AN OPTIMAL CONTROL PROBLEM FOR THE STATIONARY NAVIER–STOKES EQUATIONS WITH POINT SOURCES*

FRANCISCO FUICA[†], FELIPE LEPE[‡], ENRIQUE OTÁROLA[§], AND DANIEL QUERO[¶]

Abstract. The aim of this work is to analyze a two dimensional optimal control problem for the Navier–Stokes equations where the control variable corresponds to the amplitude of forces modeled as point sources; control constraints are also considered. This particular setting leads to solutions to the state equation exhibiting reduced regularity properties. We operate within the theory of Muckenhoupt weights, Muckenhoupt–weighted Sobolev spaces, and the corresponding weighted norm inequalities and derive the existence of optimal solutions and first and, necessary and sufficient, second order optimality conditions.

Key words. optimal control problems, Navier–Stokes equations, Dirac measures, Muckenhoupt weights, first and second order optimality conditions.

AMS subject classifications. 35Q30, 35Q35, 49J20, 49K20.

1. Introduction.

In this work we are interested in the analysis of an optimal control problem that involves the stationary Navier–Stokes equations and Dirac measures. The control variable corresponds to the amplitude of forces modeled as point sources supported at some prescribed points of the underlying spatial domain; control constraints are also considered. The singular forcing appears in the right hand side of the momentum equation. Since Dirac measures are supported at points and points have Lebesgue measure zero, the aforementioned optimization setting can be seen as an instance of *sparse* PDE-constrained optimization.

The analysis of PDE-constrained optimization problems that induce a sparse structure in the involved optimal control variables have been widely studied in the literature over the last decade. This analysis has been mainly motivated by applications due to the fact that sparsity is a desirable feature in practice; for instance, in the optimal placement of discrete actuators [36]. To the best of our knowledge, the first work that provides an analysis for a particular setting inducing sparsity is [36]; the sparsity arises from the consideration of a quadratic cost functional that includes an $L^1(\Omega)$-control cost term (control constraints are also considered). The state equation is governed by a linear second order elliptic differential operator. Since the resulting problem is strictly convex, first order optimality conditions, from which the sparsity of the optimal control can be deduced, are necessary and sufficient for global optimality. Rates of convergence with respect to a suitable regularization parameter were derived in [39]. Later, the authors of [9] extend the theory to semilinear elliptic state equations and a more general cost functional. Since the involved PDE is not linear, the control problem is, in general, not convex; therefore, second order optimality conditions are

*FF is supported by UTFSM through Beca de Mantención. FL is partially supported by DIUBB through project 2120173 GI/C Universidad del Bío–Bío and ANID through FONDECYT grant 11200529. EO is partially supported by ANID through FONDECYT grant 11180193. DQ is supported by ANID through Beca de Doctorado Nacional 21210988.

[†]Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile. (francisco.fuica@sansano.usm.cl).

[‡]GIMNAP-Departamento de Matemática, Universidad del Bío Bío, Concepción, Chile. (flepe@ubiobio.cl).

[§]Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile. (enrique.otarola@usm.cl, http://eotarola.mat.utfsm.cl/).

[¶]Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile. (daniel.quero@alumnos.usm.cl).

needed in order to deal with local minima. At a continuous level, the authors of [9] derive first order optimality conditions, suitable representation/projection formulas, regularity results, and, necessary and sufficient second order optimality conditions. For an up-to-date overview of the theory, including the analysis of suitable finite element discretization and extensions to some optimal control problems governed by semilinear parabolic equations, we refer the interested reader to [8].

Another instance that induces sparsity in PDE-constrained optimization is the search of optimal controls within the space of regular Borel measures. To the best of our knowledge, the first work that provides an analysis within this setting is [13]; the state equation is governed by a linear second order elliptic differential operator. In this work, the authors consider control problems with measures and functions of bounded variation as controls and analyze the existence and uniqueness of solutions to the corresponding predual problems; the solution of the corresponding optimality systems by a semismooth Newton method is also investigated. Subsequently, the authors of [14] address the feasibility of optimal source placement by optimal control in measure spaces and extend the results of [13] by including partial observation, control on subdomains, and non-negativity of control variables. Later, the authors of [10] extend the theory by considering a semilinear elliptic equation as state equation. In particular, necessary and sufficient second order optimality conditions are derived. We refer the reader to [8] for an up-to-date discussion that also includes the analysis of suitable finite element discretizations and extensions for parabolic equations.

In this work, we are particularly interested in the case where the control variable corresponds to the amplitude of singular forces modeled as point sources supported at some prescribed points. This particular setting is of relevance in applications where one can specify the position of actuators at finitely many described prespecified points. References [7] and [26] discuss applications within the context of the active control of sound and vibrations, respectively. An analysis of the corresponding PDE-constrained optimization problem for when the state equation is a Poisson problem can be found in [23, 6, 4]. These references also analyze some suitable finite element discretizations. Extensions of the theory to the Stokes and semilinear elliptic equations have been recently investigated in [20] and [31], respectively.

To the best of our knowledge, the only work available in the literature that considers an optimal control problem for the stationay Navier–Stokes equations with a control that is measure valued is [11]. The two dimensional analysis developed in [11] assumes that the underlying domain is of class C^2 and seeks solutions to the Navier–Stokes equations in $\mathbf{W}_0^{1,q}(\Omega) \times L^q(\Omega)/\mathbb{R}$ with $q \in [4/3, 2)$. In this setting a complete existence theory for the state equation is provided and the corresponding optimal control problem is analyzed. In particular, first and, necessary and sufficient, second order optimality conditions are derived.

In the present paper, we analyze an optimal control problem for the stationary Navier–Stokes equations with a control variable that corresponds to the amplitude of prescribed point sources. Since solutions to the Navier–Stokes equations under singular forcing exhibit reduced regularity properties, we follow [33, 32] and operate under the theory of Muckenhoupt weights, Muckenhoupt–weighted Sobolev spaces, and the corresponding weighted norm inequalities. We work under a suitable *smallness assumption on data* and under the assumption that $\Omega \subset \mathbb{R}^2$ is *merely Lipschitz* and provide a complete analysis for the optimal control problem that includes existence of optimal solutions (Theorem 7), first order optimality conditions (Theorem 12), and necessary and sufficient second order optimality conditions (Theorems 15 and 16). As instrumental results, we analyze the differentiability properties of the so-called control to state map and regularity estimates for the solution to the corresponding adjoint equation. In addition to the classical difficulties that are associated to the appearance of the classical nonlinear convective term, we have to deal with the fact that solutions to the state and adjoint equations lie in different function spaces. The analysis that we provide thus requires fine properties of Muckenhoupt weights and embeddings between weighted and non-weighted spaces. This subtle intertwining of ideas is one of the highlights of our contribution.

Our presentation is organized as follows. In section 2 we introduce the PDE-constrained optimization problem that is under consideration. We collect background information and the main assumptions under which we shall operate in section 3, where we also analyze differentiability properties of a suitable solution operator. In section 4, we introduce a weak formulation for our optimal control problem and prove the existence of solutions. Section 5 is dedicated to the analysis of optimality conditions. As a first step, in section 5.1, we analyze the so-called adjoint equations and derive regularity properties of solutions to subsequently derive first order conditions in section 5.2 and necessary and sufficient second order conditions in section 5.3.

2. Statement of the problem. To describe our optimal control problem, we let $\Omega \subset \mathbb{R}^2$ be an open and bounded polygonal domain with Lipschitz boundary $\partial\Omega$ and let $\mathcal{D} \subset \Omega$ be a nonempty and finite ordered set with cardinality $\#\mathcal{D} = \ell$. Given a desired velocity $\mathbf{y}_\Omega \in \mathbf{L}^2(\Omega)$ and a regularization parameter $\eta > 0$, we introduce the cost functional

$$(1) \qquad J(\mathbf{y},\mathcal{U}) := \frac{1}{2}\|\mathbf{y}-\mathbf{y}_\Omega\|^2_{\mathbf{L}^2(\Omega)} + \frac{\eta}{2}\sum_{t\in\mathcal{D}}|\mathbf{u}_t|^2, \quad \mathcal{U} = (\mathbf{u}_1,\ldots,\mathbf{u}_\ell), \quad \mathbf{u}_t \in \mathbb{R}^2.$$

Within this setting, we are interested in the following PDE-constrained optimization problem: Find $\min J(\mathbf{y},\mathcal{U})$ subject to

$$(2) \quad -\nu\Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \sum_{t\in\mathcal{D}} \mathbf{u}_t\delta_t \text{ in } \Omega, \qquad \text{div } \mathbf{y} = 0 \text{ in } \Omega, \qquad \mathbf{y} = \mathbf{0} \text{ on } \partial\Omega,$$

and the control constraints

$$(3) \qquad \mathcal{U} \in \mathbb{U}_{ad}, \qquad \mathbb{U}_{ad} := \{\mathcal{V} = (\mathbf{v}_1,\ldots,\mathbf{v}_\ell) \in [\mathbb{R}^2]^\ell : \mathbf{a}_t \leq \mathbf{v}_t \leq \mathbf{b}_t \text{ for all } t \in \mathcal{D}\},$$

with $\mathbf{a}_t, \mathbf{b}_t \in \mathbb{R}^2$ satisfying $\mathbf{a}_t < \mathbf{b}_t$ for every $t \in \mathcal{D}$. We immediately comment that, throughout this work, vector inequalities must be understood componentwise and that $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$. In problem (2), $\mathbf{y}$ represents the velocity of the fluid, p represents the pressure, and $\nu > 0$ denotes the kinematic viscosity; δ_t corresponds to the Dirac delta supported at the interior point $t \in \mathcal{D}$.

3. Notation and preliminaries. Let us set notation and describe the setting we shall operate with.

3.1. Notation. If $\mathfrak{X}$ and $\mathfrak{Y}$ are normed vector spaces, we write $\mathfrak{X} \hookrightarrow \mathfrak{Y}$ to denote that $\mathfrak{X}$ is continuously embedded in $\mathfrak{Y}$. We denote by $\mathfrak{X}'$ and $\|\cdot\|_{\mathfrak{X}}$ the dual and the norm of $\mathfrak{X}$, respectively. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{X}$. We denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong and weak convergence, respectively, of $\{x_n\}_{n\in\mathbb{N}}$ to x in $\mathfrak{X}$.

For $E \subset \mathbb{R}^2$ of finite Hausdorff i-dimension, $i \in \{1,2\}$, and $f : E \to \mathbb{R}$, we set

$$\fint_E f = \frac{1}{|E|}\int_E f, \qquad |E| = \int_E .$$

We shall use standard notation for Lebesgue and Sobolev spaces. To denote vector-valued functions we shall use lowercase bold letters, whereas to denote function spaces we shall use uppercase bold letters. For instance, for a bounded domain $G \subset \mathbb{R}^d$, with $d \in \{1, 2\}$, we denote $\mathbf{L}^2(G) = [L^2(G)]^2$ and equip $\mathbf{L}^2(G)$ with the following inner product and norm, respectively,

$$(\mathbf{w}, \mathbf{v})_{\mathbf{L}^2(G)} = \int_G \mathbf{w} \cdot \mathbf{v}, \qquad \|\mathbf{v}\|_{\mathbf{L}^2(G)} = (\mathbf{v}, \mathbf{v})^{\frac{1}{2}}_{\mathbf{L}^2(G)} \qquad \forall\, \mathbf{w}, \mathbf{v} \in \mathbf{L}^2(G).$$

Given $q \in (1, \infty)$, we denote by q' its Hölder conjugate, i.e., the real number such that $1/q + 1/q' = 1$. The relation $a \lesssim b$ indicates that $a \leq Cb$, with a positive constant that depends neither on a, b. The value of C might change at each occurrence. If the particular value of a constant is of relevance, then we will assign it a name.

3.2. Muckenhoupt weights and weighted Sobolev spaces. A notion which will be fundamental for further discussions is that of a weight. A weight is an almost everywhere positive function defined on $\mathbb{R}^2$ that is locally integrable. A particular class of weights that will be of importance in our analysis is the so-called Muckenhoupt class A_2 [15, 17, 30, 38]: We say that a weight ω belongs to the class A_2 if

$$[\omega]_{A_2} := \sup_B \left(\fint_B \omega \right) \left(\fint_B \omega^{-1} \right) < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^2$. We call $[\omega]_{A_2}$ the Muckenhoupt characteristic of ω.

Distances to lower dimensional objects are prototypical examples of Muckenhoupt weights. In particular, if $\mathcal{K}$ denotes a smooth compact submanifold of dimension $k \in \{0, 1\}$, then owing to [3] and [19, Lemma 2.3(vi)], we have that $\mathrm{d}^\alpha_{\mathcal{K}}(x) := \mathrm{dist}(x, \mathcal{K})^\alpha \in A_2$ provided $\alpha \in (-(2-k), (2-k))$. This allows us to identify two particular cases:

1. Let $z \in \Omega$. Then, the weight $\mathrm{d}^\alpha_z \in A_2$ if $\alpha \in (-2, 2)$.
2. Let $\gamma \subset \Omega$ be a smooth closed curve without self-intersections. Then, the weight $\mathrm{d}^\alpha_\gamma \in A_2$ if $\alpha \in (-1, 1)$.

Since the aforementioned examples of Muckenhoupt weights are based on lower dimensional objects that are strictly contained in Ω, there is a neighborhood of $\partial\Omega$ where the weight has no degeneracies or singularities; the weight is continuous and strictly positive. This observation motivates the following restricted class of Muckenhoupt weights [19, Definition 2.5].

Definition 1 (Class $A_2(G)$). *Let $G \subset \mathbb{R}^2$ be a Lipschitz domain. We say that $\omega \in A_2$ belongs to $A_2(G)$ if there is an open set $\mathcal{G} \subset G$ and $\varepsilon, \omega_l > 0$ such that $\{x \in G : \mathrm{dist}(x, \partial G) < \varepsilon\} \subset \mathcal{G}$, $\omega \in C(\bar{\mathcal{G}})$, and $\omega(x) \geq \omega_l$ for all $x \in \bar{\mathcal{G}}$.*

Let $G \subset \mathbb{R}^2$ be an open set and $\omega \in A_2$. We define the weighted space $L^2(\omega, G)$ as the space of Lebesgue square–integrable functions with respect to the measure $\omega \mathrm{d}x$. The weighted Sobolev space $H^1(\omega, G)$ is defined as the set of functions $v \in L^2(\omega, G)$ with weak derivatives $D^\alpha v \in L^2(\omega, G)$ for $|\alpha| \leq 1$. We equip $H^1(\omega, G)$ with the norm

$$\|v\|_{H^1(\omega,G)} := \sqrt{\|v\|^2_{L^2(\omega,G)} + \|\nabla v\|^2_{L^2(\omega,G)}}.$$

We define $H^1_0(\omega, G)$ as the closure of $C^\infty_0(G)$ in $H^1(\omega, G)$. Due to the fact that the weight ω belongs to A_2, most of the properties of classical Sobolev spaces have a weighted counterpart. In particular, $L^2(\omega, G)$ and $H^1(\omega, G)$ are Hilbert spaces [38, Proposition 2.1.2] and smooth functions are dense [38, Corollary 2.1.6]; see also [22, Theorem 1]. In view of a weighted Poincaré inequality, that follows from [17, Theorem 1.3], we have that in $H_0^1(\omega,G)$ the seminorm $\|\nabla v\|_{L^2(\omega,G)}$ is equivalent to $\|v\|_{H^1(\omega,G)}$.

We introduce the vector space $\mathbf{H}_0^1(\omega,G)$ and the semi-norm $|\cdot|_{\mathbf{H}^1(\omega,G)}$ as follows:

$$\mathbf{H}_0^1(\omega,G) := [H_0^1(\omega,G)]^2, \qquad |\mathbf{v}|_{\mathbf{H}^1(\omega,G)} := \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\omega,G)} = \left(\sum_{i=1}^{2} \|\nabla v_i\|_{L^2(\omega,G)}^2\right)^{\frac{1}{2}}.$$

The following fundamental result, which is known as *reverse Hölder inequality* will be of importance for our analysis; see [15, Theorem 7.4].

Proposition 2 (reverse Hölder inequality). *If $\omega \in A_2$, then there exists $\epsilon > 0$ such that, for every ball $B \subset \mathbb{R}^2$, we have*

$$\fint_B \omega^{1+\epsilon} \lesssim \left(\fint_B \omega\right)^{1+\epsilon}.$$

The hidden constant only depends on the Muckenhoupt characteristic $[\omega]_{A_2}$.

We now present the following embedding results.

Theorem 3 (continuous embeddings). *Let $\omega \in A_2$. Then, we have that*
(i) there exists $\varepsilon > 0$ such that $\mathbf{H}_0^1(\omega,\Omega) \hookrightarrow \mathbf{L}^{2+\varepsilon}(\Omega)$, and that
(ii) there exists $\kappa > 1$ such that $\mathbf{H}_0^1(\omega,\Omega) \hookrightarrow \mathbf{W}^{1,\kappa}(\Omega)$.

