol{u}_{\mathscr{T}} - \nabla \pi_{\mathscr{T}}) \cdot (\boldsymbol{v} - \mathcal{I}_{\mathscr{T}}\boldsymbol{v}) \\ = \mathbf{I} + \mathbf{II} + \mathbf{III}, \quad (5.13)$$

where we have used that, for $S \in \mathscr{S}$, the term $\int_S [\![(\nabla \boldsymbol{u}_{\mathscr{T}} - \mathbb{I}\pi_{\mathscr{T}}) \cdot \boldsymbol{n}]\!] \cdot (\boldsymbol{v} - \mathcal{I}_{\mathscr{T}}\boldsymbol{v})$ vanishes. This is a consequence of the fact that the discrete functions belonging to our finite element velocity spaces are continuous functions.

We now control each term separately. We begin by bounding the term $\mathbf{I}$. Notice that, if $z \in T$ is such that (i) or (ii) do not hold, then $(\boldsymbol{v} - \mathcal{I}_{\mathscr{T}}\boldsymbol{v})|_z = \mathbf{0}$. Otherwise, we invoke the $\mathbf{W}_0^{1,p\prime}(\Omega)$-regularity of $\boldsymbol{v}$ and a basic $\mathbf{L}^\infty$–error estimate for $\mathcal{I}_{\mathscr{T}}$ to obtain

$$\mathbf{I} = \boldsymbol{f} \cdot (\boldsymbol{v} - \mathcal{I}_{\mathscr{T}}\boldsymbol{v})(z) \lesssim |\boldsymbol{f}| \|\boldsymbol{v} - \mathcal{I}_{\mathscr{T}}\boldsymbol{v}\|_{\mathbf{L}^\infty(T)} \lesssim h_T^{1-2/p\prime} |\boldsymbol{f}| \|\nabla \boldsymbol{v}\|_{\mathbf{L}^{p\prime}(T)}.$$

To control $\mathbf{II}$, we invoke Hölder's inequality and estimate (4.13). We thus obtain

$$\mathbf{II} \lesssim \sum_{S\in\mathscr{S}} \| [\![(\nabla \boldsymbol{u}_{\mathscr{T}} - \mathbb{I}\pi_{\mathscr{T}}]\!]) \cdot \boldsymbol{n} \|_{\mathbf{L}^p(S)} \|\boldsymbol{v} - \mathcal{I}_{\mathscr{T}}\boldsymbol{v}\|_{\mathbf{L}^{p\prime}(S)} \\ \lesssim \sum_{S\in\mathscr{S}} h_S^{1-1/p\prime} \| [\![(\nabla \boldsymbol{u}_{\mathscr{T}} - \mathbb{I}\pi_{\mathscr{T}}) \cdot \boldsymbol{n}]\!] \|_{\mathbf{L}^p(S)} \|\nabla \boldsymbol{v}\|_{\mathbf{L}^{p\prime}(\mathcal{N}_S)}.$$

Finally, we bound the remaining term **III**. To accomplish this task, we invoke, again, Hölder's inequality and then the error estimate (4.12). These arguments yield

$$\begin{aligned}\mathbf{III} &\lesssim \sum_{T\in\mathscr{T}} \|\Delta \boldsymbol{u}_{\mathscr{T}} - (\boldsymbol{u}_{\mathscr{T}}\cdot\nabla)\boldsymbol{u}_{\mathscr{T}} - \operatorname{div}\boldsymbol{u}_{\mathscr{T}}\boldsymbol{u}_{\mathscr{T}} - \nabla\pi_{\mathscr{T}}\|_{\mathbf{L}^p(T)}\|\boldsymbol{v}-\mathcal{I}_{\mathscr{T}}\boldsymbol{v}\|_{\mathbf{L}^{p'}(T)},\\ &\lesssim \sum_{T\in\mathscr{T}} h_T\|\Delta \boldsymbol{u}_{\mathscr{T}} - (\boldsymbol{u}_{\mathscr{T}}\cdot\nabla)\boldsymbol{u}_{\mathscr{T}} - \operatorname{div}\boldsymbol{u}_{\mathscr{T}}\boldsymbol{u}_{\mathscr{T}} - \nabla\pi_{\mathscr{T}}\|_{\mathbf{L}^p(T)}\|\nabla\boldsymbol{v}\|_{\mathbf{L}^{p'}(T)}.\end{aligned}$$

Now, since $p \in (4/3-\varepsilon, 2)$, with $\varepsilon = \varepsilon(\Omega) > 0$, we are allowed to invoke the inf–sup condition (3.13) to arrive at

$$\|\nabla\boldsymbol{\Phi}\|_{\mathbf{L}^p(\Omega)} \lesssim \sup_{\mathbf{0}\neq\boldsymbol{v}\in\mathbf{W}_0^{1,p'}(\Omega)} \frac{a(\boldsymbol{\Phi},\boldsymbol{v})}{\|\nabla\boldsymbol{v}\|_{\mathbf{L}^{p'}(\Omega)}}.$$

Utilize identity (5.13), in conjunction with the bounds for **I**, **II**, and **III**, that have been previously obtained, and the finite overlapping of stars, to conclude that

$$\|\nabla\boldsymbol{\Phi}\|_{\mathbf{L}^p(\Omega)} \lesssim \eta_p.$$

We recall that η_p is defined as in (5.10).

Step 2. The goal of this step is to bound $\|\psi\|_{L^p(\Omega)}$. To accomplish this task, we first define $q := |\psi|^{p-1}\mathrm{sign}(\psi) - |\Omega|^{-1}\int_\Omega |\psi|^{p-1}\mathrm{sign}(\psi)$. Notice that $\int_\Omega q = 0$ and

$$\int_\Omega ||\psi|^{p-1}\mathrm{sign}(\psi)|^{p'} = \|\psi\|^p_{L^p(\Omega)}, \quad \int_\Omega \left|\int_\Omega ||\psi|^{p-1}\mathrm{sign}(\psi)|\right|^{p'} \leq C\|\psi\|^p_{L^p(\Omega)},$$

where $C = |\Omega|^{\frac{p}{p-1}}$. Consequently, $q \in L^{p'}(\Omega)/\mathbb{R}$ and $\|q\|_{L^{p'}(\Omega)} \lesssim \|\psi\|^{\frac{p}{p'}}_{L^p(\Omega)}$. We are thus allow to set $q = |\psi|^{p-1}\mathrm{sign}(\psi) - |\Omega|^{-1}\int_\Omega |\psi|^{p-1}\mathrm{sign}(\psi)$ in the second equation of (5.1). This yields

$$\begin{aligned}\|\psi\|^p_{L^p(\Omega)} &= (\psi,q)_{L^2(\Omega)} = b_+(\mathbf{e}_{\boldsymbol{u}},q)\\ &= -b_+(\boldsymbol{u}_{\mathscr{T}},q) \leq \|\operatorname{div}\boldsymbol{u}_{\mathscr{T}}\|_{L^p(\Omega)}\|q\|_{L^{p'}(\Omega)} \lesssim \|\operatorname{div}\boldsymbol{u}_{\mathscr{T}}\|_{L^p(\Omega)}\|\psi\|^{\frac{p}{p'}}_{L^p(\Omega)}.\end{aligned}$$

Notice that we have used that $\int_\Omega \psi = 0$, which yields $\|\psi\|^p_{L^p(\Omega)} = (\psi,q)_{L^2(\Omega)}$. Utilize that $p - p/p\prime = 1$ to arrive at $\|\psi\|_{L^p(\Omega)} \lesssim \|\operatorname{div}\boldsymbol{u}_{\mathscr{T}}\|_{L^p(\Omega)}$.