Proof. We prove the first embedding result; the second one follows from similar considerations. Let $\omega \in A_2$ and $\mathbf{\Phi} \in \mathbf{H}_0^1(\omega,\Omega)$. An application of [17, Theorem 1.3] immediately implies that $\mathbf{\Phi} \in \mathbf{L}^4(\omega,\Omega)$. We thus invoke Hölder's inequality to arrive at

$$(4) \qquad \int_\Omega \mathbf{\Phi}^{2+\varepsilon} = \int_\Omega \mathbf{\Phi}^{2+\varepsilon}\omega^{\frac{2+\varepsilon}{4}}\omega^{-\frac{2+\varepsilon}{4}} \leq \left(\int_\Omega \mathbf{\Phi}^4\omega\right)^{\frac{2+\varepsilon}{4}} \left(\int_\Omega \omega^{-\frac{2+\varepsilon}{2-\varepsilon}}\right)^{\frac{2-\varepsilon}{4}}$$

for some $\varepsilon > 0$. We observe that, since $\omega \in A_2$ and $\frac{2+\varepsilon}{2-\varepsilon} = 1+\delta$, with $\delta = \frac{2\varepsilon}{2-\varepsilon}$, the reverse Hölder inequality of Proposition 2 allows us to obtain

$$\int_\Omega \omega^{-\frac{2+\varepsilon}{2-\varepsilon}} = \int_\Omega \omega^{-(1+\delta)} \lesssim |\Omega|^{-\delta}\left(\int_\Omega \omega^{-1}\right)^{1+\delta} = |\Omega|^{-\frac{2\varepsilon}{2-\varepsilon}}\left(\int_\Omega \omega^{-1}\right)^{\frac{2+\varepsilon}{2-\varepsilon}}.$$

Here, $\varepsilon > 0$ is sufficiently small such that previously defined parameter δ is less or equal that the one dictated by the reverse Hölder inequality. Since $\int_\Omega \omega^{-1}$ is uniformly bounded, the previous bound combined with estimate (4) allow us to conclude. ◻

3.2.1. A particular weight. In this section, we introduce a particular weight in the class A_2 that will be of fundamental importance to analyze our optimal control problem. With the finite ordered set $\mathcal{D} \subset \Omega$ at hand, we define

$$(5) \qquad d_{\mathcal{D}} := \begin{cases} \operatorname{dist}(\mathcal{D},\partial\Omega), & \text{if } \ell = 1, \\ \min\left\{\operatorname{dist}(\mathcal{D},\partial\Omega), \min\{|t-t'| : t,t' \in \mathcal{D},\ t \neq t'\}\right\}, & \text{otherwise.} \end{cases}$$

We recall that $\ell = \#\mathcal{D}$. Since $\mathcal{D} \subset \Omega$ is finite, we immediately conclude that $d_{\mathcal{D}} > 0$. With this notation at hand, we define the weight ρ as follows:

$$(6) \qquad \text{if } \ell = 1,\ \rho(x) = \mathsf{d}_t^\alpha(x), \qquad \text{otherwise, } \rho(x) = \begin{cases} \mathsf{d}_t^\alpha(x), & \exists t \in \mathcal{D} : \mathsf{d}_t(x) < \frac{d_{\mathcal{D}}}{2}, \\ 1, & \mathsf{d}_t(x) \geq \frac{d_{\mathcal{D}}}{2}\ \forall t \in \mathcal{D}, \end{cases}$$

where $\mathsf{d}_t(x) := |x - t|$ and $\alpha \in (0, 2)$. Since $(0, 2) \subset (-2, 2)$, owing to [3, Theorem 6] and [19, Lemma 2.3 (vi)], the weight ρ belongs to the class A_2. The extra restriction on α, namely, $\alpha > 0$, is needed in order to guarantee that for $t \in \mathcal{D}$ and $\mathbf{v}_t \in \mathbb{R}^2$, $\mathbf{v}_t \delta_t \in \mathbf{H}_0^1(\rho^{-1}, \Omega)'$; see [25, Remark 21.19] and [16, Proposition 5.2] for details.

3.3. The Navier–Stokes equations under singular forcing. In this section, we follow the approach on weighted spaces developed in [33] and review existence and uniqueness results for the following stationary Navier–Stokes equations under singular forcing:

$$-\nu\Delta\mathbf{\Phi} + (\mathbf{\Phi}\cdot\nabla)\mathbf{\Phi} + \nabla\zeta = \mathbf{f} \text{ in } \Omega, \qquad \text{div } \mathbf{\Phi} = 0 \text{ in } \Omega, \qquad \mathbf{\Phi} = \mathbf{0} \text{ on } \partial\Omega. \tag{7}$$

Essentially, by introducing a suitable weight, we can allow for forces such that $\mathbf{f} \notin \mathbf{H}^{-1}(\Omega)$. In particular, for a fixed $\mathbf{F} \in \mathbb{R}^2$, we can set $\mathbf{f} = \mathbf{F}\delta_z$, where $z \in \Omega$ denotes an interior point. Similarly, if Γ denotes a smooth closed curve contained in Ω, we can allow the components of $\mathbf{f}$ to be measures supported in Γ. We must remark that the study of (7) in a nonenergy setting is not a new idea and refer the reader to [35, 18, 5, 34, 28, 11] for related results.

3.3.1. Weak formulation. We begin by observing that, if $\mathbf{v}$ is sufficiently smooth and solenoidal, then the term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ can be rewritten as $\text{div}(\mathbf{v}\otimes\mathbf{v})$. On the basis of this observation, we introduce the following weak formulation for problem (7) for $\mathbf{f} \in \mathbf{H}_0^1(\omega^{-1}, \Omega)'$ [33, Section 2]: Find $(\mathbf{\Phi}, \zeta) \in \mathbf{H}_0^1(\omega, \Omega) \times L^2(\omega, \Omega)/\mathbb{R}$ such that

$$\int_\Omega (\nu\nabla\mathbf{\Phi} : \nabla\mathbf{v} - \mathbf{\Phi}\otimes\mathbf{\Phi} : \nabla\mathbf{v} - \zeta \text{div } \mathbf{v}) = \langle \mathbf{f}, \mathbf{v}\rangle, \quad \int_\Omega q \text{div } \mathbf{\Phi} = 0 \tag{8}$$

for all $(\mathbf{v}, q) \in \mathbf{H}_0^1(\omega^{-1}, \Omega) \times L^2(\omega^{-1}, \Omega)/\mathbb{R}$. Here, $\nu > 0$, ω denotes a weight in the class A_2, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathbf{H}_0^1(\omega^{-1}, \Omega)'$ and $\mathbf{H}_0^1(\omega^{-1}, \Omega)$.

3.3.2. Existence and uniqueness of solutions. In an attempt to simplify notation and the presentation of the material, we set $\mathcal{X} := \mathbf{H}_0^1(\omega, \Omega) \times L^2(\omega, \Omega)/\mathbb{R}$ and $\mathcal{Y} := \mathbf{H}_0^1(\omega^{-1}, \Omega) \times L^2(\omega^{-1}, \Omega)/\mathbb{R}$ and introduce the mappings $\mathfrak{S} : \mathcal{X} \to \mathcal{Y}'$, $\mathfrak{NL} : \mathcal{X} \to \mathcal{Y}'$, and $\mathfrak{F} \in \mathcal{Y}'$ by

$$\langle \mathfrak{S}(\mathbf{\Phi}, \zeta), (\mathbf{v}, q)\rangle = \int_\Omega (\nabla\mathbf{\Phi} : \nabla\mathbf{v} - \zeta \text{div } \mathbf{v} - q \text{div } \mathbf{\Phi}),$$
$$\langle \mathfrak{NL}(\mathbf{\Phi}, \zeta), (\mathbf{v}, q)\rangle = -\int_\Omega \mathbf{\Phi}\otimes\mathbf{\Phi} : \nabla\mathbf{v},$$

and $\langle \mathfrak{F}, (\mathbf{v}, q)\rangle = \langle \mathbf{f}, \mathbf{v}\rangle$, respectively. With these operators at hand, we rewrite problem (8) as the following nonlinear operator equation in $\mathcal{Y}'$: $\mathfrak{S}(\nu\mathbf{\Phi}, \zeta) + \mathfrak{NL}(\mathbf{\Phi}, \zeta) = \mathfrak{F}$.

An application of the Cauchy–Schwarz inequality shows that $\mathfrak{S}$ is a linear and bounded operator. In addition, if Ω is Lipschitz and $\omega \in A_2(\Omega)$, then $\mathfrak{S}$ has a bounded inverse [32, Theorem 17], which we denote by $\mathfrak{S}^{-1}$. Within this setting, we present the following existence and uniqueness result: Let Ω be Lipschitz and $\omega \in A_2(\Omega)$. Assume that $\mathbf{f}$ is sufficiently small, or the viscosity is sufficiently large, so that

$$\nu^{-2}\mathcal{C}_\omega^2 \|\mathfrak{S}^{-1}\|^2 \|\mathbf{f}\|_{\mathbf{H}_0^1(\omega^{-1},\Omega)'} < \tfrac{1}{6} \tag{9}$$

holds. In this setting, there is a unique solution of (8) [33, Corollary 1]. Here, $\mathcal{C}_\omega$ denotes the best constant in the weighted Sobolev embedding $\mathbf{H}_0^1(\omega, \Omega) \hookrightarrow \mathbf{L}^4(\omega, \Omega)$ and $\|\mathfrak{S}^{-1}\|$ denotes the $\mathcal{Y}' \to \mathcal{X}$ norm of $\mathfrak{S}^{-1}$.

The existence of solutions for (8), without smallness conditions, follows from [33, Theorem 1]: Let Ω be Lipschitz and $\omega \in A_2(\Omega)$. For every $\nu > 0$ and $\mathbf{f} \in \mathbf{H}_0^1(\omega^{-1},\Omega)'$, problem (8) has at least one solution $(\mathbf{\Phi},\zeta) \in \mathbf{H}_0^1(\omega,\Omega) \times L^2(\omega,\Omega)/\mathbb{R}$, which satisfies

$$\|\nabla \mathbf{\Phi}\|_{\mathbf{L}^2(\omega,\Omega)} + \|\zeta\|_{L^2(\omega,\Omega)/\mathbb{R}} \lesssim \|\mathbf{f}\|_{\mathbf{H}_0^1(\omega^{-1},\Omega)'}. \tag{10}$$

3.3.3. Differentiability properties of a solution operator. In this section, we analyze differentiability properties for a solution operator associated to (8).

Let $\mathbf{f} \in \mathbf{H}_0^1(\omega^{-1},\Omega)'$ be such that assumption (9) holds. Let us introduce the solution operator $\mathcal{S} : \mathbf{H}_0^1(\omega^{-1},\Omega)' \to \mathbf{H}_0^1(\omega,\Omega) \times L^2(\omega,\Omega)/\mathbb{R}$, which associates to $\mathbf{f}$ the unique pair $(\mathbf{\Phi},\zeta) \in \mathbf{H}_0^1(\omega,\Omega) \times L^2(\omega,\Omega)/\mathbb{R}$ that solves (8). Let us also introduce

$$\mathcal{G} : \mathbf{H}_0^1(\omega^{-1},\Omega)' \to \mathbf{H}_0^1(\omega,\Omega), \qquad \mathbf{f} \mapsto \mathbf{\Phi}, \tag{11}$$

where $\mathbf{\Phi}$ corresponds to the velocity component of the pair $(\mathbf{\Phi},\zeta)$, which is such that $(\mathbf{\Phi},\zeta) = \mathcal{S}(\mathbf{f})$.

In order to study differentiability properties for the operator $\mathcal{S}$, we introduce the following problem: Find $(\boldsymbol{\theta},\xi) \in \mathbf{H}_0^1(\omega,\Omega) \times L^2(\omega,\Omega)/\mathbb{R}$ such that

$$\int_\Omega (\nu \nabla \boldsymbol{\theta} : \nabla \mathbf{v} - \mathbf{\Phi} \otimes \boldsymbol{\theta} : \nabla \mathbf{v} - \boldsymbol{\theta} \otimes \mathbf{\Phi} : \nabla \mathbf{v} - \xi \text{div } \mathbf{v}) = \langle \mathbf{g}, \mathbf{v} \rangle, \ \int_\Omega q \text{div } \boldsymbol{\theta} = 0 \tag{12}$$

for all $(\mathbf{v},q) \in \mathbf{H}_0^1(\omega^{-1},\Omega) \times L^2(\omega^{-1},\Omega)/\mathbb{R}$. Here, $\mathbf{g} \in \mathbf{H}_0^1(\omega^{-1},\Omega)'$ and $\mathbf{\Phi} = \mathcal{G}(\mathbf{f})$. We also introduce the map

$$\mathcal{K} : \mathbf{H}_0^1(\omega^{-1},\Omega)' \to \mathbf{H}_0^1(\omega,\Omega) \times L^2(\omega,\Omega)/\mathbb{R}, \qquad \mathbf{g} \mapsto (\boldsymbol{\theta},\xi), \tag{13}$$

where $(\boldsymbol{\theta},\xi)$ denotes the solution to (12).

The following result establishes the well-posedness of problem (12).

LEMMA 4 (well-posedness). *Let $\mathbf{f} \in \mathbf{H}_0^1(\omega^{-1},\Omega)'$ be such that the smallness assumption (9) holds. Let $(\mathbf{\Phi},\zeta)$ be the unique solution to (8) with $\mathbf{f}$ as a forcing term. If $\mathbf{g} \in \mathbf{H}_0^1(\omega^{-1},\Omega)'$, then problem (12) has a unique solution $(\boldsymbol{\theta},\xi) \in \mathbf{H}_0^1(\omega,\Omega) \times L^2(\omega,\Omega)/\mathbb{R}$. In addition, we have the stability estimate*

$$\|(\boldsymbol{\theta},\xi)\|_{\mathcal{X}} \lesssim \|\mathbf{g}\|_{\mathbf{H}_0^1(\omega^{-1},\Omega)'}, \tag{14}$$

with a hidden constant that only depends on $\|\mathfrak{S}^{-1}\|_{\mathcal{L}(\mathcal{Y}',\mathcal{X})}$ and ν. This, in particular, implies that the operator $\mathcal{K}$, defined in (13), is an isomorphism from $\mathbf{H}_0^1(\omega^{-1},\Omega)'$ into $\mathbf{H}_0^1(\omega,\Omega) \times L^2(\omega,\Omega)/\mathbb{R}$.

Proof. We begin the proof by introducing the linear map $\mathfrak{L}_{\mathbf{\Phi}} : \mathcal{X} \to \mathcal{Y}'$ by

$$\langle \mathfrak{L}_{\mathbf{\Phi}}(\boldsymbol{\theta},\xi), (\mathbf{v},q) \rangle := -\nu^{-1} \int_\Omega (\mathbf{\Phi} \otimes \boldsymbol{\theta} : \nabla \mathbf{v} + \boldsymbol{\theta} \otimes \mathbf{\Phi} : \nabla \mathbf{v}).$$

We recall that $\mathcal{X} = \mathbf{H}_0^1(\omega,\Omega) \times L^2(\omega,\Omega)/\mathbb{R}$ and $\mathcal{Y} = \mathbf{H}_0^1(\omega^{-1},\Omega) \times L^2(\omega^{-1},\Omega)/\mathbb{R}$. We immediately notice that the map $\mathfrak{L}_{\mathbf{\Phi}}$ is linear and, in view of Hölder's inequality, bounded. In fact, we have

$$\begin{aligned} \langle \mathfrak{L}_{\mathbf{\Phi}}(\boldsymbol{\theta},\xi), (\mathbf{v},q) \rangle &\leq 2\nu^{-1} \|\mathbf{\Phi}\|_{\mathbf{L}^4(\omega,\Omega)} \|\boldsymbol{\theta}\|_{\mathbf{L}^4(\omega,\Omega)} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\omega^{-1},\Omega)} \\ &\leq 2\nu^{-1} \mathcal{C}_\omega^2 \|\nabla \mathbf{\Phi}\|_{\mathbf{L}^2(\omega,\Omega)} \|\nabla \boldsymbol{\theta}\|_{\mathbf{L}^2(\omega,\Omega)} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\omega^{-1},\Omega)}. \end{aligned}$$

Consequently, $\|\mathfrak{L}_{\mathbf{\Phi}}\|_{\mathcal{L}(\mathcal{X},\mathcal{Y}')} \leq 2\nu^{-1} \mathcal{C}_\omega^2 \|\nabla \mathbf{\Phi}\|_{\mathbf{L}^2(\omega,\Omega)}$.

With the linear and bounded map $\mathfrak{L}_{\boldsymbol{\Phi}}$ at hand, we invoke the operator $\mathfrak{S}$, defined in section 3.3.2, to rewrite problem (12) as the following operator equation in $\mathcal{Y}'$: $\mathfrak{S}(\nu\boldsymbol{\theta},\xi) + \mathfrak{L}_{\boldsymbol{\Phi}}(\nu\boldsymbol{\theta},\xi) = \mathfrak{G}$. Here, for every $(\mathbf{v},q) \in \mathbf{H}_0^1(\omega^{-1},\Omega) \times L^2(\omega^{-1},\Omega)/\mathbb{R}$, $\langle \mathfrak{G}, (\mathbf{v},q)\rangle := \langle \mathbf{g},\mathbf{v}\rangle$. We thus utilize the fact that the map $\mathfrak{S}$ is invertible to arrive at $(\nu\boldsymbol{\theta},\xi) + \mathfrak{S}^{-1}\mathfrak{L}_{\boldsymbol{\Phi}}(\nu\boldsymbol{\theta},\xi) = \mathfrak{S}^{-1}\mathfrak{G}$ in $\mathcal{X}$.

Let us now observe that the boundedness of $\mathfrak{S}^{-1}$ and $\mathfrak{L}_{\boldsymbol{\Phi}}$ yields

$$\|\mathfrak{S}^{-1}\mathfrak{L}_{\boldsymbol{\Phi}}\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \leq \|\mathfrak{S}^{-1}\|_{\mathcal{L}(\mathcal{Y}',\mathcal{X})}\|\mathfrak{L}_{\boldsymbol{\Phi}}\|_{\mathcal{L}(\mathcal{X},\mathcal{Y}')} \leq 2\nu^{-1}\mathcal{C}_\omega^2\|\mathfrak{S}^{-1}\|_{\mathcal{L}(\mathcal{Y}',\mathcal{X})}\|\nabla\boldsymbol{\Phi}\|_{\mathbf{L}^2(\omega,\Omega)}.$$

Hence, we invoke the smallness assumption (9) and the stability estimate of [33, Corollary 1] to arrive at $\|\mathfrak{S}^{-1}\mathfrak{L}_{\boldsymbol{\Phi}}\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} < 2^{-1} < 1$. This bound, in view of [40, Theorem 1.B], allows us to conclude that problem (12) admits a unique solution.