Step 3. Apply estimate (5.7) and the bounds for $\|\nabla\boldsymbol{\Phi}\|_{\mathbf{L}^p(\Omega)}$ and $\|\psi\|_{L^p(\Omega)}$ obtained in Steps 1 and 2, respectively, to arrive at the a posteriori error estimate

$$\|\nabla\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\Omega)} + \|e_\pi\|_{L^p(\Omega)} \lesssim \|\nabla\boldsymbol{\Phi}\|_{\mathbf{L}^p(\Omega)} + \|\psi\|_{L^p(\Omega)} \lesssim \eta_p.$$

This is where the smallness assumption (5.6) is needed. This concludes the proof.

5.5 A posteriori error estimates: local estimates

We now proceed to investigate local estimates for the indicators $\eta_{p,T}$ defined in (5.8)–(5.9). To accomplish this task, we introduce the following notation: for an edge or triangle G, we denote by $\mathcal{V}(G)$ the set of vertices of G. With this notation at hand, we define, for $T \in \mathscr{T}$ and $S \in \mathscr{S}$, the standard element and edge bubble functions [22, 24]

$$\varphi_T = 27 \prod_{\mathrm{v} \in \mathcal{V}(T)} \lambda_{\mathrm{v}}, \qquad \varphi_S = 4 \prod_{\mathrm{v} \in \mathcal{V}(S)} \lambda_{\mathrm{v}}|_{T'}, \quad T' \subset \mathcal{N}_S,$$

where λ_{v} corresponds to the barycentric coordinates of T. Recall that $\mathcal{N}_S$ corresponds to the patch composed of the two elements of $\mathscr{T}$ sharing S. We also introduce the following bubble functions, whose construction we owe to [5]. Given $T \in \mathscr{T}$, we define

$$\phi_T(x) := \begin{cases} \varphi_T(x) \frac{|x-z|^2}{h_T^2}, & \text{if } z \in T, \\ \varphi_T(x), & \text{if } z \notin T. \end{cases} \tag{5.14}$$

Given $S \in \mathscr{S}$, we introduce the bubble function ϕ_S as

$$\phi_S(x) := \begin{cases} \varphi_S(x) \frac{|x-z|^2}{h_S^2}, & \text{if } z \in \mathring{\mathcal{N}}_S, \\ \varphi_S(x), & \text{if } z \notin \mathring{\mathcal{N}}_S. \end{cases} \tag{5.15}$$

Here, $\mathring{\mathcal{N}}_S$ denotes the interior of $\mathcal{N}_S$. We recall that the Dirac measure δ_z is supported at the point $z \in \Omega$. It can thus be supported on the interior, an edge, or a vertex of an element $T \in \mathscr{T}$.

Given $S \in \mathscr{S}$, we introduce the continuation operator $\Pi : L^\infty(S) \to L^\infty(\mathcal{N}_S)$ as defined in [23, Section 3]. This operator maps polynomials onto piecewise polynomials of the same degree. With this operator at hand, we present the following result whose proof can be found in [9, Lemma 3]. We notice that, due to the presence of the convective term, the polynomial degree needs to be suitably modified in order to be able to handle both the mini–element and the lowest degree Taylor–Hood elements.

Lemma 5 (bubble function properties) *Let $r \in (1, \infty)$, $T \in \mathscr{T}$, $S \in \mathscr{S}$, and $m \in \mathbb{N}$. Then, the bubble functions ϕ_T and ϕ_S satisfy*

$$\|\phi_T\|_{\mathrm{W}^{m,r}(T)} \lesssim h_T^{2/r-m}. \tag{5.16}$$

In addition, if $\boldsymbol{v}_{\mathscr{T}}|_T \in [\mathbb{P}_5(T)]^2$ and $\boldsymbol{w}_{\mathscr{T}}|_S \in [\mathbb{P}_3(S)]^2$, then

$$\|\boldsymbol{v}_{\mathscr{T}}\|_{\mathbf{L}^r(T)} \lesssim \|\boldsymbol{v}_{\mathscr{T}} \phi_T^{\frac{1}{r}}\|_{\mathbf{L}^r(T)} \lesssim \|\boldsymbol{v}_{\mathscr{T}}\|_{\mathbf{L}^r(T)}, \tag{5.17}$$

$$\|\boldsymbol{w}_{\mathscr{T}}\|_{\mathbf{L}^r(S)} \lesssim \|\boldsymbol{w}_{\mathscr{T}} \phi_S^{\frac{1}{r}}\|_{\mathbf{L}^r(S)} \lesssim \|\boldsymbol{w}_{\mathscr{T}}\|_{\mathbf{L}^r(S)}, \tag{5.18}$$

$$\|\phi_S \Pi \boldsymbol{w}_{\mathscr{T}}\|_{\mathbf{L}^r(T)} \lesssim h_T^{\frac{1}{r}} \|\boldsymbol{w}_{\mathscr{T}}\|_{\mathbf{L}^r(S)}. \tag{5.19}$$

With all these ingredients at hand, we are in position to investigate local estimates for the indicators $\eta_{p,T}$.

Theorem 3 (local estimates) *Let* $p \in [4/3, 2)$. *Let* $(\boldsymbol{u}, \pi) \in \mathcal{X}$ *be the solution of problem* (3.5) *and* $(\boldsymbol{u}_{\mathscr{T}}, \pi_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ *its finite element approximation obtained as the solution to* (4.7). *Assume that the forcing term* $\boldsymbol{f}\delta_z$ *is sufficiently small so that* (5.6) *holds. Then*

$$\eta_{p,T}^p \lesssim \|\nabla \mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\mathcal{N}_T)}^p + \|e_\pi\|_{L^p(\mathcal{N}_T)}^p + h_T^{-2/3}\|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\mathcal{N}_T)}^p, \tag{5.20}$$

where the hidden constant is independent of the continuous and discrete solutions, the size of the elements in the mesh $\mathscr{T}$, *and* $\#\mathscr{T}$.