We conclude the proof by observing that a direct computation reveals

$$\|(\nu\boldsymbol{\theta},\xi)\|_{\mathcal{X}} \leq \frac{\|\mathfrak{S}^{-1}\|_{\mathcal{L}(\mathcal{Y}',\mathcal{X})}}{1-\|\mathfrak{S}^{-1}\mathfrak{L}_{\boldsymbol{\Phi}}\|_{\mathcal{L}(\mathcal{X},\mathcal{X})}}\|\mathbf{g}\|_{\mathbf{H}_0^1(\omega^{-1},\Omega)'} \lesssim \|\mathbf{g}\|_{\mathbf{H}_0^1(\omega^{-1},\Omega)'}.$$

This yields (14) and concludes the proof. ∎

We now present the following Lipschitz property.

THEOREM 5 (Lipschitz property). *If* $\mathbf{f}$ *and* $\hat{\mathbf{f}}$ *belong to* $\mathbf{H}_0^1(\omega^{-1},\Omega)'$ *and both satisfy the smallness assumption* (9)*, then we have the following Lipschitz property*

$$\|\nabla(\boldsymbol{\Phi}-\hat{\boldsymbol{\Phi}})\|_{\mathbf{L}^2(\omega,\Omega)} \lesssim \|\mathbf{f}-\hat{\mathbf{f}}\|_{\mathbf{H}_0^1(\omega^{-1},\Omega)'}, \tag{15}$$

where $(\boldsymbol{\Phi},\zeta) = \mathcal{S}(\mathbf{f})$ *and* $(\hat{\boldsymbol{\Phi}},\hat{\zeta}) = \mathcal{S}(\hat{\mathbf{f}})$.

Proof. We begin the proof by noticing that $(\boldsymbol{\Phi}-\hat{\boldsymbol{\Phi}},\zeta-\hat{\zeta})$ satisfies the following weak problem: Find $(\boldsymbol{\Phi}-\hat{\boldsymbol{\Phi}},\zeta-\hat{\zeta}) \in \mathbf{H}_0^1(\omega,\Omega)\times L^2(\omega,\Omega)/\mathbb{R}$ such that

$$\int_\Omega \Big(\nu\nabla(\boldsymbol{\Phi}-\hat{\boldsymbol{\Phi}}):\nabla\mathbf{v} - \boldsymbol{\Phi}\otimes(\boldsymbol{\Phi}-\hat{\boldsymbol{\Phi}}):\nabla\mathbf{v} - (\boldsymbol{\Phi}-\hat{\boldsymbol{\Phi}})\otimes\hat{\boldsymbol{\Phi}}:\nabla\mathbf{v}\Big) \\ - \int_\Omega(\zeta-\hat{\zeta})\text{div }\mathbf{v} = \langle \mathbf{f}-\hat{\mathbf{f}},\mathbf{v}\rangle, \qquad \int_\Omega q\text{div }(\boldsymbol{\Phi}-\hat{\boldsymbol{\Phi}}) = 0, \tag{16}$$

for all $(\mathbf{v},q)\in\mathbf{H}_0^1(\omega^{-1},\Omega)\times L^2(\omega^{-1},\Omega)/\mathbb{R}$. Observe that (16) is a slightly different problem to the one posed in (12). Consequently, similar arguments to the ones utilized to obtain the results of Lemma 4 combined with the fact that both $\mathbf{f}$ and $\hat{\mathbf{f}}$ satisfy (9), reveal that problem (16) is well posed. In particular, we arrive at (15) as a stability bound for problem (16). This concludes the proof. ∎

We are now in position to analyze differentiability properties for the map $\mathcal{S}$.

THEOREM 6 (differentiability properties). *Let* $\mathbf{f}\in\mathbf{H}_0^1(\omega^{-1},\Omega)'$ *be such that the smallness assumption* (9) *holds. Let* $(\boldsymbol{\Phi},\zeta)$ *be the unique solution to* (8) *with* $\mathbf{f}$ *as a forcing term. The map* $\mathcal{S}$ *is first order Fréchet differentiable at* $\mathbf{f}$ *and* $\mathcal{S}'(\mathbf{f}):\mathbf{H}_0^1(\omega^{-1},\Omega)'\to\mathbf{H}_0^1(\omega,\Omega)\times L^2(\omega,\Omega)/\mathbb{R}$ *is an isomorphism. If* $\mathbf{g}\in\mathbf{H}_0^1(\omega^{-1},\Omega)'$*, then* $(\boldsymbol{\theta},\xi)=\mathcal{S}'(\mathbf{f})\mathbf{g}$ *corresponds to the unique solution to* (12)*. In addition, there exists an open neighborhood of* $\mathbf{f}$ *such that* $\mathcal{S}$ *is second order Fréchet differentiable on it. If*

$\mathbf{g}_1, \mathbf{g}_2 \in \mathbf{H}_0^1(\omega^{-1}, \Omega)'$, *then* $(\boldsymbol{\psi}, \gamma) = \mathcal{S}''(\mathbf{f})\mathbf{g}_1\mathbf{g}_2$ *corresponds to the unique solution to*

$$(17)\qquad \int_\Omega ([\nu \nabla \boldsymbol{\psi} - \boldsymbol{\Phi} \otimes \boldsymbol{\psi} - \boldsymbol{\psi} \otimes \boldsymbol{\Phi}] : \nabla \mathbf{v} - \gamma \text{div } \mathbf{v}) \\ = \int_\Omega (\boldsymbol{\theta}_{\mathbf{g}_1} \otimes \boldsymbol{\theta}_{\mathbf{g}_2} + \boldsymbol{\theta}_{\mathbf{g}_2} \otimes \boldsymbol{\theta}_{\mathbf{g}_1}) : \nabla \mathbf{v}, \qquad \int_\Omega q \text{div } \boldsymbol{\psi} = 0$$

for all $(\mathbf{v}, q) \in \mathbf{H}_0^1(\omega^{-1}, \Omega) \times L^2(\omega^{-1}, \Omega)/\mathbb{R}$. *Here,* $(\boldsymbol{\theta}_{\mathbf{g}_i}, \xi_{\mathbf{g}_i}) = \mathcal{S}'(\mathbf{f})\mathbf{g}_i$, *with* $i \in \{1, 2\}$, *corresponds to the unique solution to* (12) *with* $\mathbf{g}$ *being replaced by* $\mathbf{g}_i$.

Proof. We begin the proof by obtaining, inspired by [37, Theorem 4.17], the first order Fréchet differentiability of $\mathcal{S}$. Let $\mathbf{h} \in \mathbf{H}_0^1(\omega^{-1}, \Omega)'$ be such that $\mathbf{f} + \mathbf{h}$ satisfies the smallness assumption (9). To simplify the presentation of the material, we define

$$(\tilde{\boldsymbol{\Phi}}, \tilde{\zeta}) := \mathcal{S}(\mathbf{f} + \mathbf{h}), \qquad \mathbf{Y} := \tilde{\boldsymbol{\Phi}} - \boldsymbol{\Phi}, \qquad P = \tilde{\zeta} - \zeta.$$

We recall that $(\boldsymbol{\Phi}, \zeta) = \mathcal{S}(\mathbf{f})$ denotes the unique solution to (8) with $\mathbf{f}$ as a forcing term. Since $\mathbf{f}+\mathbf{h}$ satisfies (9), the existence and uniqueness of $(\tilde{\boldsymbol{\Phi}}, \tilde{\zeta})$ is also guaranteed. Let us now introduce the pair $(r_{\mathbf{Y}}, r_P) \in \mathbf{H}_0^1(\omega, \Omega) \times L^2(\omega, \Omega)/\mathbb{R}$ as the solution to

$$(18)\qquad \int_\Omega (\nu \nabla r_{\mathbf{Y}} : \nabla \mathbf{v} - [r_{\mathbf{Y}} \otimes \boldsymbol{\Phi} + \boldsymbol{\Phi} \otimes r_{\mathbf{Y}}] : \nabla \mathbf{v} - r_P \text{div } \mathbf{v}) \\ = \int_\Omega \mathbf{Y} \otimes \mathbf{Y} : \nabla \mathbf{v}, \quad \int_\Omega q \text{div } r_{\mathbf{Y}} = 0 \quad \forall (\mathbf{v}, q) \in \mathbf{H}_0^1(\omega^{-1}, \Omega) \times L^2(\omega^{-1}, \Omega)/\mathbb{R}.$$

Since $\mathbf{f}$ satisfies (9), Lemma 4 reveals that problem (18) admits a unique solution upon observing that the involved forcing term belongs to $\mathbf{H}_0^1(\omega^{-1}, \Omega)'$; see (20) below.

We now observe that $(\mathbf{Y}, P)$ can be written as $(\mathbf{Y}, P) = \mathcal{K}(\mathbf{h}) + (r_{\mathbf{Y}}, r_P)$. Here, $\mathcal{K}(\mathbf{h}) = (\boldsymbol{\theta}, \xi)$, $\mathcal{K}$ is defined in (13), and the pair $(\boldsymbol{\theta}, \xi)$ solves (12) with $\mathbf{g}$ being replaced by $\mathbf{h}$. Notice that, in light of Lemma 4, the operator $\mathcal{K}$ is linear and continuous. To obtain the first order Fréchet differentiability of $\mathcal{S}$ it thus suffices to prove

$$(19)\qquad \|\mathbf{h}\|_{\mathbf{H}_0^1(\omega^{-1}, \Omega)'} \to 0 \implies \frac{\|\nabla r_{\mathbf{Y}}\|_{\mathbf{L}^2(\omega, \Omega)} + \|r_P\|_{L^2(\omega, \Omega)}}{\|\mathbf{h}\|_{\mathbf{H}_0^1(\omega^{-1}, \Omega)'}} \to 0.$$

To accomplish this task, we first invoke the stability estimate (14), Hölder's inequality, and the weighted Sobolev embedding $\mathbf{H}_0^1(\omega, \Omega) \hookrightarrow \mathbf{L}^4(\omega, \Omega)$ to arrive at

$$(20)\qquad \|\nabla r_{\mathbf{Y}}\|_{\mathbf{L}^2(\omega, \Omega)} + \|r_P\|_{L^2(\omega, \Omega)} \lesssim \sup_{\mathbf{v} \in \mathbf{H}_0^1(\omega^{-1}, \Omega)} \frac{\langle \mathbf{Y} \otimes \mathbf{Y}, \nabla \mathbf{v} \rangle}{\|\nabla \mathbf{v}\|_{\mathbf{L}^2(\omega^{-1}, \Omega)}} \lesssim \|\nabla \mathbf{Y}\|^2_{\mathbf{L}^2(\omega, \Omega)}.$$

Since $(\mathbf{Y}, P) = (\tilde{\boldsymbol{\Phi}} - \boldsymbol{\Phi}, \tilde{\zeta} - \zeta)$, we have that $\|\nabla \mathbf{Y}\|_{\mathbf{L}^2(\omega, \Omega)} \lesssim \|\mathbf{h}\|_{\mathbf{H}_0^1(\omega^{-1}, \Omega)'}$, upon utilizing the Lipschitz property derived in Theorem 5. Replacing this estimate into (20) allows us to conclude (19).

We now prove, on the basis of the implicit function theorem, the second order Fréchet differentiability of $\mathcal{S}$. To accomplish this task, we follow [37, Theorem 4.24] and [11, Theorem 2.10] and define the nonlinear map $\mathfrak{F} : \mathbf{H}_0^1(\omega, \Omega) \times L^2(\omega, \Omega)/\mathbb{R} \times \mathbf{H}_0^1(\omega^{-1}, \Omega)' \to \mathbf{H}_0^1(\omega^{-1}, \Omega)'$ by

$$\mathfrak{F}(\boldsymbol{\Phi}, \zeta, \mathbf{f}) := -\nu \Delta \boldsymbol{\Phi} + \text{div } (\boldsymbol{\Phi} \otimes \boldsymbol{\Phi}) + \nabla \zeta - \mathbf{f}.$$

The map $\mathfrak{F}$ is well defined. Direct computations reveal that $\mathfrak{F}$ is of class C^2. In addition, by definition of $\mathcal{S}$, we have $\mathfrak{F}(\mathcal{S}(\mathbf{f}),\mathbf{f}) = 0$. We also observe that

$$\frac{\partial \mathfrak{F}}{\partial(\mathbf{\Phi},\zeta)}(\mathcal{S}(\mathbf{f}),\mathbf{f})(\boldsymbol{\theta},\xi) = -\nu\Delta\boldsymbol{\theta} + \text{div }(\mathbf{\Phi}\otimes\boldsymbol{\theta}) + \text{div }(\boldsymbol{\theta}\otimes\mathbf{\Phi}) + \nabla\xi = \mathcal{K}^{-1}(\boldsymbol{\theta},\xi).$$

Lemma 4 guarantees that $\mathcal{K}^{-1}$ is an isomorphism from $\mathbf{H}_0^1(\omega,\Omega)\times L^2(\omega,\Omega)/\mathbb{R}$ into $\mathbf{H}_0^1(\omega^{-1},\Omega)'$. We can thus invoke the implicit function theorem, to conclude that $\mathcal{S}$ is of class C^2 on a suitable neighborhood of $\mathbf{f}$. The characterization (17) follows from differentiating the relation $\mathfrak{F}(\mathcal{S}(\mathbf{f}),\mathbf{f}) = 0$ in the spirit of [37, Theorem 4.24]. We finally observe that the forcing term of problem (17) belongs to $\mathbf{H}_0^1(\omega^{-1},\Omega)'$ because $\boldsymbol{\theta}_{\mathbf{g}_1},\boldsymbol{\theta}_{\mathbf{g}_2}\in\mathbf{H}_0^1(\omega,\Omega)$ and $\mathbf{H}_0^1(\omega,\Omega)\hookrightarrow\mathbf{L}^4(\omega,\Omega)$. We can thus apply the results of Lemma 4 to conclude that problem (17) is well posed. ☐

4. The optimal control problem. In this section, we propose and analyze the following weak formulation of the optimal control problem (1)–(3): Find

$$\min\{J(\mathbf{y},\mathcal{U}) : (\mathbf{y},p,\mathcal{U})\in\mathbf{H}_0^1(\rho,\Omega)\times L^2(\rho,\Omega)/\mathbb{R}\times\mathbb{U}_{ad}\}, \tag{21}$$

subject to the weak formulation of the stationary Navier–Stokes equations

$$\int_\Omega(\nu\nabla\mathbf{y}:\nabla\mathbf{v} - \mathbf{y}\otimes\mathbf{y}:\nabla\mathbf{v} - p\text{div }\mathbf{v}) = \sum_{t\in\mathcal{D}}\langle\mathbf{u}_t\delta_t,\mathbf{v}\rangle, \quad \int_\Omega q\text{div }\mathbf{y} = 0 \tag{22}$$

for all $(\mathbf{v},q)\in\mathbf{H}_0^1(\rho^{-1},\Omega)\times L^2(\rho^{-1},\Omega)/\mathbb{R}$. The weight ρ is defined as in (6) and the parameter α is such that $\alpha\in(0,2)$. We comment that, since the velocity component $\mathbf{y}$ of a solution to the state equation is sought in $\mathbf{H}_0^1(\rho,\Omega)$, an application of Theorem 3 guarantees that $\mathbf{y}\in\mathbf{L}^2(\Omega)$. Consequently, all the terms involved in the definition of the cost functional J are well defined.

4.1. Existence of optimal solutions. The existence of an optimal solution $(\bar{\mathbf{y}},\bar{p},\bar{\mathcal{U}})$ is as follows.

THEOREM 7 (existence of an optimal solution). *The control problem* (21)–(22) *admits at least one global solution* $(\bar{\mathbf{y}},\bar{p},\bar{\mathcal{U}})\in\mathbf{H}_0^1(\rho,\Omega)\times L^2(\rho,\Omega)/\mathbb{R}\times\mathbb{U}_{ad}$.

Proof. Let $\{(\mathbf{y}_k,p_k,\mathcal{U}_k)\}_{k\in\mathbb{N}}$ be a minimizing sequence, i.e., for $k\in\mathbb{N}$, the pair $(\mathbf{y}_k,p_k)\in\mathbf{H}_0^1(\rho,\Omega)\times L^2(\rho,\Omega)/\mathbb{R}$ solves

$$\int_\Omega(\nu\nabla\mathbf{y}_k:\nabla\mathbf{v} - \mathbf{y}_k\otimes\mathbf{y}_k:\nabla\mathbf{v} - p_k\text{div }\mathbf{v}) = \sum_{t\in\mathcal{D}}\langle\mathbf{u}_t^k\delta_t,\mathbf{v}\rangle, \quad \int_\Omega q\text{div }\mathbf{y}_k = 0, \tag{23}$$

for all $(\mathbf{v},q)\in\mathbf{H}_0^1(\rho^{-1},\Omega)\times L^2(\rho^{-1},\Omega)/\mathbb{R}$, and the pair $(\mathbf{y}_k,p_k)$ together with $\mathcal{U}_k$ are such that $J(\mathbf{y}_k,p_k,\mathcal{U}_k)\to\mathfrak{i}:=\inf\{J(\mathbf{y},p,\mathcal{U}):(\mathbf{y},p,\mathcal{U})\in\mathbf{H}_0^1(\rho,\Omega)\times L^2(\rho,\Omega)/\mathbb{R}\times\mathbb{U}_{ad}\}$ as $k\uparrow\infty$. Here, for $k\in\mathbb{N}$, we denote $\mathcal{U}_k:=\{\mathbf{u}_t^k\}_{t\in\mathcal{D}}$. We notice that the existence of solutions for problem (23) follows from the results stated in section 3.3.2.

Since $\mathbb{U}_{ad}$ is compact, we immediately conclude the existence of a nonrelabeled subsequence $\{\mathcal{U}_k\}_{k\in\mathbb{N}}$ such that $\mathcal{U}_k\to\bar{\mathcal{U}}$ in $[\mathbb{R}^2]^\ell$ with $\bar{\mathcal{U}}\in\mathbb{U}_{ad}$. On the other hand, in view of the stability bound (10), we conclude that $\{(\mathbf{y}_k,p_k)\}_{k\in\mathbb{N}}$ is uniformly bounded in $\mathbf{H}_0^1(\rho,\Omega)\times L^2(\rho,\Omega)/\mathbb{R}$. Consequently, we deduce the existence of a nonrelabeled subsequence $\{(\mathbf{y}_k,p_k)\}_{k\in\mathbb{N}}$ such that $(\mathbf{y}_k,p_k)\rightharpoonup(\bar{\mathbf{y}},\bar{p})$ in $\mathbf{H}_0^1(\rho,\Omega)\times L^2(\rho,\Omega)/\mathbb{R}$ as $k\uparrow\infty$; $(\bar{\mathbf{y}},\bar{p})$ being the natural candidate for an optimal state. The rest of the proof is dedicated to prove that $(\bar{\mathbf{y}},\bar{p})$ solves (22) with $\mathbf{u}_t$ being replaced by $\bar{\mathbf{u}}_t$ for $t\in\mathcal{D}$.

With the weak convergence $(\mathbf{y}_k, p_k) \rightharpoonup (\bar{\mathbf{y}}, \bar{p})$ in $\mathbf{H}_0^1(\rho,\Omega) \times L^2(\rho,\Omega)/\mathbb{R}$, as $k \uparrow \infty$, at hand, we obtain

$$\int_\Omega \nu \nabla \mathbf{y}_k : \nabla \mathbf{v} \to \int_\Omega \nu \nabla \bar{\mathbf{y}} : \nabla \mathbf{v}, \quad \int_\Omega p_k \text{div } \mathbf{v} \to \int_\Omega \bar{p} \text{div } \mathbf{v}, \quad \int_\Omega q \text{div } \mathbf{y}_k \to \int_\Omega q \text{div } \bar{\mathbf{y}},$$

as $k \uparrow \infty$, for every $\mathbf{v} \in \mathbf{H}_0^1(\rho^{-1},\Omega)$ and $q \in L^2(\rho^{-1},\Omega)/\mathbb{R}$. On the other hand, the convergence $\mathcal{U}_k \to \bar{\mathcal{U}}$ in $[\mathbb{R}^2]^\ell$ yields $\sum_{t\in\mathcal{D}} \langle \mathbf{u}_t^k \delta_t, \mathbf{v}\rangle \to \sum_{t\in\mathcal{D}} \langle \bar{\mathbf{u}}_t \delta_t, \mathbf{v}\rangle$ as $k \uparrow \infty$. It thus suffices to analyze the convective term. To accomplish this task, we invoke Hölder's inequality to arrive at

$$\left| \int_\Omega (\mathbf{y}_k \otimes \mathbf{y}_k - \bar{\mathbf{y}} \otimes \bar{\mathbf{y}}) : \nabla \mathbf{v} \right| \\ \leq \left(\|\mathbf{y}_k\|_{\mathbf{L}^4(\rho,\Omega)} + \|\bar{\mathbf{y}}\|_{\mathbf{L}^4(\rho,\Omega)} \right) \|\bar{\mathbf{y}} - \mathbf{y}_k\|_{\mathbf{L}^4(\rho,\Omega)} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\rho^{-1},\Omega)}.$$

The compact embedding $\mathbf{H}_0^1(\rho,\Omega) \hookrightarrow \mathbf{L}^4(\rho,\Omega)$, which follows from [24, Theorem 4.12] (see also [33, Proposition 2]), combined with $\mathbf{y}_k \rightharpoonup \bar{\mathbf{y}}$ in $\mathbf{H}_0^1(\rho,\Omega)$, as $k \uparrow \infty$, allow us to conclude that $(\bar{\mathbf{y}}, \bar{p})$ solves (22) with $\mathbf{u}_t$ being replaced by $\bar{\mathbf{u}}_t$ for $t \in \mathcal{D}$; $\bar{\mathcal{U}} = \{\bar{\mathbf{u}}_t\}_{t\in\mathcal{D}}$.

To conclude the proof, we must prove the optimality of $\bar{\mathcal{U}}$. Observe that $\mathcal{U}_k \to \bar{\mathcal{U}}$ in $[\mathbb{R}^2]^\ell$, as $k \uparrow \infty$, and that $\mathbf{y}_k \to \bar{\mathbf{y}}$ in $\mathbf{L}^2(\Omega)$, as $k \uparrow \infty$. The latter follows from

$$\|\bar{\mathbf{y}} - \mathbf{y}_k\|_{\mathbf{L}^2(\Omega)} \leq \|\bar{\mathbf{y}} - \mathbf{y}_k\|_{\mathbf{L}^4(\rho,\Omega)} \left(\int_\Omega \rho^{-1} \right)^{\frac{1}{4}} \lesssim \|\bar{\mathbf{y}} - \mathbf{y}_k\|_{\mathbf{L}^4(\rho,\Omega)} \to 0, \quad k \uparrow \infty.$$

We can thus arrive at $J(\bar{\mathbf{y}}, \bar{\mathcal{U}}) = \lim_{k\to\infty} J(\mathbf{y}_k, \mathcal{U}_k) = \mathfrak{i}$. This concludes the proof. ∎

5. First and second order optimality conditions. In this section, we analyze first and second order optimality conditions for the control problem (21)–(22).

We begin our analysis by introducing a first smallness assumption which guarantees existence and uniqueness of solutions for the state equation (22). To present it, we introduce the map

$$\mathcal{H} : [\mathbb{R}^2]^\ell \to \mathbf{H}_0^1(\rho^{-1},\Omega)' : \quad \langle \mathcal{H}(\mathcal{U}), \mathbf{v} \rangle = \sum_{t\in\mathcal{D}} \langle \mathbf{u}_t \delta_t, \mathbf{v} \rangle, \quad \mathbf{v} \in \mathbf{H}_0^1(\rho^{-1},\Omega). \tag{24}$$

It is immediate that $\mathcal{H}$ is a linear and bounded operator. With $\mathcal{H}$ at hand, we define $\mathfrak{M}_{ad} := \sup_{\mathcal{U}\in\mathbb{U}_{ad}} \|\mathcal{H}(\mathcal{U})\|_{\mathbf{H}_0^1(\rho^{-1},\Omega)'}$. The aforementioned assumption reads as follows:

$$\nu^{-2} C_\rho^2 \|\mathfrak{S}^{-1}\|^2 \mathfrak{M}_{ad} < \tfrac{1}{6}, \tag{25}$$

where C_ρ denotes the best constant in the embedding $\mathbf{H}_0^1(\rho,\Omega) \hookrightarrow \mathbf{L}^4(\rho,\Omega)$.

Remark 8 (smallness assumption). Assumption (25) is an additional constraint that restricts the admissible set $\mathbb{U}_{ad}$. Under assumption (25), problem (22) admits a unique solution $(\mathbf{y}_\mathcal{U}, p_\mathcal{U}) \in \mathbf{H}_0^1(\rho,\Omega) \times L^2(\rho,\Omega)/\mathbb{R}$.

5.1. Adjoint equation. The adjoint problem reads as follows: Find $(\mathbf{z}, r) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ such that

$$\int_\Omega (\nu \nabla \mathbf{z} : \nabla \mathbf{w} - (\mathbf{y}_\mathcal{U} \cdot \nabla)\mathbf{z}\mathbf{w} + \nabla \mathbf{y}_\mathcal{U}^\intercal \mathbf{z} \cdot \mathbf{w} - r \text{div } \mathbf{w}) \\ = \int_\Omega (\mathbf{y}_\mathcal{U} - \mathbf{y}_\Omega) \cdot \mathbf{w}, \qquad \int_\Omega s \text{div } \mathbf{z} = 0, \tag{26}$$

for all $(\mathbf{w}, s) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$. Here, $\mathbf{y}_{\mathcal{U}}$ denotes the velocity component of the solution $(\mathbf{y}_{\mathcal{U}}, p_{\mathcal{U}}) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ to problem (22).

To derive the well posedness of the adjoint equation, we shall assume that

$$\|\nabla \mathbf{y}_{\mathcal{U}}\|_{\mathbf{L}^2(\rho,\Omega)} \leq \theta \nu \mathfrak{C}^{-1}, \tag{27}$$

for $\theta < 1$; arbitrarily close to 1. Here, $\mathfrak{C} = \mathcal{C}_e \mathfrak{C}_{\rho,\Omega}$, where $\mathcal{C}_e$ denotes the best constant in the standard Sobolev embedding $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^\sigma(\Omega)$, which holds for every $\sigma < \infty$ (see [2, Theorem 4.12]), and $\mathfrak{C}_{\rho,\Omega}$ denotes the constant defined in (31) below.

Theorem 9 (well-posedness). *If the smallness assumptions (25) and (27) hold, then there exists a unique solution $(\mathbf{z}, r) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ to problem (26). In addition, we have the stability bound*

$$\|\nabla \mathbf{z}\|_{\mathbf{L}^2(\Omega)} + \|r\|_{L^2(\Omega)} \lesssim \|\mathbf{y} - \mathbf{y}_\Omega\|_{\mathbf{H}^{-1}(\Omega)}. \tag{28}$$

The hidden constant in this estimate is independent of $(\mathbf{z}, r)$, $(\mathbf{y}_{\mathcal{U}}, p_{\mathcal{U}})$, and $\mathbf{y}_\Omega$.

Proof. To simplify the presentation of the material, we introduce the form

$$\mathcal{B} : \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \to \mathbb{R}, \quad \mathcal{B}(\mathbf{v}, \mathbf{w}) := \int_\Omega (\nu \nabla \mathbf{v} : \nabla \mathbf{w} - (\mathbf{y}_{\mathcal{U}} \cdot \nabla)\mathbf{v}\mathbf{w} + \nabla \mathbf{y}_{\mathcal{U}}^{\intercal} \mathbf{v} \cdot \mathbf{w}).$$

It is immediate that $\mathcal{B}$ is bilinear. To prove that $\mathcal{B}$ is bounded, it suffices to control the convective terms. To accomplish this task, we first notice that

$$\begin{aligned} \left| \int_\Omega (\mathbf{y}_{\mathcal{U}} \cdot \nabla)\mathbf{v}\mathbf{w} \right| &\leq \|\mathbf{y}_{\mathcal{U}}\|_{\mathbf{L}^\mu(\Omega)} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{w}\|_{\mathbf{L}^\kappa(\Omega)} \\ &\leq \mathcal{C} \|\nabla \mathbf{y}_{\mathcal{U}}\|_{\mathbf{L}^2(\rho,\Omega)} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \|\nabla \mathbf{w}\|_{\mathbf{L}^2(\Omega)}, \quad \mu^{-1} + \kappa^{-1} = \tfrac{1}{2}. \end{aligned} \tag{29}$$

Here, $\mathcal{C} := \mathcal{C}_{2+\epsilon}\mathcal{C}_e$. $\mathcal{C}_e$ denotes the best constant in the Sobolev embedding $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^\sigma(\Omega)$, which holds for every $\sigma < \infty$ [2, Theorem 4.12]. $\mathcal{C}_{2+\epsilon}$ denotes the best constant in $\mathbf{H}_0^1(\rho, \Omega) \hookrightarrow \mathbf{L}^{2+\epsilon}(\Omega)$, where $\epsilon > 0$ is dictated by Theorem 3. We now control the remaining convective term as follows: We first observe that

$$\left| \int_\Omega \nabla \mathbf{y}_{\mathcal{U}}^{\intercal} \mathbf{v} \cdot \mathbf{w} \right| \leq \|\nabla \mathbf{y}_{\mathcal{U}}\|_{\mathbf{L}^2(\rho,\Omega)} \|\mathbf{v}\|_{\mathbf{L}^\mu(\Omega)} \|\mathbf{w}\|_{\mathbf{L}^\kappa(\Omega)} \|\rho^{-\frac{1}{2}}\|_{\mathbf{L}^\varsigma(\Omega)}, \quad \mu^{-1} + \kappa^{-1} + \varsigma^{-1} = \tfrac{1}{2}.$$

We thus utilize similar arguments to those used to derive (29) and obtain

$$\left| \int \nabla \mathbf{y}_{\mathcal{U}}^{\intercal} \mathbf{v} \cdot \mathbf{w} \right| \leq \mathfrak{C} \|\nabla \mathbf{y}_{\mathcal{U}}\|_{\mathbf{L}^2(\rho,\Omega)} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \|\nabla \mathbf{w}\|_{\mathbf{L}^2(\Omega)}. \tag{30}$$

Here, $\mathfrak{C} = \mathcal{C}_e^2 \mathfrak{C}_{\rho,\Omega}$ and $\mathfrak{C}_{\rho,\Omega}$ denotes the constant defined as follows:

$$\|\rho^{-\frac{1}{2}}\|_{\mathbf{L}^\varsigma(\Omega)}^\varsigma = \left(\int_\Omega \rho^{-1-\frac{\delta}{2}} \right) \leq |\Omega|^{-\frac{\delta}{2}} \left(\int_\Omega \rho^{-1} \right)^{1+\frac{\delta}{2}} =: \mathfrak{C}_{\rho,\Omega}^\varsigma. \tag{31}$$

To derive the inequality in (31), we have utilized the reverse Hölder's inequality of Proposition 2 and the fact that ς can be written as $\varsigma = 2 + \delta$, for $\delta > 0$ arbitrarily small.

We now prove that $\mathcal{B}$ is coercive on $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$. To accomplish this task, let us first observe that, for $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, $\int_\Omega (\mathbf{y}_{\mathcal{U}} \cdot \nabla)\mathbf{v}\mathbf{v} = 0$. Consequently, for $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$,

$$\mathcal{B}(\mathbf{v}, \mathbf{v}) = \int_\Omega (\nu \nabla \mathbf{v} : \nabla \mathbf{v} + \nabla \mathbf{y}_{\mathcal{U}}^{\intercal} \mathbf{v} \cdot \mathbf{v}) \geq \nu \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2 - \mathfrak{C} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2 \|\nabla \mathbf{y}_{\mathcal{U}}\|_{\mathbf{L}^2(\rho,\Omega)}, \tag{32}$$

upon utilizing (30). We now invoke the smallness assumption (27) to obtain that $\mathcal{B}$ is coercive on $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$: $\mathcal{B}(\mathbf{v},\mathbf{v}) \geq \nu(1-\theta)\|\nabla \mathbf{v}\|^2_{\mathbf{L}^2(\Omega)}$ for every $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$.

The standard inf-sup condition for saddle point problems yields, on the basis of the surjectivity of the divergence operator, as a map from $\mathbf{H}_0^1(\Omega)$ to $\mathbf{L}^2(\Omega)/\mathbb{R}$ [1], the existence and uniqueness of a solution $(\mathbf{z}, r) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ and the stability estimate (28). This concludes the proof. ∎

The following result guarantees that point evaluations of the velocity component $\mathbf{z}$ of the adjoint pair $(\mathbf{z}, r)$ are well defined.

THEOREM 10 (regularity estimates). *If $(\mathbf{z}, r)$ solves (26), then the velocity component $\mathbf{z}$ satisfies $\mathbf{z} \in \mathbf{W}^{1,q}(\Omega)$ for some $q > 2$. Consequently, $\mathbf{z} \in \mathbf{C}(\bar{\Omega})$.*

Proof. We begin the proof by rewriting the adjoint equation as a Stokes problem:

$$\begin{aligned} \int_\Omega (\nu \nabla \mathbf{z} : \nabla \mathbf{w} - r \text{div } \mathbf{w}) &= \int_\Omega (\mathbf{y}_{\mathcal{U}} - \mathbf{y}_\Omega) \cdot \mathbf{w} + \int_\Omega \left[(\mathbf{y}_{\mathcal{U}} \cdot \nabla)\mathbf{z}\mathbf{w} - \nabla \mathbf{y}_{\mathcal{U}}^{\intercal} \mathbf{z} \cdot \mathbf{w} \right], \\ \int_\Omega s \text{div } \mathbf{z} &= 0, \end{aligned}$$

for all $(\mathbf{w}, s) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$.