Proof To simplify notation, we define

$$\mathbf{R}_T := (\Delta \boldsymbol{u}_{\mathscr{T}} - (\boldsymbol{u}_{\mathscr{T}} \cdot \nabla)\boldsymbol{u}_{\mathscr{T}} - \operatorname{div} \boldsymbol{u}_{\mathscr{T}}\boldsymbol{u}_{\mathscr{T}} - \nabla \pi_{\mathscr{T}})|_T, \quad \boldsymbol{\Upsilon}_T := \phi_T \mathbf{R}_T, \tag{5.21}$$

where ϕ_T denotes the bubble function defined in (5.14). With these definitions at hand, we proceed on the basis of five steps.

Step 1. Let $T \in \mathscr{T}$. The goal of this step is to bound $h_T^p\|\mathbf{R}_T\|_{\mathbf{L}^p(T)}^p$ in (5.8)–(5.9). As a first step, we invoke (5.17) to obtain the basic estimate

$$\|\mathbf{R}_T\|_{\mathbf{L}^2(T)}^2 \lesssim \|\mathbf{R}_T \phi_T^{\frac{1}{2}}\|_{\mathbf{L}^2(T)}^2 = \int_T \mathbf{R}_T \cdot \boldsymbol{\Upsilon}_T. \tag{5.22}$$

To explore the term $\int_T \mathbf{R}_T \cdot \boldsymbol{\Upsilon}_T$, we observe that $\boldsymbol{\Upsilon}_T(z) = \phi_T(z)\mathbf{R}_T(z) = \mathbf{0}$ and that, for $S \in \mathscr{S}_T$, $\boldsymbol{\Upsilon}_{T|S} = \mathbf{0}$. We thus set $\boldsymbol{v} = \boldsymbol{\Upsilon}_T$ in (5.12) to obtain

$$\int_T \mathbf{R}_T \cdot \boldsymbol{\Upsilon}_T = a(\boldsymbol{\Phi}, \boldsymbol{\Upsilon}_T).$$

It thus suffices to bound the term $a(\boldsymbol{\Phi}, \boldsymbol{\Upsilon}_T)$. To accomplish this task, we first notice that from assumption (5.6), the following estimate can be derived:

$$\|\nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|\nabla \boldsymbol{u}_{\mathscr{T}}\|_{\mathbf{L}^p(\Omega)} \leq \frac{1-\eta}{\|\mathcal{S}^{-1}\| C_{2p\to p}^2}. \tag{5.23}$$

We thus invoke the first equation in problem (5.1) with $\boldsymbol{v} = \boldsymbol{\Upsilon}_T$ to obtain

$$\begin{aligned} a(\boldsymbol{\Phi}, \boldsymbol{\Upsilon}_T) \lesssim & \left(\|\nabla \mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T)} + \|e_\pi\|_{L^p(T)}\right) \|\nabla \boldsymbol{\Upsilon}_T\|_{\mathbf{L}^{p'}(T)} \\ & + \left(\|\boldsymbol{u}\|_{\mathbf{L}^{3p}(T)}\|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T)} + \|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T)}\|\boldsymbol{u}_{\mathscr{T}}\|_{\mathbf{L}^{3p}(T)}\right) \|\nabla \boldsymbol{\Upsilon}_T\|_{\mathbf{L}^s(T)}, \end{aligned} \tag{5.24}$$

where we have also used that $\operatorname{supp}\boldsymbol{\Upsilon}_T \subset T$. Here, s is such that $1/s + 1/3p + 1/p = 1$. Notice that, since $p \geq 4/3$, we have $\mathbf{W}^{1,p}(\Omega) \hookrightarrow \mathbf{L}^{3p}(\Omega)$ [11, Theorem II.3.2]. In what follows we control $\|\nabla \boldsymbol{\Upsilon}_T\|_{\mathbf{L}^{p'}(T)}$, where $\boldsymbol{\Upsilon}_T = \phi_T \mathbf{R}_T$. Invoke Lemma 5 and standard inverse inequalities to arrive at

$$\begin{aligned} \|\nabla \boldsymbol{\Upsilon}_T\|_{\mathbf{L}^{p'}(T)} \lesssim & \|\phi_T \nabla \mathbf{R}_T\|_{\mathbf{L}^{p'}(T)} + \|\nabla \phi_T \mathbf{R}_T\|_{\mathbf{L}^{p'}(T)} \\ & \lesssim h_T^{-1}\|\mathbf{R}_T\|_{\mathbf{L}^{p'}(T)} \lesssim h_T^{-1} h_T^{2/p'-1}\|\mathbf{R}_T\|_{\mathbf{L}^2(T)}. \end{aligned} \tag{5.25}$$

Similarly, $\|\nabla \boldsymbol{\Upsilon}_T\|_{\mathbf{L}^s(T)} \lesssim h_T^{-1} h_T^{2/s-1} \|\mathbf{R}_T\|_{\mathbf{L}^2(T)}$. Replacing (5.25) and the previous estimate into (5.24) and the obtained estimate into (5.22), we obtain

$$\|\mathbf{R}_T\|^2_{\mathbf{L}^2(T)} \lesssim (\|\nabla \mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T)} + \|e_\pi\|_{L^p(T)}) h_T^{-1} h_T^{2/p\prime-1} \|\mathbf{R}_T\|_{\mathbf{L}^2(T)} \\ + \|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T)} h_T^{2/s-2} \|\mathbf{R}_T\|_{\mathbf{L}^2(T)}. \quad (5.26)$$

The inverse inequality $\|\mathbf{R}_T\|_{\mathbf{L}^p(T)} \lesssim h_T^{2/p-1} \|\mathbf{R}_T\|_{\mathbf{L}^2(T)}$ thus yield

$$\|\mathbf{R}_T\|_{\mathbf{L}^p(T)} \lesssim h_T^{-1}(\|\nabla \mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T)} + \|e_\pi\|_{L^p(T)}) + h_T^{-1-2/3p} \|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T)}, \quad (5.27)$$

which immediately implies that

$$h_T^p \|\mathbf{R}_T\|^p_{\mathbf{L}^p(T)} \lesssim \|\nabla \mathbf{e}_{\boldsymbol{u}}\|^p_{\mathbf{L}^p(T)} + \|e_\pi\|^p_{L^p(T)} + h_T^{-2/3} \|\mathbf{e}_{\boldsymbol{u}}\|^p_{\mathbf{L}^p(T)}. \quad (5.28)$$