Define the linear functional $\mathfrak{F} = \mathfrak{F}_1 - \mathfrak{F}_2$, where $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are defined as follows:

$$\mathfrak{F}_1, \mathfrak{F}_2 : \mathbf{H}_0^1(\Omega) \to \mathbb{R}, \qquad \mathfrak{F}_1(\mathbf{w}) := \int_\Omega (\mathbf{y}_{\mathcal{U}} \cdot \nabla)\mathbf{z}\mathbf{w}, \qquad \mathfrak{F}_2(\mathbf{w}) := \int_\Omega \nabla \mathbf{y}_{\mathcal{U}}^{\intercal} \mathbf{z} \cdot \mathbf{w}.$$

Let us prove that the functional $\mathfrak{F} \in \mathbf{W}^{-1,q}(\Omega) := \mathbf{W}_0^{1,q'}(\Omega)'$ for some $q > 2$. To accomplish this task, we first study the functional $\mathfrak{F}_1$. Observe that

$$\begin{aligned} \|\mathfrak{F}_1\|_{\mathbf{W}^{-1,q}(\Omega)} &:= \sup_{\mathbf{w} \in \mathbf{W}_0^{1,q'}(\Omega)} \frac{\mathfrak{F}_1(\mathbf{w})}{\|\nabla \mathbf{w}\|_{\mathbf{L}^{q'}(\Omega)}} \\ &\leq \sup_{\mathbf{w} \in \mathbf{W}_0^{1,q'}(\Omega)} \frac{\|\mathbf{y}_{\mathcal{U}}\|_{\mathbf{L}^\mu(\Omega)} \|\nabla \mathbf{z}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{w}\|_{\mathbf{L}^\kappa(\Omega)}}{\|\nabla \mathbf{w}\|_{\mathbf{L}^{q'}(\Omega)}}, \quad \kappa^{-1} + \mu^{-1} = \tfrac{1}{2}. \end{aligned} \tag{33}$$

We now invoke Theorem 3 to guarantee the existence of $\epsilon > 0$ such that $\mathbf{H}_0^1(\rho, \Omega) \hookrightarrow \mathbf{L}^{2+\epsilon}(\Omega)$. Set $\mu = 2 + \epsilon$, where $\epsilon > 0$ is dictated by Theorem 3. Since $\kappa^{-1} + \mu^{-1} = \frac{1}{2}$, we immediately arrive at $\kappa = 2(2+\epsilon)/\epsilon$. Observe that once ϵ is fixed, so is κ. Consequently, for some $q' < 2$, we have

$$\mathbf{W}_0^{1,q'}(\Omega) \hookrightarrow \mathbf{L}^\kappa(\Omega), \qquad \|\mathbf{w}\|_{\mathbf{L}^\kappa(\Omega)} \lesssim \|\nabla \mathbf{w}\|_{\mathbf{L}^{q'}(\Omega)}.$$

These arguments reveal the existence of $q > 2$ such that $\mathfrak{F}_1 \in \mathbf{W}^{-1,q}(\Omega)$, together with $\|\mathfrak{F}_1\|_{\mathbf{W}^{-1,q}(\Omega)} \lesssim \|\nabla \mathbf{y}_{\mathcal{U}}\|_{\mathbf{L}^2(\rho,\Omega)} \|\nabla \mathbf{z}\|_{\mathbf{L}^2(\Omega)}$. Similar arguments to the ones used to obtain (30), (31), and the previous bound yield $\mathfrak{F}_2 \in \mathbf{W}^{-1,q}(\Omega)$ for some $q > 2$.

Having obtained the existence of $q > 2$ such that $\mathfrak{F} \in \mathbf{W}^{-1,q}(\Omega)$, it suffices to invoke [29, (1.52)] to conclude that $\mathbf{z} \in \mathbf{W}^{1,q}(\Omega)$. This concludes the proof. ∎

THEOREM 11 (weighted integrability). *If the pair $(\mathbf{z}, r)$ solves (26), then $(\mathbf{z}, r) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)$.*

Proof. We prove that $\mathbf{z} \in \mathbf{H}_0^1(\rho^{-1}, \Omega)$; the proof of $r \in L^2(\rho^{-1}, \Omega)$ being similar. For each $t \in \mathcal{D}$, we denote by B_t the ball with center t and radius $d_{\mathcal{D}}/2$; $d_{\mathcal{D}}$ being defined as in (5). Set $A = \Omega \setminus \cup_{t \in \mathcal{D}} B_t$ and observe that

$$\int_\Omega \rho^{-1} |\nabla \mathbf{z}|^2 = \int_A \rho^{-1} |\nabla \mathbf{z}|^2 + \sum_{t \in \mathcal{D}} \int_{B_t} \rho^{-1} |\nabla \mathbf{z}|^2.$$

The definition of the weight ρ, given by (6), combined with the stability estimate (28) for problem (26) allow us to control the integral over A:

$$\int_A \rho^{-1}|\nabla \mathbf{z}|^2 \lesssim \int_A |\nabla \mathbf{z}|^2 \lesssim \int_\Omega |\mathbf{y}_{\mathcal{U}} - \mathbf{y}_\Omega|^2.$$

We now bound the integral near the support of the Dirac measures on the basis of an interior regularity result for the Stokes problem [21]. Since $\ell = \#\mathcal{D}$ is finite, it suffices to consider a single ball B_t. Let us first invoke Theorem 3 to obtain that, form some $\epsilon > 0$, $\mathbf{y}_{\mathcal{U}}$ belongs to $\mathbf{L}^{2+\epsilon}(\Omega)$. The assumption on $\mathbf{y}_\Omega$ thus implies that $\mathbf{y}_{\mathcal{U}} - \mathbf{y}_\Omega \in \mathbf{L}^2(\Omega)$. To control the term $(\mathbf{y}_{\mathcal{U}} \cdot \nabla)\mathbf{z}$ we invoke a finer result: $\mathbf{y}_{\mathcal{U}} \in \mathbf{L}^q(\Omega)$ for every $q < \infty$; see [27, Section 3] for details. Thus, since $\mathbf{z} \in \mathbf{W}^{1,q}(\Omega)$, for some $q > 2$, we have that $(\mathbf{y}_{\mathcal{U}} \cdot \nabla)\mathbf{z}$ belongs to $\mathbf{L}^2(\Omega)$. It suffices to control the term $\nabla \mathbf{y}_{\mathcal{U}}^{\intercal}\mathbf{z}$. To accomplish this task, we invoke the regularity results of Theorem 10 and the results of [27, Section 3], again, to conclude that, for every $q < 2$,

$$\|\nabla \mathbf{y}_{\mathcal{U}}^{\intercal}\mathbf{z}\|_{\mathbf{L}^q(\Omega)}^q \leq \|\mathbf{z}\|_{\mathbf{L}^\infty(\Omega)} \int_\Omega |\nabla \mathbf{y}_{\mathcal{U}}|^q < \infty.$$

Consequently, the interior regularity result for the Stokes problem derived in [21, Theorem IV.4.1] allow us to obtain that $\mathbf{z} \in \mathbf{W}^{2,q}_{\mathrm{loc}}(\Omega)$ for every $q < 2$. This, on the basis of a standard Sobolev embedding, reveals that $\nabla \mathbf{z} \in \mathbf{W}^{1,\vartheta}_{\mathrm{loc}}(\Omega)$ for $\vartheta \leq 2q/(2-q)$.

Let $\epsilon > 0$. Invoke Hölder's inequality to obtain

$$\int_{B_t} \rho^{-1}|\nabla \mathbf{z}|^2 \leq \left(\int_{B_t} \rho^{-(1+\epsilon)}\right)^{\frac{1}{1+\epsilon}} \left(\int_{B_t} |\nabla \mathbf{z}|^{\frac{2(1+\epsilon)}{\epsilon}}\right)^{\frac{\epsilon}{1+\epsilon}}. \tag{34}$$

Choosing the value of ϵ given by Proposition 2, the reverse Hölder's inequality yields the control of the first integral on the right hand side of the previous expression:

$$\left(\int_{B_t} \rho^{-(1+\epsilon)}\right)^{\frac{1}{1+\epsilon}} \lesssim |B_t|^{-\frac{\epsilon}{1+\epsilon}} \int_{B_t} \rho^{-1} < \infty.$$

The hidden constant only depends on $[\omega]_{A_2}$. To control the second integral, we utilize the previously derived regularity result for $\nabla \mathbf{z}$, namely, $\nabla \mathbf{z} \in \mathbf{W}^{1,\vartheta}_{\mathrm{loc}}(\Omega)$ for $\vartheta \leq 2q/(2-q)$. Observe that once $\epsilon > 0$ is fixed, so is $2(1+\epsilon)/\epsilon$. Therefore, since

$$\lim_{q \uparrow 2} \frac{2q}{2-q} = +\infty \implies \exists q_0 < 2 : \frac{2q_0}{2-q_0} \geq \frac{2(1+\epsilon)}{\epsilon}.$$

This allows us to control the second integral on the right hand side of (34). This concludes the proof. ◻

5.2. First order optimality conditions. With the regularity results of Theorems 10 and 11 at hand, we derive first order optimality conditions for our optimal control problem. As customary, we begin by introducing the reduced cost functional:

$$j : \mathbb{U}_{ad} \to \mathbb{R}, \qquad j(\mathcal{U}) := \frac{1}{2}\|\mathcal{G}(\mathcal{U}) - \mathbf{y}_\Omega\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\eta}{2}\sum_{t \in \mathcal{D}} |\mathbf{u}_t|^2, \quad \mathcal{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_\ell). \tag{35}$$

Here, $\mathcal{G}$ denotes the operator defined in (11). Strictly speaking, in (35), $\mathcal{G}(\mathcal{U})$ should be replaced by $\mathcal{G} \circ \mathcal{H}(\mathcal{U})$; $\mathcal{H}$ being defined in (24). By abuse of notation and for the sake of simplicity, we will simply utilize the notation $\mathcal{G}(\mathcal{U})$ to denote $\mathcal{G} \circ \mathcal{H}(\mathcal{U})$.

The following result is standard: If $\bar{\mathcal{U}}$ denotes a locally optimal control for problem (21)–(22), then we have the variational inequality [37, Lemma 4.18]

$$j'(\bar{\mathcal{U}})(\mathcal{U}-\bar{\mathcal{U}}) \geq 0 \quad \forall \mathcal{U} \in \mathbb{U}_{ad}. \tag{36}$$

We now proceed to show first order optimality conditions for our optimal control problem (21)–(22).

THEOREM 12 (first order optimality conditions). *Assume that the smallness assumptions* (25) *and* (27) *hold. If* $(\bar{\mathbf{y}}, \bar{p}, \bar{\mathcal{U}}) \in \mathbf{H}_0^1(\rho,\Omega) \times L^2(\rho,\Omega)/\mathbb{R} \times \mathbb{U}_{ad}$ *is optimal for the control problem* (21)–(22)*, then* $\bar{\mathcal{U}} \in \mathbb{U}_{ad}$ *satisfies the variational inequality*

$$\sum_{t\in\mathcal{D}} (\bar{\mathbf{z}}(t) + \eta \bar{\mathbf{u}}_t) \cdot (\mathbf{u}_t - \bar{\mathbf{u}}_t) \geq 0 \qquad \forall \mathcal{U} = (\mathbf{u}_1, \dots, \mathbf{u}_\ell) \in \mathbb{U}_{ad}, \tag{37}$$

where $(\bar{\mathbf{z}}, \bar{r}) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ *denotes the optimal adjoint pair that solves the adjoint problem* (26) *with* $\mathbf{y}_{\mathcal{U}}$ *replaced by* $\bar{\mathbf{y}}$*.*

Proof. We begin the proof by computing the expression $j'(\bar{\mathcal{U}})(\mathcal{U}-\bar{\mathcal{U}})$ and rewriting the basic variational inequality (36) as follows:

$$\int_\Omega (\bar{\mathbf{y}} - \mathbf{y}_\Omega) \cdot \mathcal{G}'(\bar{\mathcal{U}})(\mathcal{U}-\bar{\mathcal{U}}) + \eta \sum_{t\in\mathcal{D}} \bar{\mathbf{u}}_t \cdot (\mathbf{u}_t - \bar{\mathbf{u}}_t) \geq 0 \qquad \forall \mathcal{U} = (\mathbf{u}_1, \dots, \mathbf{u}_l) \in \mathbb{U}_{ad}. \tag{38}$$

Define $\boldsymbol{\theta} := \mathcal{G}'(\bar{\mathcal{U}})(\mathcal{U}-\bar{\mathcal{U}})$ and observe that $(\boldsymbol{\theta}, \xi) \in \mathbf{H}_0^1(\rho,\Omega) \times L^2(\rho,\Omega)/\mathbb{R}$ solves

$$\begin{aligned} \int_\Omega (\nu \nabla \boldsymbol{\theta} : \nabla \mathbf{v} - \bar{\mathbf{y}} \otimes \boldsymbol{\theta} : \nabla \mathbf{v} - \boldsymbol{\theta} \otimes \bar{\mathbf{y}} : \nabla \mathbf{v} - \xi \text{div } \mathbf{v}) \\ = \sum_{t\in\mathcal{D}} \langle (\mathbf{u}_t - \bar{\mathbf{u}}_t)\delta_t, \mathbf{v} \rangle, \qquad \int_\Omega q \text{div } \boldsymbol{\theta} = 0, \end{aligned} \tag{39}$$

for all $(\mathbf{v}, q) \in \mathbf{H}_0^1(\rho^{-1},\Omega) \times L^2(\rho^{-1},\Omega)/\mathbb{R}$. Having introduced the pair $(\boldsymbol{\theta}, \xi)$, the variational inequality (38) becomes

$$\int_\Omega (\bar{\mathbf{y}} - \mathbf{y}_\Omega) \cdot \boldsymbol{\theta} + \eta \sum_{t\in\mathcal{D}} \bar{\mathbf{u}}_t \cdot (\mathbf{u}_t - \bar{\mathbf{u}}_t) \geq 0 \qquad \forall \mathcal{U} = (\mathbf{u}_1, \dots, \mathbf{u}_l) \in \mathbb{U}_{ad}. \tag{40}$$

Since the second term on the right hand side of the previous expression is already present in the desired variational inequality (37), we focus on the first term.

In view of the results of Theorem 11, we are allowed to set $(\bar{\mathbf{z}}, \bar{r}) \in \mathbf{H}_0^1(\rho^{-1},\Omega) \times L^2(\rho^{-1},\Omega)/\mathbb{R}$ as a test pair in problem (39). This yields

$$\int_\Omega (\nu \nabla \boldsymbol{\theta} : \nabla \bar{\mathbf{z}} - \bar{\mathbf{y}} \otimes \boldsymbol{\theta} : \nabla \bar{\mathbf{z}} - \boldsymbol{\theta} \otimes \bar{\mathbf{y}} : \nabla \bar{\mathbf{z}}) = \sum_{t\in\mathcal{D}} (\mathbf{u}_t - \bar{\mathbf{u}}_t) \cdot \bar{\mathbf{z}}(t), \tag{41}$$

upon utilizing the fact that $\int_\Omega \xi \text{div } \bar{\mathbf{z}}$ vanishes and that there exists $q > 2$ such that $\bar{\mathbf{z}} \in \mathbf{W}^{1,q}(\Omega) \hookrightarrow \mathbf{C}(\bar{\Omega})$ (cf. Theorem 10). We would like to set now $\mathbf{w} = \boldsymbol{\theta}$ in the first equation of problem (26) to obtain

$$\int_\Omega (\nu \nabla \bar{\mathbf{z}} : \nabla \boldsymbol{\theta} - (\bar{\mathbf{y}} \cdot \nabla)\bar{\mathbf{z}}\boldsymbol{\theta} + (\nabla \bar{\mathbf{y}})^\intercal \bar{\mathbf{z}} \cdot \boldsymbol{\theta} - \bar{r} \text{div } \boldsymbol{\theta}) = \int_\Omega (\bar{\mathbf{y}} - \mathbf{y}_\Omega) \cdot \boldsymbol{\theta}. \tag{42}$$

However, $\boldsymbol{\theta} \in \mathbf{H}_0^1(\rho,\Omega) \setminus \mathbf{H}_0^1(\Omega)$, i.e., $\boldsymbol{\theta}$ is not a valid test function for the first equation of problem (26). Consequently, (42) must be justified by utilizing different arguments.

We proceed on the basis of a density argument as in [4, Theorem 3.4]. Let $\{\boldsymbol{\theta}_n\}_{n\geq 0} \subset \mathbf{C}_0^\infty(\Omega)$ be a sequence such that $\boldsymbol{\theta}_n \to \boldsymbol{\theta}$ in $\mathbf{H}_0^1(\rho,\Omega)$ as $n \uparrow \infty$. Set $\mathbf{w} = \boldsymbol{\theta}_n$ in the first equation of problem (26) to obtain

$$\int_\Omega (\nu \nabla \bar{\mathbf{z}} : \nabla \boldsymbol{\theta}_n - (\bar{\mathbf{y}} \cdot \nabla)\bar{\mathbf{z}}\boldsymbol{\theta}_n + \nabla \bar{\mathbf{y}}^\intercal \bar{\mathbf{z}} \cdot \boldsymbol{\theta}_n - \bar{r}\text{div } \boldsymbol{\theta}_n) = \int_\Omega (\bar{\mathbf{y}} - \mathbf{y}_\Omega) \cdot \boldsymbol{\theta}_n. \tag{43}$$

Since the weight $\rho \in A_2$, Theorem 3 guarantees the existence of $\epsilon > 0$ such that the weighted Sobolev embedding $\mathbf{H}_0^1(\rho,\Omega) \hookrightarrow \mathbf{L}^{2+\epsilon}(\Omega)$ holds. Consequently,

$$\left| \int_\Omega (\bar{\mathbf{y}} - \mathbf{y}_\Omega) \cdot (\boldsymbol{\theta} - \boldsymbol{\theta}_n) \right| \lesssim \|\bar{\mathbf{y}} - \mathbf{y}_\Omega\|_{\mathbf{L}^2(\Omega)} \|\nabla(\boldsymbol{\theta} - \boldsymbol{\theta}_n)\|_{\mathbf{L}^2(\rho,\Omega)} \to 0, \quad n \uparrow \infty.$$

To control the convective terms, we utilize the continuous embedding $\mathbf{H}_0^1(\rho,\Omega) \hookrightarrow \mathbf{L}^4(\rho,\Omega)$ [17, Theorem 1.3] and proceed as follows:

$$\begin{aligned} \left| \int_\Omega (\bar{\mathbf{y}} \cdot \nabla)\bar{\mathbf{z}}(\boldsymbol{\theta} - \boldsymbol{\theta}_n) \right| &\leq \|\bar{\mathbf{y}}\|_{\mathbf{L}^4(\rho,\Omega)} \|\nabla \bar{\mathbf{z}}\|_{\mathbf{L}^2(\rho^{-1},\Omega)} \|\boldsymbol{\theta} - \boldsymbol{\theta}_n\|_{\mathbf{L}^4(\rho,\Omega)} \\ &\lesssim \|\nabla \bar{\mathbf{y}}\|_{\mathbf{L}^2(\rho,\Omega)} \|\nabla \bar{\mathbf{z}}\|_{\mathbf{L}^2(\rho^{-1},\Omega)} \|\nabla(\boldsymbol{\theta} - \boldsymbol{\theta}_n)\|_{\mathbf{L}^2(\rho,\Omega)} \to 0 \end{aligned}$$

as $n \uparrow \infty$. Notice that Theorem 11 guarantees $\mathbf{z} \in \mathbf{H}_0^1(\rho^{-1},\Omega)$. Similarly,

$$\begin{aligned} \left| \int_\Omega (\nabla \bar{\mathbf{y}})^\intercal \bar{\mathbf{z}} \cdot (\boldsymbol{\theta} - \boldsymbol{\theta}_n) \right| &\leq \|\nabla \bar{\mathbf{y}}\|_{\mathbf{L}^\mu(\Omega)} \|\bar{\mathbf{z}}\|_{\mathbf{L}^\kappa(\Omega)} \|\boldsymbol{\theta} - \boldsymbol{\theta}_n\|_{\mathbf{L}^\varsigma(\Omega)} \\ &\lesssim \|\nabla \bar{\mathbf{y}}\|_{\mathbf{L}^\mu(\Omega)} \|\nabla \bar{\mathbf{z}}\|_{\mathbf{L}^2(\Omega)} \|\nabla(\boldsymbol{\theta} - \boldsymbol{\theta}_n)\|_{\mathbf{L}^2(\rho,\Omega)} \to 0 \end{aligned}$$

as $n \to \infty$. Here, μ, κ, and ς are such that $\mu^{-1} + \kappa^{-1} + \varsigma^{-1} = 1$. To obtain the previous result, we have utilized that $\nabla \bar{\mathbf{y}} \in \mathbf{L}^q(\Omega)$ for every $q < 2$ [27, Section 3]. We have also utilized that there exists $\epsilon > 0$ such that $\mathbf{H}_0^1(\rho,\Omega) \hookrightarrow \mathbf{L}^{2+\epsilon}(\Omega)$, and the Sobolev embedding $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^q(\Omega)$, which holds for every $q < \infty$. We also have

$$\left| \int_\Omega \nabla \bar{\mathbf{z}} : \nabla(\boldsymbol{\theta} - \boldsymbol{\theta}_n) \right| \leq \|\nabla \bar{\mathbf{z}}\|_{\mathbf{L}^2(\rho^{-1},\Omega)} \|\nabla(\boldsymbol{\theta} - \boldsymbol{\theta}_n)\|_{\mathbf{L}^2(\rho,\Omega)} \to 0$$

as $n \to \infty$. Finally, we have $\left|\int_\Omega \bar{r}\text{div}(\boldsymbol{\theta} - \boldsymbol{\theta}_n)\right| \lesssim \|\bar{r}\|_{L^2(\rho^{-1},\Omega)} \|\nabla(\boldsymbol{\theta} - \boldsymbol{\theta}_n)\|_{\mathbf{L}^2(\rho,\Omega)} \to 0$ as $n \uparrow \infty$.

Therefore, upon utilizing (41), (42) and the fact that $\int_\Omega \bar{r}\text{div } \boldsymbol{\theta} = 0$, together with an integration by parts in the convective terms of (42), we arrive at the needed relation $\sum_{t\in\mathcal{D}} (\mathbf{u}_t - \bar{\mathbf{u}}_t) \cdot \bar{\mathbf{z}}(t) = \int_\Omega (\bar{\mathbf{y}} - \mathbf{y}_\Omega) \cdot \boldsymbol{\theta}$. This identity, in view of (38), concludes the proof. ∎

5.3. Second order optimality conditions. In this section, we follow [12, 9] and analyze necessary and sufficient second order optimality conditions.

5.3.1. Preliminaries. We begin our studies with the following estimate.

LEMMA 13 (auxiliary estimate). *Let $\mathcal{U}, \hat{\mathcal{U}} \in \mathbb{U}_{ad}$ and $\mathcal{V} \in [\mathbb{R}^2]^\ell$. Let $\mathbf{y} = \mathcal{G}(\mathcal{U})$, $\hat{\mathbf{y}} = \mathcal{G}(\hat{\mathcal{U}})$, $(\boldsymbol{\theta}, \xi) = \mathcal{S}'(\mathcal{U})\mathcal{V}$, and $(\hat{\boldsymbol{\theta}}, \hat{\xi}) = \mathcal{S}'(\hat{\mathcal{U}})\mathcal{V}$. Then, we have the estimate*

$$\|\nabla(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})\|_{\mathbf{L}^2(\rho,\Omega)} \lesssim \|\mathcal{U} - \hat{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell} \|\mathcal{V}\|_{[\mathbb{R}^2]^\ell}. \tag{44}$$

Proof. We begin the proof by noticing that the pair $(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}},\xi-\hat{\xi})$ solves the following problem: Find $(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}},\xi-\hat{\xi})\in\mathbf{H}_0^1(\rho,\Omega)\times L^2(\rho,\Omega)/\mathbb{R}$ such that

$$\int_\Omega(\nu\nabla(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}):\nabla\mathbf{v}-\hat{\mathbf{y}}\otimes(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}):\nabla\mathbf{v}-(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})\otimes\mathbf{y}:\nabla\mathbf{v}-(\xi-\hat{\xi})\text{div }\mathbf{v}) \\ =\int_\Omega[(\mathbf{y}-\hat{\mathbf{y}})\otimes\boldsymbol{\theta}+\hat{\boldsymbol{\theta}}\otimes(\mathbf{y}-\hat{\mathbf{y}})]:\nabla\mathbf{v},\quad\int_\Omega q\text{div }(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})=0 \tag{45}$$

for all $(\mathbf{v},q)\in\mathbf{H}_0^1(\rho^{-1},\Omega)\times L^2(\rho^{-1},\Omega)/\mathbb{R}$. Since $\mathcal{U},\hat{\mathcal{U}}\in\mathbb{U}_{ad}$ and the smallness assumption (25) holds, similar arguments to the ones utilized to obtain the results of Lemma 4 yield that problem (45) is well posed. In particular, we have the bound

$$\begin{aligned}\|\nabla(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})\|_{\mathbf{L}^2(\rho,\Omega)}&\lesssim\|(\mathbf{y}-\hat{\mathbf{y}})\otimes\boldsymbol{\theta}+\hat{\boldsymbol{\theta}}\otimes(\mathbf{y}-\hat{\mathbf{y}})\|_{\mathbf{H}_0^1(\rho^{-1},\Omega)'}\\&\lesssim\left[\|\nabla\boldsymbol{\theta}\|_{\mathbf{L}^2(\rho,\Omega)}+\|\nabla\hat{\boldsymbol{\theta}}\|_{\mathbf{L}^2(\rho,\Omega)}\right]\|\nabla(\mathbf{y}-\hat{\mathbf{y}})\|_{\mathbf{L}^2(\rho,\Omega)}\\&\lesssim\|\mathcal{V}\|_{[\mathbb{R}^2]^\ell}\|\nabla(\mathbf{y}-\hat{\mathbf{y}})\|_{\mathbf{L}^2(\rho,\Omega)}\lesssim\|\mathcal{V}\|_{[\mathbb{R}^2]^\ell}\|\mathcal{U}-\hat{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell},\end{aligned}$$

upon utilizing Hölder's inequality, the weighted embedding $\mathbf{H}_0^1(\rho,\Omega)\hookrightarrow\mathbf{L}^4(\rho,\Omega)$, the stability estimate (14), and the Lipschitz property of Theorem 5 for the difference $\mathbf{y}-\hat{\mathbf{y}}=\mathcal{G}(\mathcal{U})-\mathcal{G}(\hat{\mathcal{U}})$. This concludes the proof. ∎

The following result is instrumental.

THEOREM 14 (properties of j''). *The reduced cost functional $j:\mathbb{U}_{ad}\to\mathbb{R}$ is of class C^2. In addition, for $\mathcal{U}\in\mathbb{U}_{ad}$ and $\mathcal{V},\mathcal{W}\in[\mathbb{R}^2]^\ell$, we have the identity*

$$j''(\mathcal{U})(\mathcal{V},\mathcal{W})=\int_\Omega\boldsymbol{\theta}_\mathcal{V}\cdot\boldsymbol{\theta}_\mathcal{W}+\int_\Omega[\boldsymbol{\theta}_\mathcal{V}\otimes\boldsymbol{\theta}_\mathcal{W}+\boldsymbol{\theta}_\mathcal{W}\otimes\boldsymbol{\theta}_\mathcal{V}]:\nabla\mathbf{z}+\eta\sum_{t\in\mathcal{D}}\mathbf{v}_t\cdot\mathbf{w}_t. \tag{46}$$

Here, $(\boldsymbol{\theta}_\mathcal{V},\zeta_\mathcal{V})$ and $(\boldsymbol{\theta}_\mathcal{W},\zeta_\mathcal{W})$ denote the unique solutions to (12) with forcing terms $\mathbf{g}=\sum_{t\in D}\delta_t\mathbf{v}_t$ and $\mathbf{g}=\sum_{t\in D}\delta_t\mathbf{w}_t$, respectively, and $(\mathbf{z},r)$ denotes the unique solution to problem (26). If $\mathcal{U},\hat{\mathcal{U}}\in\mathbb{U}_{ad}$ and $\mathcal{V}\in[\mathbb{R}^2]^\ell$, we have the bound

$$|j''(\mathcal{U})\mathcal{V}^2-j''(\hat{\mathcal{U}})\mathcal{V}^2|\lesssim\|\mathcal{U}-\hat{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell}\|\mathcal{V}\|^2_{[\mathbb{R}^2]^\ell}. \tag{47}$$

Proof. Since Theorem 6 guarantees that $\mathcal{S}$ is second order Fréchet differentiable, it is immediate that $\mathcal{G}$, defined in (11), is also second order Fréchet differentiable as a map from $\mathbf{H}_0^1(\rho^{-1},\Omega)'$ into $\mathbf{H}_0^1(\rho,\Omega)$. This implies that j is of class C^2. It thus suffices to derive the identity (46) and the bound (47). To accomplish this task, we begin with a basic computation, which reveals, for $\mathcal{U}\in\mathbb{U}_{ad}$ and $\mathcal{V},\mathcal{W}\in[\mathbb{R}^2]^\ell$, that

$$j''(\mathcal{U})(\mathcal{V},\mathcal{W})=\int_\Omega\mathcal{G}'(\mathcal{U})\mathcal{V}\cdot\mathcal{G}'(\mathcal{U})\mathcal{W}+\int_\Omega(\mathcal{G}(\mathcal{U})-\mathbf{y}_\Omega)\cdot\mathcal{G}''(\mathcal{U})\mathcal{V}\mathcal{W}+\eta\sum_{t\in\mathcal{D}}\mathbf{v}_t\cdot\mathbf{w}_t. \tag{48}$$

Let $(\boldsymbol{\psi},\gamma)=\mathcal{S}''(\mathcal{U})\mathcal{V}\mathcal{W}$ be the unique solution to (17) with $\boldsymbol{\theta}_{\mathbf{g}_1}=\boldsymbol{\theta}_\mathcal{V}$, $\boldsymbol{\theta}_{\mathbf{g}_2}=\boldsymbol{\theta}_\mathcal{W}$, and $\boldsymbol{\Phi}=\mathbf{y}_\mathcal{U}=\mathcal{G}(\mathcal{U})$. Notice that $\boldsymbol{\psi}=\mathcal{G}''(\mathcal{U})\mathcal{V}\mathcal{W}$. Since the adjoint pair $(\mathbf{z},r)$, i.e., the solution to (26), is such that $(\mathbf{z},r)\in\mathbf{H}_0^1(\rho^{-1},\Omega)\times L^2(\rho^{-1},\Omega)$ (cf. Theorem 11), we are allowed to set $\mathbf{v}=\mathbf{z}$ in the first equation of the problem that $(\boldsymbol{\psi},\gamma)$ solves. On the other hand, a similar approximation argument to that used in the proof of Theorem 12 allows us to set $\mathbf{w}=\boldsymbol{\psi}$ in the first equation of (26). The derived identities yield

$$\int_\Omega(\mathcal{G}(\mathcal{U})-\mathbf{y}_\Omega)\cdot\boldsymbol{\psi}=\int_\Omega[(\boldsymbol{\theta}_\mathcal{V}\otimes\boldsymbol{\theta}_\mathcal{W}+\boldsymbol{\theta}_\mathcal{W}\otimes\boldsymbol{\theta}_\mathcal{V}]:\nabla\mathbf{z}.$$

Replacing the previous identity into (48) yields (46).

We now proceed to prove the bound (47). Define $(\boldsymbol{\theta},\zeta) = \mathcal{S}'(\mathcal{U})\mathcal{V}$ and $(\hat{\boldsymbol{\theta}},\hat{\zeta}) = \mathcal{S}'(\hat{\mathcal{U}})\mathcal{V}$. We notice that $(\boldsymbol{\theta},\zeta)$ and $(\hat{\boldsymbol{\theta}},\hat{\zeta})$ solve problem (12) with $\boldsymbol{\Phi} = \mathcal{G}(\mathcal{U})$ and $\boldsymbol{\Phi} = \mathcal{G}(\hat{\mathcal{U}})$, respectively. In view of the identity (46), we thus obtain

$$(49)\quad j''(\mathcal{U})\mathcal{V}^2 - j''(\hat{\mathcal{U}})\mathcal{V}^2 = \int_\Omega (\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})\cdot(\boldsymbol{\theta}+\hat{\boldsymbol{\theta}}) + 2\left[\int_\Omega \boldsymbol{\theta}\otimes(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) : \nabla\mathbf{z} + \int_\Omega \boldsymbol{\theta}\otimes\hat{\boldsymbol{\theta}} : \nabla(\mathbf{z}-\hat{\mathbf{z}}) + \int_\Omega (\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})\otimes\hat{\boldsymbol{\theta}} : \nabla\hat{\mathbf{z}}\right].$$

Here, $(\mathbf{z},r)$ and $(\hat{\mathbf{z}},\hat{r})$ denote the unique solutions to problem (26) with $\mathbf{y}_{\mathcal{U}}$ and $\mathbf{y}_{\hat{\mathcal{U}}}$, respectively. Invoke Hölder's inequality and the Sobolev embeddings $\mathbf{H}_0^1(\rho,\Omega)\hookrightarrow \mathbf{L}^4(\rho,\Omega)$ (cf. [17, Theorem 1.3]) and $\mathbf{H}_0^1(\rho,\Omega)\hookrightarrow \mathbf{L}^2(\Omega)$ (cf. Theorem 3), to obtain

$$|j''(\mathcal{U})\mathcal{V}^2 - j''(\hat{\mathcal{U}})\mathcal{V}^2| \lesssim \|\nabla(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})\|_{\mathbf{L}^2(\rho,\Omega)}\big(\|\nabla\boldsymbol{\theta}\|_{\mathbf{L}^2(\rho,\Omega)}\left[1+\|\nabla\mathbf{z}\|_{\mathbf{L}^2(\rho^{-1},\Omega)}\right] + \|\nabla\hat{\boldsymbol{\theta}}\|_{\mathbf{L}^2(\rho,\Omega)}\left[1+\|\nabla\hat{\mathbf{z}}\|_{\mathbf{L}^2(\rho^{-1},\Omega)}\right]\big) + \|\boldsymbol{\theta}\|_{\mathbf{L}^4(\Omega)}\|\hat{\boldsymbol{\theta}}\|_{\mathbf{L}^4(\Omega)}\|\nabla(\mathbf{z}-\hat{\mathbf{z}})\|_{\mathbf{L}^2(\Omega)}.$$

This bound, combined with the stability estimate (14), the auxiliary estimate (44), and the boundedness of $\hat{\mathbf{z}},\mathbf{z}$ in $\mathbf{H}_0^1(\rho^{-1},\Omega)$ yield

$$(50)\qquad |j''(\mathcal{U})\mathcal{V}^2 - j''(\hat{\mathcal{U}})\mathcal{V}^2| \lesssim \|\mathcal{U}-\hat{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell}\|\mathcal{V}\|^2_{[\mathbb{R}^2]^\ell} + \|\nabla(\mathbf{z}-\hat{\mathbf{z}})\|_{\mathbf{L}^2(\Omega)}\|\mathcal{V}\|^2_{[\mathbb{R}^2]^\ell},$$

where we have also used that $\boldsymbol{\theta},\hat{\boldsymbol{\theta}}\in\mathbf{W}^{1,p}(\Omega)$ and $\|\nabla\boldsymbol{\theta}\|_{\mathbf{L}^p(\Omega)}, \|\nabla\hat{\boldsymbol{\theta}}\|_{\mathbf{L}^p(\Omega)} \lesssim \|\mathcal{V}\|_{[\mathbb{R}^2]^\ell}$. The latter follows from an adaption of the arguments developed in the proof of Lemma 4 to a $\mathbf{W}^{1,p}(\Omega)$ setting for $p<2$. Therefore, it suffices to bound $\|\nabla(\mathbf{z}-\hat{\mathbf{z}})\|_{\mathbf{L}^2(\Omega)}$. To accomplish this goal, we notice that $(\mathbf{z}-\hat{\mathbf{z}}, r-\hat{r})\in\mathbf{H}_0^1(\Omega)\times L^2(\Omega)/\mathbb{R}$ solves

$$(51)\quad \int_\Omega (\nu\nabla(\mathbf{z}-\hat{\mathbf{z}}) : \nabla\mathbf{w} - (\mathbf{y}_{\mathcal{U}}\cdot\nabla)(\mathbf{z}-\hat{\mathbf{z}})\mathbf{w} + \nabla\mathbf{y}_{\mathcal{U}}^{\intercal}(\mathbf{z}-\hat{\mathbf{z}})\cdot\mathbf{w} - (r-\hat{r})\mathrm{div}\,\mathbf{w})$$
$$= \int_\Omega (\mathbf{y}_{\mathcal{U}}-\mathbf{y}_{\hat{\mathcal{U}}})\cdot\mathbf{w} + \int_\Omega ([\mathbf{y}_{\mathcal{U}}-\mathbf{y}_{\hat{\mathcal{U}}}]\cdot\nabla)\hat{\mathbf{z}}\mathbf{w} + \int_\Omega [\nabla(\mathbf{y}_{\hat{\mathcal{U}}}-\mathbf{y}_{\mathcal{U}})]^{\intercal}\hat{\mathbf{z}}\cdot\mathbf{w}, \quad \int_\Omega s\,\mathrm{div}(\mathbf{z}-\hat{\mathbf{z}}) = 0,$$

for all $(\mathbf{w},s)\in\mathbf{H}_0^1(\Omega)\times L^2(\Omega)/\mathbb{R}$. In view of the stability estimate (28), we bound the $\mathbf{H}^{-1}(\Omega)$-norm of the right hand side of the first equation of (51) and obtain

$$\|\nabla(\mathbf{z}-\hat{\mathbf{z}})\|_{\mathbf{L}^2(\Omega)} \lesssim \|\nabla(\mathbf{y}_{\mathcal{U}}-\mathbf{y}_{\hat{\mathcal{U}}})\|_{\mathbf{W}^{1,\kappa}(\Omega)} \lesssim \|\nabla(\mathbf{y}_{\mathcal{U}}-\mathbf{y}_{\hat{\mathcal{U}}})\|_{\mathbf{L}^2(\rho,\Omega)},$$

where $\kappa>1$ is sufficiently close to 1. To obtain the first estimate above, we have used the fact that $\hat{\mathbf{z}}\in\mathbf{W}^{1,q}(\Omega)$ for some $q>2$ (cf. Theorem 10) and basic Sobolev embeddings. The second bound follows from the embedding $\mathbf{H}_0^1(\rho,\Omega)\hookrightarrow\mathbf{W}^{1,\kappa}(\Omega)$, which follows from utilizing the reverse Hölder inequality of Proposition 2 and the fact that $\kappa>1$ is sufficiently close to 1 (cf. Theorem 3). The desired estimate (47) thus follows from the Lipschitz property of Theorem 5, which yields the bound $\|\nabla(\mathbf{y}_{\mathcal{U}}-\mathbf{y}_{\hat{\mathcal{U}}})\|_{\mathbf{L}^2(\rho,\Omega)} \lesssim \|\mathcal{U}-\hat{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell}$. This concludes the proof. ∎