Step 2. We now control the term $h_T \|[\![(\nabla \boldsymbol{u}_{\mathscr{T}} - \mathbb{I}\pi_{\mathscr{T}}) \cdot \boldsymbol{n}]\!]\|_{\mathbf{L}^p(S)}$ for $S \in \mathscr{S}_T$ and $T \in \mathscr{T}$. To simplify the presentation of the material, we define $\mathbf{J}_S := [\![(\nabla \boldsymbol{u}_{\mathscr{T}} - \mathbb{I}\pi_{\mathscr{T}}) \cdot \boldsymbol{n}]\!]$ and $\boldsymbol{\Lambda}_S := \phi_S \mathbf{J}_S$, where ϕ_S is the bubble function defined in (5.15). As a first step, we invoke (5.18) to obtain

$$\|\mathbf{J}_S\|^2_{\mathbf{L}^2(S)} \lesssim \|\mathbf{J}_S \phi_S^{\frac{1}{2}}\|^2_{\mathbf{L}^2(S)} = \int_S \mathbf{J}_S \cdot \boldsymbol{\Lambda}_S. \quad (5.29)$$

Set $\boldsymbol{v} = \boldsymbol{\Lambda}_S$ in (5.12) and use that $\boldsymbol{\Lambda}_S(z) = \mathbf{0}$ to arrive at

$$\int_S \mathbf{J}_S \cdot \boldsymbol{\Lambda}_S = a(\boldsymbol{\Phi}, \boldsymbol{\Lambda}_S) - \sum_{T' \in \mathcal{N}_S} \int_{T'} \mathbf{R}_{T'} \cdot \boldsymbol{\Lambda}_S,$$

where $\mathbf{R}_{T'}$ is defined as in (5.21). Notice that $\operatorname{supp}(\boldsymbol{\Lambda}_S) \subseteq R_S := \operatorname{supp}(\phi_S) \subset \{T' : T' \in \mathcal{N}_S\}$. We thus invoke similar arguments to the ones that yield (5.24) to obtain

$$\int_S \mathbf{J}_S \cdot \boldsymbol{\Lambda}_S \lesssim \sum_{T' \subset \mathcal{N}_S} \|\nabla \boldsymbol{\Phi}\|_{\mathbf{L}^p(T')} \|\nabla \boldsymbol{\Lambda}_S\|_{\mathbf{L}^{p\prime}(T')} + \|\mathbf{R}_{T'}\|_{\mathbf{L}^p(T')} \|\boldsymbol{\Lambda}_S\|_{\mathbf{L}^{p\prime}(T')}$$

$$\lesssim \sum_{T' \subset \mathcal{N}_S} \Big[(\|\nabla \mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T')} + \|e_\pi\|_{L^p(T')}) \|\nabla \boldsymbol{\Lambda}_S\|_{\mathbf{L}^{p\prime}(T')} + \|\mathbf{R}_{T'}\|_{\mathbf{L}^p(T')} \|\boldsymbol{\Lambda}_S\|_{\mathbf{L}^{p\prime}(T')} \\ + \left(\|\boldsymbol{u}\|_{\mathbf{L}^{3p}(T')} \|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T')} + \|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T')} \|\boldsymbol{u}_{\mathscr{T}}\|_{\mathbf{L}^{3p}(T')} \right) \|\nabla \boldsymbol{\Lambda}_S\|_{\mathbf{L}^s(T')} \Big].$$

Inverse inequalities and (5.27) yield the estimate

$$\int_S \mathbf{J}_S \cdot \boldsymbol{\Lambda}_S \lesssim \sum_{T' \in \mathcal{N}_S} \Big[h_{T'}^{-1} (\|\nabla \mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T')} + \|e_\pi\|_{L^p(T')}) \|\boldsymbol{\Lambda}_S\|_{\mathbf{L}^{p'}(T')} \\ + \left(h_T^{-1-2/3p} \|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T')} + h_{T'}^{2/s-2/p'-1} \|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T')} \right) \|\boldsymbol{\Lambda}_S\|_{\mathbf{L}^{p'}(T')}. \Big] \quad (5.30)$$

Next, $\|\boldsymbol{\Lambda}_S\|_{\mathbf{L}^{p'}(T')} \lesssim h_{T'}^{1/p'}\|\mathbf{J}_S\|_{\mathbf{L}^{p'}(S)}$, which follows from (5.19), and $\|\mathbf{J}_S\|_{\mathbf{L}^{p'}(S)} \lesssim h_{T'}^{1/p'-1/2}\|\mathbf{J}_S\|_{\mathbf{L}^2(S)}$ yield, on the basis of (5.29) and (5.30), the estimate

$$\|\mathbf{J}_S\|_{\mathbf{L}^2(S)} \lesssim \sum_{T'\in\mathcal{N}_S}\Big[h_{T'}^{-1/p}h_{T'}^{1/p'-1/2}\left(\|\nabla\mathbf{e}_{\boldsymbol{u}}\|^p_{\mathbf{L}^p(T')} + \|e_\pi\|^p_{L^{p'}(T')}\right)^{1/p} \\ + h_{T'}^{-3/2-2/3p+2/p'}\|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T')} + h_{T'}^{2/s-3/2}\|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T')}\Big].$$

Finally, we invoke the inequality $\|\mathbf{J}_S\|_{\mathbf{L}^p(S)} \lesssim h_{T'}^{1/p-1/2}\|\mathbf{J}_S\|_{\mathbf{L}^2(S)}$ to obtain

$$h_T\|\mathbf{J}_S\|^p_{\mathbf{L}^p(S)} \lesssim \sum_{T'\in\mathcal{N}_S}\left[\|\nabla\mathbf{e}_{\boldsymbol{u}}\|^p_{\mathbf{L}^p(T')} + \|e_\pi\|^p_{L^{p'}(T')} + h_{T'}^{-2/3}\|\mathbf{e}_{\mathbf{u}}\|^p_{\mathbf{L}^p(T')}\right]. \quad (5.31)$$

Step 3. Let $T \in \mathscr{T}$. The estimate of the term $\|\operatorname{div}\boldsymbol{u}_{\mathscr{T}}\|_{L^p(T)}$ is as follows:

$$\|\operatorname{div}\boldsymbol{u}_{\mathscr{T}}\|_{L^p(T)} = \|\operatorname{div}\mathbf{e}_{\boldsymbol{u}}\|_{L^p(T)} \lesssim \|\nabla\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T)}. \quad (5.32)$$

Step 4. We now bound the term $h_T^{2-p}|\boldsymbol{f}|^p$ in (5.8). Let $T \in \mathscr{T}$. If $T\cap\{z\} = \emptyset$, the desired estimate (5.20) follows directly from (5.28), (5.31), and (5.32). If $T\cap\{z\} \neq \emptyset$, and (i) and (ii) hold, $\eta_{p,T}$ contains the term $h_T^{2-p}|\boldsymbol{f}|^p$. To control such a term, we invoke the smooth function μ constructed in [5, Section 3], which is such that

$$\mathcal{S}_\mu := \operatorname{supp}(\mu) \subset \mathcal{N}_T, \quad \mu(z) = 1, \quad \|\mu\|_{L^\infty(\mathcal{S}_\mu)} = 1, \quad \|\nabla\mu\|_{L^\infty(\mathcal{S}_\mu)} \lesssim h_T^{-1}.$$