Before starting with the analysis of necessary and sufficient second order optimality conditions, we introduce the last two ingredients. We define

$$(52)\qquad \boldsymbol{\Psi} := (\boldsymbol{\Psi}_1,\ldots,\boldsymbol{\Psi}_\ell)\in[\mathbb{R}^2]^\ell, \qquad \boldsymbol{\Psi}_t := \bar{\mathbf{z}}(t)+\eta\bar{\mathbf{u}}_t, \qquad t\in\mathcal{D}.$$

Let $s\in\mathcal{D}$ and $\mathcal{U}\in\mathbb{U}_{ad}$ be such that $\mathbf{u}_t=\bar{\mathbf{u}}_t$ for $t\in\mathcal{D}\setminus\{s\}$. Set $\mathcal{U}$ into the variational inequality (37). This yields

$$(53)\qquad 0 \le (\bar{\mathbf{z}}(s)+\eta\bar{\mathbf{u}}_s)\cdot(\mathbf{u}_s-\bar{\mathbf{u}}_s) = \boldsymbol{\Psi}_s\cdot(\mathbf{u}_s-\bar{\mathbf{u}}_s).$$

Let $i, j \in \{1,2\}$ be such that $i \neq j$. Set $(\mathbf{u}_s)_i = (\bar{\mathbf{u}}_s)_i$. If $(\bar{\mathbf{u}}_s)_j = (\mathbf{a}_s)_j$, then inequality (53) reveals that

$$0 \leq \mathbf{\Psi}_s \cdot (\mathbf{u}_s - \bar{\mathbf{u}}_s) = (\mathbf{\Psi}_s)_j[(\mathbf{u}_s)_j - (\mathbf{a}_s)_j] \implies (\mathbf{\Psi}_s)_j \geq 0.$$

Similarly,

- if $(\mathbf{a}_s)_j < (\bar{\mathbf{u}}_s)_j < (\mathbf{b}_s)_j$, then $(\mathbf{\Psi}_s)_j = 0$, and
- if $(\bar{\mathbf{u}}_s)_j = (\mathbf{b}_s)_j$, then $(\mathbf{\Psi}_s)_j \leq 0$.

We also introduce the cone of critical directions

$$\mathbf{C}_{\bar{\mathcal{U}}} := \{\mathcal{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_\ell) \in [\mathbb{R}^2]^\ell \text{ that satisfies (54) and (55)}\},$$

where, for $t \in \mathcal{D}$ and $i \in \{1,2\}$, conditions (54) and (55) read as follows:

$$(\mathbf{v}_t)_i \begin{cases} \geq 0 & \text{if } (\bar{\mathbf{u}}_t)_i = (\mathbf{a}_t)_i \text{ and } (\mathbf{\Psi}_t)_i = 0, \\ \leq 0 & \text{if } (\bar{\mathbf{u}}_t)_i = (\mathbf{b}_t)_i \text{ and } (\mathbf{\Psi}_t)_i = 0, \end{cases} \tag{54}$$

and

$$(\mathbf{v}_t)_i = 0 \text{ if } (\mathbf{\Psi}_t)_i \neq 0. \tag{55}$$

5.3.2. Second order necessary optimality conditions. We are now in position to present a second order necessary optimality condition.

THEOREM 15 (second order necessary optimality condition). *If $\bar{\mathcal{U}} \in \mathbb{U}_{ad}$ is a local minimum for problem (21)–(22), then $j''(\bar{\mathcal{U}})\mathcal{V}^2 \geq 0$ for all $\mathcal{V} \in \mathbf{C}_{\bar{\mathcal{U}}}$.*

Proof. Let $\mathcal{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_\ell) \in \mathbf{C}_{\bar{\mathcal{U}}}$. Define, for $k \in \mathbb{N}$, $\mathcal{V}_k := (\mathbf{v}_1^k, \ldots, \mathbf{v}_\ell^k)$, where

$$(\mathbf{v}_t^k)_i = \begin{cases} 0 & \text{if } (\bar{\mathbf{u}}_t)_i \in \left((\mathbf{a}_t)_i, (\mathbf{a}_t)_i + \frac{1}{k}\right) \text{ or } (\bar{\mathbf{u}}_t)_i \in \left((\mathbf{b}_t)_i - \frac{1}{k}, (\mathbf{b}_t)_i\right), \\ \Pi_{[-k,k]}\left[(\mathbf{v}_t)_i\right] & \text{otherwise,} \end{cases}$$

$t \in \mathcal{D}$, and $i \in \{1,2\}$. We immediately notice that $\mathcal{V}_k \to \mathcal{V}$ in $[\mathbb{R}^2]^\ell$ as $k \uparrow \infty$. In addition, as a consequence of the fact that $\mathcal{V} \in \mathbf{C}_{\bar{\mathcal{U}}}$, we have that, for every $k \in \mathbb{N}$, $\mathcal{V}_k \in \mathbf{C}_{\bar{\mathcal{U}}}$. We also have that, for every μ such that $0 < \mu < k^{-2}$, with k sufficiently large, $\bar{\mathcal{U}} + \mu\mathcal{V}_k \in \mathbb{U}_{ad}$. To prove this assertion, we proceed as follows. Let $t \in \mathcal{D}$ and $i \in \{1,2\}$ be fixed but arbitrary. If $(\mathbf{\Psi}_t)_i \neq 0$, then $(\bar{\mathbf{u}}_t)_i + \mu(\mathbf{v}_t^k)_i = (\bar{\mathbf{u}}_t)_i$. Consequently, $(\bar{\mathbf{u}}_t)_i + \mu(\mathbf{v}_t^k)_i \in [(\mathbf{a}_t)_i, (\mathbf{b}_t)_i]$. Let us now assume that $(\mathbf{\Psi}_t)_i = 0$ and investigate the term $(\bar{\mathbf{u}}_t)_i + \mu(\mathbf{v}_t^k)_i$ on the basis of the following cases:

- If $(\bar{\mathbf{u}}_t)_i = (\mathbf{a}_t)_i$, then $(\mathbf{v}_t)_i \geq 0$ and $(\mathbf{v}_t^k)_i \geq 0$. Hence, $(\mathbf{a}_t)_i \leq (\bar{\mathbf{u}}_t)_i + \mu(\mathbf{v}_t^k)_i$. On the other hand, notice that $(\bar{\mathbf{u}}_t)_i + \mu(\mathbf{v}_t^k)_i < (\mathbf{a}_t)_i + k^{-2}k = (\mathbf{a}_t)_i + k^{-1} \leq (\mathbf{b}_t)_i$ for k sufficiently large. If $(\bar{\mathbf{u}}_t)_i = (\mathbf{b}_t)_i$, similar considerations reveal that $(\bar{\mathbf{u}}_t)_i + \mu(\mathbf{v}_t^k)_i \in [(\mathbf{a}_t)_i, (\mathbf{b}_t)_i]$.
- If $(\bar{\mathbf{u}}_t)_i \in ((\mathbf{a}_t)_i, (\mathbf{a}_t)_i + k^{-1})$ or $(\bar{\mathbf{u}}_t)_i \in ((\mathbf{b}_t)_i - k^{-1}, (\mathbf{b}_t)_i)$, then $(\bar{\mathbf{u}}_t)_i + \mu(\mathbf{v}_t^k)_i = (\bar{\mathbf{u}}_t)_i \in [(\mathbf{a}_t)_i, (\mathbf{b}_t)_i]$.
- If $(\bar{\mathbf{u}}_t)_i \in [(\mathbf{a}_t)_i + k^{-1}, (\mathbf{b}_t)_i - k^{-1}]$, we have, on the one hand

  $$(\bar{\mathbf{u}}_t)_i + \mu(\mathbf{v}_t^k)_i > (\mathbf{a}_t)_i + k^{-1} - k^{-2}k = (\mathbf{a}_t)_i.$$

  On the other hand, we have $(\bar{\mathbf{u}}_t)_i + \mu(\mathbf{v}_t^k)_i < (\mathbf{b}_t)_i - k^{-1} + k^{-2}k = (\mathbf{b}_t)_i$. Consequently, $(\bar{\mathbf{u}}_t)_i + \mu(\mathbf{v}_t^k)_i \in [(\mathbf{a}_t)_i, (\mathbf{b}_t)_i]$ for every μ such that $0 < \mu < k^{-2}$.

The previous arguments show that $(\bar{\mathbf{u}}_t)_i + \mu(\mathbf{v}_t^k)_i$ belongs to $[(\mathbf{a}_t)_i, (\mathbf{b}_t)_i]$ for every μ such that $0 < \mu < k^{-2}$, with k sufficiently large. Since $i \in \{1,2\}$ and $t \in \mathcal{D}$ are arbitrary, we can thus conclude the desired claim: $\bar{\mathcal{U}} + \mu \mathcal{V}_k \in \mathbb{U}_{ad}$.

We now apply the local optimality of $\bar{\mathcal{U}}$, the fact that $\bar{\mathcal{U}} + \mu\mathcal{V}_k$ is admissible for $0 < \mu < k^{-2}$ and k sufficiently large, Taylor's theorem and that, for every $k \in \mathbb{N}$, $j'(\bar{\mathcal{U}})\mathcal{V}_k = 0$, which follows from the fact that $\mathcal{V}_k \in \mathbf{C}_{\bar{\mathcal{U}}}$, to arrive at

$$0 \leq \tfrac{1}{\mu}\left[j(\bar{\mathcal{U}} + \mu\mathcal{V}_k) - j(\bar{\mathcal{U}})\right] = j'(\bar{\mathcal{U}})\mathcal{V}_k + \tfrac{\mu}{2} j''(\bar{\mathcal{U}} + \theta_k \mu \mathcal{V}_k)\mathcal{V}_k^2 = \tfrac{\mu}{2} j''(\bar{\mathcal{U}} + \theta_k \mu \mathcal{V}_k)\mathcal{V}_k^2,$$

where $\theta_k \in (0,1)$. Divide by μ and pass to limit as $\mu \downarrow 0$ to conclude, in view of (47), that $j''(\bar{\mathcal{U}})\mathcal{V}_k^2 \geq 0$. We thus take, on the basis of (46), the limit as $k \uparrow \infty$ to conclude. ☐

5.3.3. Second order sufficient optimality conditions. We now provide a sufficient second order optimality condition with a minimal gap with respect to the necessary one derived in Theorem 15.

THEOREM 16 (second order sufficient optimality condition). *Let $(\bar{\mathbf{y}}, \bar{p})$, $(\bar{\mathbf{z}}, \bar{r})$ and $\bar{\mathcal{U}}$ satisfy the first order optimality conditions (22), (26), and (37). If $j''(\bar{\mathcal{U}})\mathcal{V}^2 > 0$ for all $\mathcal{V} \in \mathbf{C}_{\bar{\mathcal{U}}} \setminus \{0\}$, then there exist $\mu > 0$ and $\sigma > 0$ such that*

$$j(\mathcal{U}) \geq j(\bar{\mathcal{U}}) + \tfrac{\mu}{2}\|\mathcal{U} - \bar{\mathcal{U}}\|^2_{[\mathbb{R}^2]^\ell} \qquad \forall \mathcal{U} \in \mathbb{U}_{ad} : \|\mathcal{U} - \bar{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell} \leq \sigma. \tag{56}$$

In particular, $\bar{\mathcal{U}}$ is a locally optimal control.

Proof. We proceed by contradiction and assume that for every $k \in \mathbb{N}$ there exists an element $\mathcal{U}_k \in \mathbb{U}_{ad}$ such that

$$\|\bar{\mathcal{U}} - \mathcal{U}_k\|_{[\mathbb{R}^2]^\ell} < k^{-1}, \qquad j(\mathcal{U}_k) < j(\bar{\mathcal{U}}) + (2k)^{-1}\|\bar{\mathcal{U}} - \mathcal{U}_k\|^2_{[\mathbb{R}^2]^\ell}. \tag{57}$$

Define, for $k \in \mathbb{N}$, $\rho_k := \|\bar{\mathcal{U}} - \mathcal{U}_k\|_{[\mathbb{R}^2]^\ell}$ and $\mathcal{V}_k := \rho_k^{-1}(\mathcal{U}_k - \bar{\mathcal{U}})$. Since $\|\mathcal{V}_k\|_{[\mathbb{R}^2]^\ell} = 1$, there exists a non relabeled subsequence $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$ such that $\mathcal{V}_k \to \mathcal{V}$ in $[\mathbb{R}^2]^\ell$ as $k \uparrow \infty$.

We now proceed on the basis of three steps.

Step 1. We prove that $\mathcal{V} \in \mathbf{C}_{\bar{\mathcal{U}}}$. As a first step, we notice that the set of elements in $[\mathbb{R}^2]^\ell$ that satisfy (54) is closed. Since $\mathcal{V}_k$, for $k \in \mathbb{N}$, satisfies (54) and $\mathcal{V}$ is the limit of $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$, in $[\mathbb{R}^2]^\ell$, we deduce that $\mathcal{V}$ satisfies (54) as well. We must thus prove that $(\mathbf{\Psi}_t)_i \neq 0$ implies $(\mathbf{v}_t)_i = 0$ to conclude that $\mathcal{V} \in \mathbf{C}_{\bar{\mathcal{U}}}$. Here, $i \in \{1,2\}$. Since $\mathcal{V}_k = \rho_k^{-1}(\mathcal{U}_k - \bar{\mathcal{U}})$, the variational inequality (37) and definition (52) yield

$$\sum_{t\in\mathcal{D}} \mathbf{\Psi}_t \cdot \mathbf{v}_t = \lim_{k\uparrow\infty} \sum_{t\in\mathcal{D}} \mathbf{\Psi}_t \cdot \mathbf{v}_t^k = \lim_{k\uparrow\infty} \rho_k^{-1} \sum_{t\in\mathcal{D}} \mathbf{\Psi}_t \cdot (\mathbf{u}_t^k - \bar{\mathbf{u}}_t) \geq 0.$$

We now prove that $\sum_{t\in\mathcal{D}} \mathbf{\Psi}_t \cdot \mathbf{v}_t \leq 0$. To accomplish this task, we invoke the mean value theorem and the right hand side inequality in (57) to obtain

$$j(\mathcal{U}_k) - j(\bar{\mathcal{U}}) = j'(\bar{\mathcal{U}} + \theta_k(\mathcal{U}_k - \bar{\mathcal{U}}))(\mathcal{U}_k - \bar{\mathcal{U}}) < \tfrac{1}{2k}\|\bar{\mathcal{U}} - \mathcal{U}_k\|^2_{[\mathbb{R}^2]^\ell} = \tfrac{\rho_k^2}{2k}, \quad \theta_k \in (0,1).$$

Divide by ρ_k and let $k \uparrow \infty$. This yields $j'(\bar{\mathcal{U}} + \theta_k(\mathcal{U}_k - \bar{\mathcal{U}}))\mathcal{V}_k < (2k)^{-1}\rho_k \to 0$ as $k \uparrow \infty$. Let us now define, for $k \in \mathbb{N}$, $\hat{\mathcal{U}}_k := \bar{\mathcal{U}} + \theta_k(\mathcal{U}_k - \bar{\mathcal{U}})$ and observe that the Lipschitz property provided in Theorem 5 reveals the following convergence property:

$$\hat{\mathbf{y}}_k := \mathcal{G}(\hat{\mathcal{U}}_k) \to \mathcal{G}(\bar{\mathcal{U}}) =: \mathbf{y}_{\bar{\mathcal{U}}} \text{ in } \mathbf{H}_0^1(\rho,\Omega), \qquad k \uparrow \infty. \tag{58}$$

The continuous Sobolev embeddings $\mathbf{H}_0^1(\rho,\Omega) \hookrightarrow \mathbf{L}^4(\rho,\Omega)$ [17, Theorem 1.3] and $\mathbf{H}_0^1(\rho,\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ (cf. Theorem 3) thus guarantee that $\hat{\mathbf{y}}_k \to \mathbf{y}_{\bar{\mathcal{U}}}$ in $\mathbf{L}^4(\rho,\Omega)$ and

$\mathbf{L}^2(\Omega)$, respectively, as $k \uparrow \infty$. Let $(\hat{\mathbf{z}}_k, \hat{r}_k)$ be the solution to (26) with $\mathbf{y}_{\mathcal{U}}$ being replaced by $\hat{\mathbf{y}}_k$. To analyze the convergence of the sequence $\{\hat{\mathbf{z}}_k\}_{k\in\mathbb{N}}$, we proceed on the basis of the arguments developed at the end of the proof of Theorem 14. These arguments reveal that $\hat{\mathbf{z}}_k \to \bar{\mathbf{z}}$ in $\mathbf{W}^{1,q}(\Omega)$, for some $q > 2$, as $k \uparrow \infty$. Consequently,

$$\sum_{t\in\mathcal{D}} \mathbf{\Psi}_t \cdot \mathbf{v}_t = \lim_{k\uparrow\infty} \sum_{t\in\mathcal{D}} (\hat{\mathbf{z}}_k(t) + \eta\hat{\mathbf{u}}_t^k) \cdot \mathbf{v}_t^k = \lim_{k\uparrow\infty} j'(\hat{\mathcal{U}}_k)\mathcal{V}_k \leq 0.$$

We have thus deduced that $0 = \sum_{t\in\mathcal{D}} \mathbf{\Psi}_t \cdot \mathbf{v}_t = \sum_{t\in\mathcal{D}} \sum_{i=1}^{2} |(\mathbf{\Psi}_t)_i(\mathbf{v}_t)_i|$. To obtain the second equality, we have used the properties that $\mathbf{\Psi}_t$ satisfies combined with the fact that $\mathcal{V}$ verifies the sign condition (54). Consequently, $(\mathbf{\Psi}_t)_i \neq 0$ implies $(\mathbf{v}_t)_i = 0$; $i \in \{1, 2\}$. We have thus proved that $\mathcal{V} \in \mathbf{C}_{\bar{\mathcal{U}}}$.