In addition, the function μ satisfies the following estimates:

$$\|\mu\|_{L^{p'}(\mathcal{N}_T)} \lesssim h_T^{2/p'}, \quad \|\nabla\mu\|_{L^{p'}(\mathcal{N}_T)} \lesssim h_T^{2/p'-1}, \quad \|\mu\|_{L^{p'}(S)} \lesssim h_T^{1/p'}. \quad (5.33)$$

Set $\boldsymbol{v} = \boldsymbol{v}_\mu := \mu|\boldsymbol{f}|^{p-1}\operatorname{sign}(\boldsymbol{f})$, where $\operatorname{sign}(\boldsymbol{f})$ must be understood as the componentwise sign function of $\boldsymbol{f}$, as a test function in (3.5). Since $(\boldsymbol{u},\pi)$ and $(\boldsymbol{\Phi},\psi)$ solve problem (3.5) and (5.1) respectively, we obtain

$$\begin{aligned}|\boldsymbol{f}|^p &= a(\boldsymbol{u},\boldsymbol{v}_\mu) + b_-(\boldsymbol{v}_\mu,\pi) + c(\boldsymbol{u},\boldsymbol{u};\boldsymbol{v}_\mu)\\ &= a(\boldsymbol{\Phi},\boldsymbol{v}_\mu) + a(\boldsymbol{u}_{\mathscr{T}},\boldsymbol{v}_\mu) + b_-(\boldsymbol{v}_\mu,\pi_{\mathscr{T}}) + c(\boldsymbol{u}_{\mathscr{T}},\boldsymbol{u}_{\mathscr{T}};\boldsymbol{v}_\mu).\end{aligned}$$

Since $\operatorname{supp}(\mu) \subset \mathcal{N}_T$, similar arguments to the ones used in the previous steps allow us to derive the estimate

$$|\boldsymbol{f}|^p \lesssim (\|\nabla\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\mathcal{N}_T)} + \|e_\pi\|_{L^p(\mathcal{N}_T)})\|\nabla\boldsymbol{v}_\mu\|_{\mathbf{L}^{p'}(\mathcal{N}_T)} + \|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\mathcal{N}_T)}\|\nabla\boldsymbol{v}_\mu\|_{\mathbf{L}^s(\mathcal{N}_T)} \\ + \sum_{\substack{T'\in\mathscr{T}\\ T'\subset\mathcal{N}_T}}\left[\|\mathbf{R}_{T'}\|_{\mathbf{L}^p(T')}\|\boldsymbol{v}_\mu\|_{\mathbf{L}^{p'}(T')} + \sum_{\substack{S\in\mathscr{S}\\ S\subset\partial\mathcal{N}_T}}\|\mathbf{J}_S\|_{\mathbf{L}^p(S)}\|\boldsymbol{v}_\mu\|_{\mathbf{L}^{p'}(S)}\right]. \quad (5.34)$$

We then apply the properties of μ stated in (5.33) to obtain

$$|\boldsymbol{f}|^p \lesssim \Big[{h_T}^{2/p'-1}(\|\nabla \mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\mathcal{N}_T)} + \|e_\pi\|_{L^p(\mathcal{N}_T)})) + {h_T}^{2/s-1}\|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(\mathcal{N}_T)} \Big] |\boldsymbol{f}|^{p-1} + \sum_{\substack{T'\in\mathscr{T}\\ T'\subset\mathcal{N}_T}} \left[h_{T'}^{2/p'}\|\mathbf{R}_{T'}\|_{\mathbf{L}^p(T')} + \sum_{\substack{S\in\mathscr{S}\\ S\subset\partial\mathcal{N}_T}} h_{T'}^{1/p'}\|\mathbf{J}_S\|_{\mathbf{L}^p(S)} \right] |\boldsymbol{f}|^{p-1}. \quad (5.35)$$

Invoke (5.27) and (5.31) to conclude

$$h_T^{1-2/p'}|\boldsymbol{f}| \lesssim \sum_{T'\in\mathcal{N}_T} \Big(\|\nabla \mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T')} + \|e_\pi\|_{L^p(T')} + h_{T'}^{2/s-2/p'}\|\mathbf{e}_{\boldsymbol{u}}\|_{\mathbf{L}^p(T')} \Big),$$

which immediately implies the desired bound

$$h_T^{2-p}|\boldsymbol{f}|^p \lesssim \sum_{T'\in\mathcal{N}_T} \Big(\|\nabla \mathbf{e}_{\boldsymbol{u}}\|^p_{\mathbf{L}^p(T')} + \|e_\pi\|^p_{L^p(T')} + h_{T'}^{-2/3}\|\mathbf{e}_{\boldsymbol{u}}\|^p_{\mathbf{L}^p(T')} \Big). \quad (5.36)$$

Step 5. Finally, by gathering estimates (5.28), (5.31), (5.32) and (5.36), we arrive at the desired estimate (5.20). This concludes the proof.

6 Numerical Experiments

In this section we conduct a series of numerical examples that illustrate the performance of the devised a posteriori error estimator (5.10). The numerical examples have been carried out with the help of a code that we implemented using `C++`. All matrices have been assembled exactly. The right–hand sides of the assembled systems, the local indicators, and the approximation errors, are computed by a quadrature formula which is exact for polynomials of degree 19. For simplicity, in all the experiments that we have performed, we have taken the kinematic viscosity to be equal to one.

For a given partition $\mathscr{T}$, we seek $(\boldsymbol{u}_{\mathscr{T}}, \pi_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ that solves (4.7) on the basis of the discrete spaces (4.4)–(4.5). We thus calculate the local error indicators $\eta_{p,T}$, defined in (5.8)–(5.9), in order to drive the adaptive mesh refinement procedure described in **Algorithm** 1. A sequence of adaptively refined meshes is thus generated from the initial meshes shown in Figure 1. The total number of degrees of freedom is $\mathsf{Ndof} := \dim(\mathbf{V}(\mathscr{T})) + \dim(\mathcal{P}(\mathscr{T}))$, where $(\mathbf{V}(\mathscr{T}), \mathcal{P}(\mathscr{T}))$ is given by (4.4)–(4.5).

In the experiments that we perform we go beyond the presented theory and include a series of Dirac delta sources on the right–hand side of the momentum equation. To be precise, we replace the forcing term in the first equation of (3.5) by $\sum_{t\in\mathcal{D}} \boldsymbol{f}_t\delta_t$. Here, $\mathcal{D}$ denotes a finite ordered subset of Ω with cardinality $\#\mathcal{D}$ and $\{\boldsymbol{f}_t\}_{t\in\mathcal{D}} \subset \mathbb{R}^2$. Within this setting, the following a posteriori error estimator can be proposed:

$$\zeta_p := \left(\sum_{T\in\mathscr{T}} \zeta_{p,T}^p \right)^{\frac{1}{p}}.$$

Algorithm 1: Adaptive algorithm.

Input: Initial mesh $\mathscr{T}_0$, set of Dirac points $\mathcal{D}$, vectors $\{\boldsymbol{f}_t\}_{t\in\mathcal{D}}$, and the index p.
Set: $i = 0$.
1: Solve the discrete system (4.7) by using a fixed point algorithm.
2: For each $T \in \mathscr{T}_i$ compute the local error indicator $\eta_{p,T}$ defined in (5.8)–(5.9);
3: Mark an element T for refinement if $\eta_{p,T}^p > \frac{1}{2}\max_{T'\in\mathscr{T}_i}\eta_{p,T'}^p$;
4: From step **3**, construct a new mesh $\mathscr{T}_{i+1}$ using a longest edge bisection algorithm. Set $i \leftarrow i+1$ and go to step **1**.

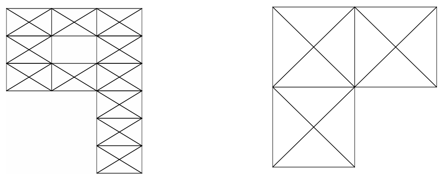

Fig. 1 The initial meshes used in the adaptive loop of **Algorithm** 1 when Ω is a two dimensional q–shaped domain (Example 1) and a two dimensional L–shaped domain (Example 2).

For each $T \in \mathscr{T}$, $\zeta_{p,T}$ is defined as follows: If $t \in \mathcal{D}\cap T$ and (i) or (ii) hold, then

$$\zeta_{p,T} := \Big(h_T^p\|\Delta\boldsymbol{u}_{\mathscr{T}} - (\boldsymbol{u}_{\mathscr{T}}\cdot\nabla)\boldsymbol{u}_{\mathscr{T}} - \operatorname{div}\boldsymbol{u}_{\mathscr{T}}\boldsymbol{u}_{\mathscr{T}} - \nabla\pi_{\mathscr{T}}\|_{\mathbf{L}^p(T)}^p + h_T\|[\![(\nabla\boldsymbol{u}_{\mathscr{T}} - \mathbb{I}\pi_{\mathscr{T}})\cdot\boldsymbol{n}]\!]\|_{\mathbf{L}^p(\partial T\setminus\partial\Omega)}^p + \|\mathrm{div}\boldsymbol{u}_{\mathscr{T}}\|_{L^p(T)}^p + \sum_{t\in\mathcal{D}\cap T} h_T^{2-p}|\boldsymbol{f}_t|^p\Big)^{\frac{1}{p}}. \quad (6.1)$$

If $t \in \mathcal{D}\cap T$ and (i) or (ii) do not hold, then

$$\zeta_{p,T} := \Big(h_T^p\|\Delta\boldsymbol{u}_{\mathscr{T}} - (\boldsymbol{u}_{\mathscr{T}}\cdot\nabla)\boldsymbol{u}_{\mathscr{T}} - \operatorname{div}\boldsymbol{u}_{\mathscr{T}}\boldsymbol{u}_{\mathscr{T}} - \nabla\pi_{\mathscr{T}}\|_{\mathbf{L}^p(T)}^p + h_T\|[\![(\nabla\boldsymbol{u}_{\mathscr{T}} - \mathbb{I}\pi_{\mathscr{T}})\cdot\boldsymbol{n}]\!]\|_{\mathbf{L}^p(\partial T\setminus\partial\Omega)}^p + \|\mathrm{div}\boldsymbol{u}_{\mathscr{T}}\|_{L^p(T)}^p\Big)^{\frac{1}{p}}. \quad (6.2)$$

If $T\cap\mathcal{D} = \emptyset$, then the indicator is defined as in (6.2). Notice that, when $\#\mathcal{D} = 1$, the total error estimator ζ_p coincides with η_p, the estimator defined in (5.10). Depending on the test, we may use (6.1)–(6.2) instead of (5.8)–(5.9) in **Algorithm** 1 in order to account for the forcing term $\sum_{t\in\mathcal{D}}\boldsymbol{f}_t\delta_t$.

We perform two–dimensional examples on polygonal but nonconvex domains with different number of source points. To accomplish this task we use the adaptive procedure described in **Algorithm** 1. To solve the discrete problem (4.7) we employ the Taylor–Hood finite element pair given as in (4.4)–(4.5).

6.1 Example 1: q-shaped domain

The first test that we report is posed on a L-shaped domain with a rectangular obstacle. We will simply refer to such a domain as a q-shaped domain. To make matters precise, we let $\Omega = (1, 1.5) \times (0, 1.5] \cup ((0, 1.5) \times (1.5, 3) \setminus [0.5, 1] \times [2, 2.5])$,

$$\mathcal{D} = \{(1.25, 2.25), (0.75, 2.75), (0.25, 2.25), (0.75, 1.75), (1.25, 0.75)\},$$

$\boldsymbol{f}_{(1.25,2.25)} = 0.02, \boldsymbol{f}_{(0.25,2.25)} = 0.02, \boldsymbol{f}_{(0.75,2.75)} = -0.03, \boldsymbol{f}_{(0.75,1.75)} = -0.03$ and $\boldsymbol{f}_{(1.25,0.75)} = 0.04$. The purpose of this example is to investigate the performance of the error estimator ζ_p for different values of the integrability index p. In particular, we consider $p \in \{1.1, 1.3, 1.5, 1.7\}$. Notice that the exact solution to this problems is unknown.

In Figure 2 we present the results obtained for Example 1. We observe, from subfigure (A), optimal experimental rates of convergence for the error estimator ζ_p for all the values of the integrability index p considered. In subfigures (B) and (C) we present the finite element approximations of $|\boldsymbol{u}_{\mathscr{T}}|$ and $\pi_{\mathscr{T}}$ obtained after 20 adaptive refinements of the corresponding initial mesh shown in Figure 1; the 20th adaptive mesh has 24238 elements and 12315 vertices. In subfigures (D), (E), and (F) we show the adaptive meshes obtained after 28 iterations of our adaptive loop for $p = 1.3$, $p = 1.5$, and $p = 1.7$, respectively. It can be appreciated that most of adaptive refinement is being concentrated around the points $t \in \mathcal{D}$ where the Dirac measures are supported. Adaptive refinement is also being performed at the re-entrant corners of the domain, specially for small values of p. We also show, in subfigures (G), (H), and (I), the streamlines of the velocity field $\boldsymbol{u}_{\mathscr{T}}$ obtained after 22 iterations of our adaptive loop with $p = 1.3$ (11922 elements and 6091 vertices), $p = 1.5$ (3562 elements and 1831 vertices), and $p = 1.7$ (1697 elements and 870 vertices), respectively.