Step 2. We prove that $\mathcal{V} = \mathbf{0}$. We begin this step with a simple application of Taylor's theorem and write

$$j(\mathcal{U}_k) = j(\bar{\mathcal{U}}) + \rho_k j'(\bar{\mathcal{U}})\mathcal{V}_k + \tfrac{\rho_k^2}{2} j''(\hat{\mathcal{U}}_k)\mathcal{V}_k^2, \qquad \hat{\mathcal{U}}_k = \bar{\mathcal{U}} + \theta_k(\mathcal{U}_k - \bar{\mathcal{U}}), \qquad \theta_k \in (0,1).$$

Since $\mathcal{V}_k = \rho_k^{-1}(\mathcal{U}_k - \bar{\mathcal{U}})$, the first order optimality condition (36) yields $j'(\bar{\mathcal{U}})\mathcal{V}_k \geq 0$. This and the right hand side estimate in (57) reveal that

$$\tfrac{\rho_k^2}{2} j''(\hat{\mathcal{U}}_k)\mathcal{V}_k^2 \leq j(\mathcal{U}_k) - j(\bar{\mathcal{U}}) < \tfrac{1}{2k}\|\bar{\mathcal{U}} - \mathcal{U}_k\|_{[\mathbb{R}^2]^\ell}^2 = \tfrac{\rho_k^2}{2k}.$$

This implies that $j''(\hat{\mathcal{U}}_k)\mathcal{V}_k^2 < k^{-1} \to 0$ as $k \uparrow \infty$.

We now prove that $j''(\hat{\mathcal{U}}_k)\mathcal{V}_k^2 \to j''(\bar{\mathcal{U}})\mathcal{V}^2$ as $k \uparrow \infty$. Invoke (46) and write

$$j''(\hat{\mathcal{U}}_k)\mathcal{V}_k^2 = \int_\Omega \hat{\boldsymbol{\theta}}_k \cdot \hat{\boldsymbol{\theta}}_k + 2\int_\Omega \hat{\boldsymbol{\theta}}_k \otimes \hat{\boldsymbol{\theta}}_k : \nabla\hat{\mathbf{z}}_k + \eta \sum_{t\in\mathcal{D}} |\mathbf{v}_t^k|^2.$$

Here, $(\hat{\boldsymbol{\theta}}_k, \hat{\xi}_k)$ denotes the solution to (12) with $\mathbf{g} = \sum_{t\in D} \mathbf{v}_t^k \delta_t$ and with $\mathbf{\Phi}$ being replaced by $\hat{\mathbf{y}}_k$. Notice that, $\hat{\mathcal{U}}_k \to \bar{\mathcal{U}}$ and $\mathcal{V}_k \to \mathcal{V}$ in $[\mathbb{R}^2]^\ell$, as $k \uparrow \infty$, guarantee (58) and

$$\hat{\mathbf{z}}_k \to \bar{\mathbf{z}}, \qquad \hat{\boldsymbol{\theta}}_k \to \boldsymbol{\theta}, \qquad k \uparrow \infty, \tag{59}$$

in $\mathbf{H}_0^1(\Omega)$ and $\mathbf{H}_0^1(\rho,\Omega)$, respectively; $(\boldsymbol{\theta}, \xi)$ denotes the solution to (12) with $\mathbf{g} = \sum_{t\in D} \mathbf{v}_t\delta_t$ and $\mathbf{\Phi}$ being replaced by $\mathbf{y}_{\bar{\mathcal{U}}}$. The left hand side convergence property in (59) follows from the arguments developed at the end of the proof of Theorem 14, while the right hand side convergence property follows from Lemma 13. We can thus obtain

$$\eta \sum_{t\in\mathcal{D}} |\mathbf{v}_t^k|^2 \to \eta \sum_{t\in\mathcal{D}} |\mathbf{v}_t|^2, \qquad \int_\Omega \hat{\boldsymbol{\theta}}_k \cdot \hat{\boldsymbol{\theta}}_k \to \int_\Omega \boldsymbol{\theta} \cdot \boldsymbol{\theta}, \qquad k \uparrow \infty,$$

upon utilizing that $\hat{\boldsymbol{\theta}}_k \to \boldsymbol{\theta}$ in $\mathbf{L}^2(\Omega)$, which follows from (59) and Theorem 3. We now observe that

$$\begin{aligned}\mathfrak{I}_k := \left| \int_\Omega \boldsymbol{\theta} \otimes \boldsymbol{\theta} : \nabla\bar{\mathbf{z}} - \int_\Omega \hat{\boldsymbol{\theta}}_k \otimes \hat{\boldsymbol{\theta}}_k : \nabla\hat{\mathbf{z}}_k \right| &\leq \|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_k\|_{\mathbf{L}^4(\rho,\Omega)} \|\boldsymbol{\theta}\|_{\mathbf{L}^4(\rho,\Omega)} \|\nabla\bar{\mathbf{z}}\|_{\mathbf{L}^2(\rho^{-1},\Omega)} \\ &+ \|\hat{\boldsymbol{\theta}}_k\|_{\mathbf{L}^4(\rho,\Omega)} \|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_k\|_{\mathbf{L}^4(\rho,\Omega)} \|\nabla\bar{\mathbf{z}}\|_{\mathbf{L}^2(\rho^{-1},\Omega)} + \|\hat{\boldsymbol{\theta}}_k\|_{\mathbf{L}^4(\Omega)}^2 \|\nabla(\bar{\mathbf{z}} - \hat{\mathbf{z}}_k)\|_{\mathbf{L}^2(\Omega)}.\end{aligned}$$

Lemma 4 guarantees that $\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}_k \in \mathbf{H}_0^1(\rho,\Omega)$. This implies that $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}_k \in \mathbf{L}^4(\rho,\Omega)$ upon utilizing [17, Theorem 1.3]. An adaption of the arguments developed in the proof of

Lemma 4 yield $\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}_k \in \mathbf{W}^{1,p}(\Omega)$, for every $p < 2$, so that $\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}_k \in \mathbf{L}^4(\Omega)$. On the other hand, in view of the regularity results of Theorem 11, we have $\nabla \bar{\mathbf{z}} \in \mathbf{L}^2(\rho^{-1}, \Omega)$. On the basis of the previous inequality, that controls $\mathfrak{J}_k$, these arguments combined with the convergence properties (59) reveal that $\mathfrak{J}_k \to 0$ as $k \uparrow \infty$.

Consequently, $j''(\hat{\mathcal{U}}_k)\mathcal{V}_k^2 \to j''(\bar{\mathcal{U}})\mathcal{V}^2$ as $k \uparrow \infty$. This, in view of $j''(\hat{\mathcal{U}}_k)\mathcal{V}_k^2 < k^{-1} \to 0$ as $k \uparrow \infty$, implies that $j''(\bar{\mathcal{U}})\mathcal{V}^2 \leq 0$. Invoke the fact that $\mathcal{V} \in \mathbf{C}_{\bar{\mathcal{U}}}$ and the optimality condition $j''(\bar{\mathcal{U}})\mathcal{W}^2 > 0$, which holds for every $\mathcal{W} \in \mathbf{C}_{\bar{\mathcal{U}}} \setminus \{0\}$, to conclude that $\mathcal{V} = \mathbf{0}$.

Step 3. In this step we arrive at a contradiction. In fact, since $\mathcal{V} = \mathbf{0}$, we have $\hat{\boldsymbol{\theta}}_k \to 0$, in $\mathbf{H}_0^1(\Omega)$, as $k \uparrow \infty$. Thus, in view of the identity

$$\eta = \eta \|\mathcal{V}_k\|_{[\mathbb{R}^2]^\ell}^2 = j''(\hat{\mathcal{U}}_k)\mathcal{V}_k^2 - \int_\Omega \hat{\boldsymbol{\theta}}_k \cdot \hat{\boldsymbol{\theta}}_k - 2\int_\Omega \hat{\boldsymbol{\theta}}_k \otimes \hat{\boldsymbol{\theta}}_k : \nabla \hat{\mathbf{z}}_k$$

and the fact that $j''(\hat{\mathcal{U}}_k)\mathcal{V}_k^2 < k^{-1} \to 0$ as $k \uparrow \infty$, we conclude that $\eta \leq 0$. Since, by assumption, $\eta > 0$, we arrive at a contradiction. This concludes the proof. ∎

5.3.4. An equivalence result. To present the following result, we introduce, for $\tau > 0$, the cone

$$\mathbf{C}_{\bar{\mathcal{U}}}^\tau := \{\mathcal{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_\ell) \in [\mathbb{R}^2]^\ell \text{ that satisfies (54) and (60)}\},$$

where, for $t \in \mathcal{D}$ and $i \in \{1, 2\}$, condition (60) reads as follows:

$$(\mathbf{v}_t)_i = 0 \text{ if } |(\boldsymbol{\Psi}_t)_i| > \tau. \tag{60}$$

THEOREM 17 (equivalent optimality conditions). *Let $(\bar{\mathbf{y}}, \bar{p})$, $(\bar{\mathbf{z}}, \bar{r})$ and $\bar{\mathcal{U}}$ satisfy the first order optimality conditions (22), (26), and (37). Then, the following statements are equivalent:*

$$j''(\bar{\mathcal{U}})\mathcal{V}^2 > 0 \quad \forall \mathcal{V} \in \mathbf{C}_{\bar{\mathcal{U}}} \setminus \{\mathbf{0}\}, \tag{61}$$

and

$$\exists \kappa, \tau > 0: \quad j''(\bar{\mathcal{U}})\mathcal{V}^2 \geq \kappa \|\mathcal{V}\|_{[\mathbb{R}^2]^\ell}^2 \quad \forall \mathcal{V} \in \mathbf{C}_{\bar{\mathcal{U}}}^\tau. \tag{62}$$

Proof. We begin the proof by observing that, for any $\tau > 0$, $\mathbf{C}_{\bar{\mathcal{U}}} = \mathbf{C}_{\bar{\mathcal{U}}}^0 \subset \mathbf{C}_{\bar{\mathcal{U}}}^\tau$. It is thus immediate that (62) implies (61).

We now prove that (61) implies (62) on the basis of a contradiction argument. Let us assume that, for any $\tau > 0$, there exists $\mathcal{V}_\tau := (\mathbf{v}_1^\tau, \ldots, \mathbf{v}_\ell^\tau) \in \mathbf{C}_{\bar{\mathcal{U}}}^\tau$ such that $j''(\bar{\mathcal{U}})\mathcal{V}_\tau^2 < \tau \|\mathcal{V}_\tau\|_{[\mathbb{R}^2]^\ell}^2$. We define

$$\mathcal{W}_\tau := (\mathbf{w}_1^\tau, \ldots, \mathbf{w}_\ell^\tau), \qquad \mathbf{w}_i^\tau := \mathbf{v}_i^\tau \|\mathcal{V}_\tau\|_{[\mathbb{R}^2]^\ell}^{-1}, \qquad i \in \{1, \cdots, \ell\}.$$

Note that, up to a nonrelabeled subsequence if necessary,

$$\mathcal{W}_\tau \in \mathbf{C}_{\bar{\mathcal{U}}}^\tau, \quad \|\mathcal{W}_\tau\|_{[\mathbb{R}^2]^\ell} = 1, \quad j''(\bar{\mathcal{U}})\mathcal{W}_\tau^2 < \tau, \quad \mathcal{W}_\tau \to \mathcal{W} \text{ in } [\mathbb{R}^2]^\ell \text{ as } \tau \downarrow 0. \tag{63}$$

We prove that $\mathcal{W} \in \mathbf{C}_{\bar{\mathcal{U}}}$. To accomplish this task, we first observe that, for any $\tau > 0$, $\mathcal{W}_\tau$ satisfies the sign condition (54). Since the set of elements satisfying such a condition is closed and $\mathcal{W}_\tau \to \mathcal{W}$ in $[\mathbb{R}^2]^\ell$ as $\tau \downarrow 0$, we deduce that $\mathcal{W}$ satisfies the sign condition (54) as well. On the other hand, we utilize the convergence $\mathcal{W}_\tau \to \mathcal{W}$ in $[\mathbb{R}^2]^\ell$, as $\tau \downarrow 0$ and the fact that $\mathcal{W}_\tau$ satisfies (60) and (63) to arrive at

$$\sum_{t\in\mathcal{D}} \boldsymbol{\Psi}_t \cdot \mathbf{w}_t = \lim_{\tau\downarrow 0} \sum_{t\in\mathcal{D}} \boldsymbol{\Psi}_t \cdot \mathbf{w}_t^\tau \lesssim \lim_{\tau\downarrow 0} \tau \sum_{t\in\mathcal{D}} \mathbf{w}_t^\tau \lesssim \lim_{\tau\downarrow 0} \tau = 0.$$

We thus conclude that $\sum_{t\in\mathcal{D}} \sum_{i=1}^{2} |(\boldsymbol{\Psi}_t)_i (\mathbf{w}_t)_i| = \sum_{t\in\mathcal{D}} \boldsymbol{\Psi}_t \cdot \mathbf{w}_t = 0$. Consequently, we have that $(\boldsymbol{\Psi}_t)_i \neq 0$ implies that $(\mathbf{v}_t)_i = 0$; $i = \{1, 2\}$. This proves that $\mathcal{W} \in \mathbf{C}_{\bar{\mathcal{U}}}$.

We now observe that, since $\mathcal{W} \in \mathbf{C}_{\bar{\mathcal{U}}}$, condition (61) implies that either $\mathcal{W} = \mathbf{0}$ or $j''(\bar{\mathcal{U}})\mathcal{W}^2 > 0$. Let us prove that $\mathcal{W} = \mathbf{0}$. To accomplish this task, we observe that the convergence property $j''(\bar{\mathcal{U}})\mathcal{W}^2 = \lim_{\tau\downarrow 0} j''(\bar{\mathcal{U}})\mathcal{W}_\tau^2$ combined with (63) yield

$$j''(\bar{\mathcal{U}})\mathcal{W}^2 = \lim_{\tau\downarrow 0} j''(\bar{\mathcal{U}})\mathcal{W}_\tau^2 \leq 0. \tag{64}$$

As a result, we have thus obtained that $\mathcal{W} = \mathbf{0}$. On the other hand, notice that $\mathcal{W}_\tau \to \mathcal{W} = \mathbf{0}$ in $[\mathbb{R}^2]^\ell$ implies the strong convergence $\boldsymbol{\theta}_{\mathcal{W}_\tau} \to 0$ in $\mathbf{H}_0^1(\rho, \Omega)$ as $\tau \downarrow 0$. Here, $(\boldsymbol{\theta}_{\mathcal{W}_\tau}, \zeta_{\mathcal{W}_\tau})$ denotes the solution to (12) with forcing term $\sum \mathbf{w}_t^\tau \delta_t$. The Sobolev embeddings $\mathbf{H}_0^1(\rho, \Omega) \hookrightarrow \mathbf{L}^4(\rho, \Omega)$ (cf. [17, Theorem 1.3]) and $\mathbf{H}_0^1(\rho, \Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ (cf. Theorem 3) guarantee that $\boldsymbol{\theta}_{\mathcal{W}_\tau} \to 0$ in $\mathbf{L}^2(\Omega)$ and $\mathbf{L}^4(\rho, \Omega)$, as $\tau \downarrow 0$, respectively. With these convergence results at hand, we now invoke (46) and (64) to obtain

$$0 < \eta = \lim_{\tau\downarrow 0} \left[\int_\Omega (\boldsymbol{\theta}_{\mathcal{W}_\tau}^2 + 2\boldsymbol{\theta}_{\mathcal{W}_\tau} \otimes \boldsymbol{\theta}_{\mathcal{W}_\tau} : \nabla \bar{\mathbf{z}}) + \eta \right] = \lim_{\tau\downarrow 0} j''(\bar{\mathcal{U}})\mathcal{W}_\tau^2 \leq 0,$$

which is a contradiction. ∎

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