6.2 Example 2 (L-shaped domain)

We let $\Omega = (0, 1)^2 \setminus [0.5, 1) \times (0, 0.5]$, $\mathcal{D} = \{(0.25, 0.25), (0.25, 0.75), (0.75, 0.75)\}$, and $\boldsymbol{f}_{(0.25,0.25)} = \boldsymbol{f}_{(0.25,0.75)} = \boldsymbol{f}_{(0.75,0.75)} = (1, 1)$. In this example we investigate, once more, the effect of varying the integrability index p by considering $p \in \{1.1, 1.3, 1.5, 1.7\}$.

In Figure 3 we present the results obtained for Example 2. From subfigure (A) we observe that, for the different values of p that we consider, optimal experimental rates of convergence are attained for ζ_p. Subfigures (B) and (C) show the finite element approximations of the magnitude of the velocity and the pressure, respectively. In subfigures (D), (E), and (F) we present the adaptive meshes obtained after 20 iterations of our adaptive loop for $p = 1.3$, $p = 1.5$, and $p = 1.7$, respectively. It can be observed that the refinement is being concentrated around the points that support the Dirac measures and to a lesser extent about the re-entrant corner.

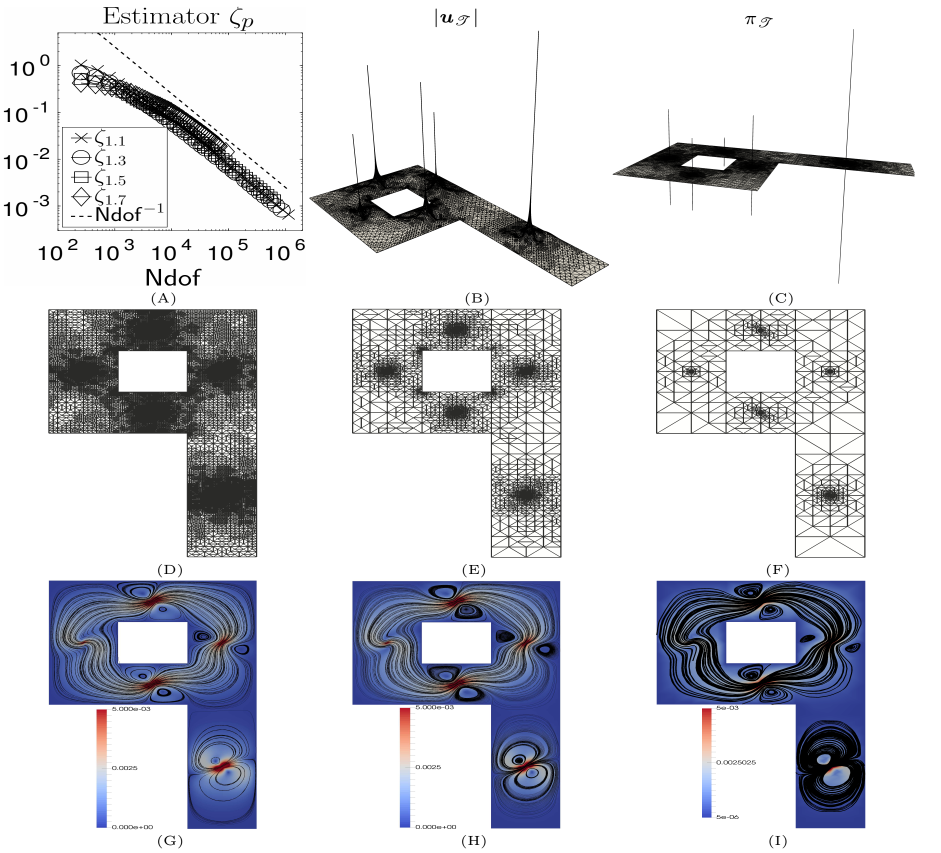

Fig. 2 Example 1: Experimental rates of convergence for the error estimator ζ_p for $p \in \{1.1, 1.3, 1.5, 1.7\}$ (A); the finite element approximations of $|\boldsymbol{u}_{\mathscr{T}}|$ (B) and $\pi_{\mathscr{T}}$ (C) obtained on the 20th adaptively refined mesh for $p = 1.1$; adaptively refined meshes obtained after 28 iterations of our adaptive loop with $p = 1.3$ (D), $p = 1.5$, (E) and $p = 1.7$ (F); and streamlines of the velocity field $\boldsymbol{u}_{\mathscr{T}}$ obtained after 22 adaptive refinements for $p = 1.3$ (G), $p = 1.5$ (H), and $p = 1.7$ (I).

References

1. Acosta, G., Durán, R.G.: Divergence operator and related inequalities. SpringerBriefs in Mathematics. Springer, New York (2017). DOI 10.1007/978-1-4939-6985-2. URL `https://doi.org/10.1007/978-1-4939-6985-2`
2. Adams, R.A., Fournier, J.J.F.: Sobolev spaces, *Pure and Applied Mathematics (Amsterdam)*, vol. 140, second edn. Elsevier/Academic Press, Amsterdam (2003)
3. Ainsworth, M., Oden, J.T.: A posteriori error estimators for the Stokes and Oseen equations. SIAM J. Numer. Anal. **34**(1), 228–245 (1997). DOI 10.1137/S0036142994264092. URL `https://doi.org/10.1137/S0036142994264092`
4. Allendes, A., Otárola, E., Salgado, A.J.: A posteriori error estimates for the stationary Navier-Stokes equations with Dirac measures. SIAM J. Sci. Comput. **42**(3), A1860–A1884 (2020). DOI 10.1137/19M1292436. URL `https://doi.org/10.1137/19M1292436`
5. Araya, R., Behrens, E., Rodríguez, R.: A posteriori error estimates for elliptic problems with Dirac delta source terms. Numer. Math. **105**(2), 193–216 (2006). DOI 10.1007/s00211-006-0041-2. URL `https://doi.org/10.1007/s00211-006-0041-2`

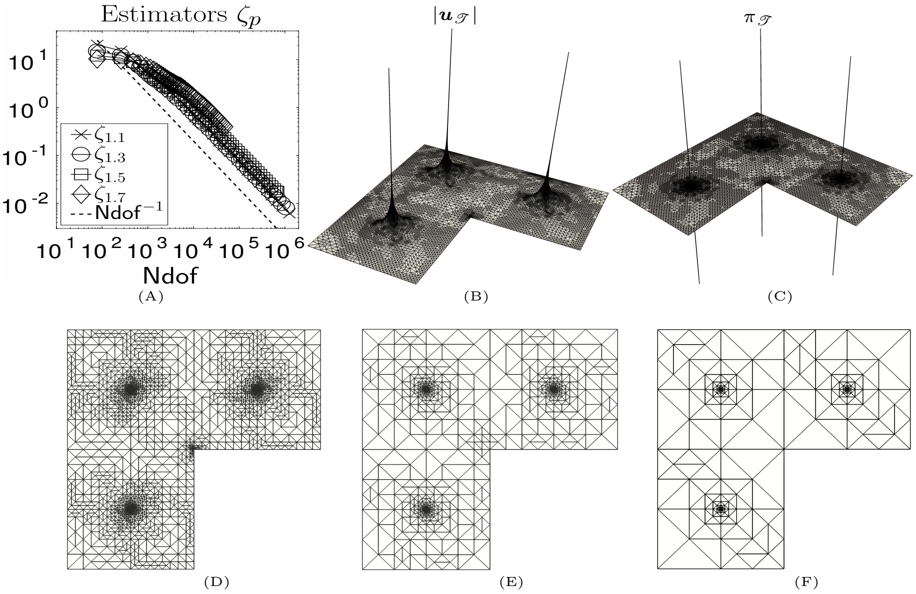

Fig. 3 Example 2: Experimental rates of convergence for the error estimator ζ_p for $p \in \{1.1, 1.3, 1.5, 1.7\}$ (A); the finite element approximations of $|\boldsymbol{u}_{\mathscr{T}}|$ (B) and $\pi_{\mathscr{T}}$ (C) obtained on the 20th adaptively refined mesh for $p = 1.1$ and adaptively refined meshes obtained after 20 iterations of our adaptive loop for $p = 1.3$ (D), $p = 1.5$ (E), and $p = 1.7$ (F).

6. Bernardi, C., Canuto, C., Maday, Y.: Generalized inf-sup conditions for Chebyshev spectral approximation of the Stokes problem. SIAM J. Numer. Anal. **25**(6), 1237–1271 (1988). DOI 10.1137/0725070. URL `https://doi.org/10.1137/0725070`
7. Casas, E., Kunisch, K.: Optimal control of the two-dimensional stationary Navier-Stokes equations with measure valued controls. SIAM J. Control Optim. **57**(2), 1328–1354 (2019). DOI 10.1137/18M1185582. URL `https://doi.org/10.1137/18M1185582`
8. Ern, A., Guermond, J.L.: Theory and practice of finite elements, *Applied Mathematical Sciences*, vol. 159. Springer-Verlag, New York (2004). DOI 10.1007/978-1-4757-4355-5. URL `https://doi.org/10.1007/978-1-4757-4355-5`
9. Fuica, F., Lepe, F., Otárola, E., Quero, D.: A posteriori error estimates in $\mathbf{W}^{1,p} \times L^p$ spaces for the Stokes system with Dirac measures. arXiv:1912.08325 (2019)
10. Fuica, F., Otárola, E., Quero, D.: Error estimates for optimal control problems involving the Stokes system and Dirac measures. Appl. Math. Optim. (2020). URL `https://doi.org/10.1007/s00245-020-09693-0`
11. Galdi, G.P.: An introduction to the mathematical theory of the Navier-Stokes equations, second edn. Springer Monographs in Mathematics. Springer, New York (2011). DOI 10.1007/978-0-387-09620-9. URL `https://doi-org.usm.idm.oclc.org/10.1007/978-0-387-09620-9`. Steady-state problems
12. Girault, V., Nochetto, R.H., Scott, L.R.: Max-norm estimates for Stokes and Navier-Stokes approximations in convex polyhedra. Numer. Math. **131**(4), 771–822 (2015). DOI 10.1007/s00211-015-0707-8. URL `https://doi.org/10.1007/s00211-015-0707-8`
13. Girault, V., Nochetto, R.H., Scott, R.: Maximum-norm stability of the finite element Stokes projection. J. Math. Pures Appl. (9) **84**(3), 279–330 (2005). DOI 10.1016/j.matpur.2004.09.017. URL `https://doi.org/10.1016/j.matpur.2004.09.017`
14. Girault, V., Raviart, P.A.: Finite element methods for Navier-Stokes equations, *Springer Series in Computational Mathematics*, vol. 5. Springer-Verlag, Berlin (1986). DOI

10.1007/978-3-642-61623-5. URL http://dx.doi.org/10.1007/978-3-642-61623-5. Theory and algorithms
15. Lacouture, L.: A numerical method to solve the Stokes problem with a punctual force in source term. Comptes Rendus Mécanique **343**(3), 187–191 (2015). DOI 10.1016/j.crme.2014.09.008. URL http://www.sciencedirect.com/science/article/pii/S1631072114001818
16. Lions, P.L.: Mathematical topics in fluid mechanics. Vol. 1, *Oxford Lecture Series in Mathematics and its Applications*, vol. 3. The Clarendon Press, Oxford University Press, New York (1996). Incompressible models, Oxford Science Publications
17. Mitrea, M., Wright, M.: Boundary value problems for the Stokes system in arbitrary Lipschitz domains. Astérisque (344), viii+241 (2012)
18. Otárola, E., Salgado, A.J.: A weighted setting for the stationary Navier Stokes equations under singular forcing. Appl. Math. Lett. **99**, 105933, 7 (2020). DOI 10.1016/j.aml.2019.06.004. URL https://doi.org/10.1016/j.aml.2019.06.004
19. Tartar, L.: An introduction to Navier-Stokes equation and oceanography, *Lecture Notes of the Unione Matematica Italiana*, vol. 1. Springer-Verlag, Berlin; UMI, Bologna (2006). DOI 10.1007/3-540-36545-1. URL https://doi.org/10.1007/3-540-36545-1
20. Temam, R.: Navier-Stokes equations. AMS Chelsea Publishing, Providence, RI (2001). DOI 10.1090/chel/343. URL https://doi.org/10.1090/chel/343. Theory and numerical analysis, Reprint of the 1984 edition
21. Tsai, T.P.: Lectures on Navier-Stokes equations, *Graduate Studies in Mathematics*, vol. 192. American Mathematical Society, Providence, RI (2018)
22. Verfürth, R.: A posteriori error estimators for the Stokes equations. Numer. Math. **55**(3), 309–325 (1989). DOI 10.1007/BF01390056. URL https://doi.org/10.1007/BF01390056
23. Verfürth, R.: A posteriori error estimators for convection-diffusion equations. Numer. Math. **80**(4), 641–663 (1998). DOI 10.1007/s002110050381. URL https://doi.org/10.1007/s002110050381
24. Verfürth, R.: A posteriori error estimation techniques for finite element methods. Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford (2013). DOI 10.1093/acprof:oso/9780199679423.001.0001. URL https://doi.org/10.1093/acprof:oso/9780199679423.001.0001