## A POSTERIORI ERROR ESTIMATES FOR A DISTRIBUTED OPTIMAL CONTROL PROBLEM OF THE STATIONARY NAVIER-STOKES EQUATIONS\*

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**Abstract.** In two and three dimensional Lipschitz, but not necessarily convex, polytopal domains, we propose and analyze an a posteriori error estimator for an optimal control problem that involves the stationary Navier–Stokes equations; control constraints are also considered. The proposed error estimator is defined as the sum of three contributions, which are related to the discretization of the state and adjoint equations and the control variable. We prove that the devised error estimator is globally reliable and locally efficient. We conclude by presenting numerical experiments which reveal the competitive performance of an adaptive loop based on the proposed error estimator.

Key words. optimal control problems, Navier–Stokes equations, finite elements, a posteriori error estimates, adaptive finite element methods

### AMS subject classifications. 35Q35, 35Q30, 49M05, 49M25, 65N15, 65N30, 65N50.

1. Introduction. In this work we shall be interested in the design and analysis of an a posteriori error estimator for a distributed optimal control problem involving the stationary Navier–Stokes equations; control constraints are also considered. To make matters precise, let  $\Omega \subset \mathbb{R}^d$ , with  $d \in \{2, 3\}$ , be an open and bounded polytopal domain with Lipschitz boundary  $\partial\Omega$ . Given a desired state  $\mathbf{y}_{\Omega} \in \mathbf{L}^2(\Omega)$  and a regularization parameter  $\alpha > 0$ , we define the quadratic cost functional

$$J(\mathbf{y}, \mathbf{u}) := \frac{1}{2} \|\mathbf{y} - \mathbf{y}_{\Omega}\|_{\mathbf{L}^{2}(\Omega)}^{2} + \frac{\alpha}{2} \|\mathbf{u}\|_{\mathbf{L}^{2}(\Omega)}^{2}$$

We shall be concerned with the following PDE-constrained optimization problem: Find

(1) 
$$\min J(\mathbf{y}, \mathbf{u})$$

subject to the stationary Navier–Stokes equations

(2) 
$$-\nu\Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{u} \text{ in } \Omega, \quad \text{div } \mathbf{y} = 0 \text{ in } \Omega, \quad \mathbf{y} = \mathbf{0} \text{ on } \partial\Omega,$$

and the control constraints

(3) 
$$\mathbf{u} \in \mathbb{U}_{ad}, \qquad \mathbb{U}_{ad} := \{ \mathbf{v} \in \mathbf{L}^2(\Omega) : \mathbf{a} \le \mathbf{v} \le \mathbf{b} \text{ a.e. in } \Omega \},$$

with  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$  satisfying  $\mathbf{a} < \mathbf{b}$ . We immediately comment that, throughout this work, vector inequalities must be understood componentwise. In (2),  $\nu > 0$  denotes the kinematic viscosity.

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The numerical analysis of optimal control problems governed by the stationary Navier–Stokes equations has been previously considered in a number of works. For a slightly different cost functional J, which in contrast to (1) measures the difference  $\mathbf{y} - \mathbf{y}_{\Omega}$  in the  $\mathbf{L}^4(\Omega)$ -norm, the authors of [19] have analyzed inf-sup stable finite element approximations of suitable distributed and boundary optimal control problems; control constraints are not considered. In two and three dimensions and under the assumptions that  $\Omega$  is a convex polytope, the mesh-size is sufficiently small, and that both the optimal state  $(\bar{\mathbf{y}}, \bar{p})$  and adjoint state  $(\bar{\mathbf{z}}, \bar{r})$  belong to  $\mathbf{H}^2(\Omega) \times H^1(\Omega)$ , the authors prove, for the distributed case, that  $\|\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(\Omega)} \lesssim h_{\mathscr{T}}^{2}$  [19, Corollary 4.5 and section 5.2]. Here,  $\bar{\mathbf{u}}_{\mathscr{T}}$  denotes the corresponding finite element approximation of the optimal control variable  $\bar{\mathbf{u}}$ . Later, the authors of [13] derived error estimates for suitable finite element approximations of (1)-(3). Notice that control constraints are considered. Under the assumption that  $\Omega$  is of class  $C^2$ , the authors show that the  $\mathbf{L}^2(\Omega)$ -norm of the error approximation of the control variable behaves as  $h^2_{\mathscr{T}}$ , when the control set is not discretized, and as  $h_{\mathcal{T}}$ , when such a set is discretized by using piecewise constant functions [13, Theorem 4.18]. These error estimates are obtained for local solutions of the optimal control problem which are nonsingular (in the sense that the linearized Navier–Stokes equations around these solutions define some isomorphisms) and satisfy a second order sufficient optimality condition.

A class of numerical methods that has proven useful for approximating the solution to PDE–constrained optimization problems, and the ones we will use in this work, are the so-called adaptive finite element methods (AFEMs). AFEMs are iterative methods that improve the quality of the finite element approximation to a partial differential equation (PDE) while striving to keep an optimal distribution of computational resources measured in terms of degrees of freedom. An essential ingredient of an AFEM is an a posteriori error estimator, which is a computable quantity that depends on the discrete solution and data and provides information about the local quality of the approximate solution. The a posteriori error analysis for optimal control problems that are based on the minimization of a quadratic functional subject to a linear PDE and control constraints has achieved several advances in recent years. We refer the reader to [10, 21, 22, 23] for a discussion. As opposed to these advances, the analysis of AFEMs for optimal control problems involving nonlinear equations is rather scarce. We mention the approach introduced in [10] for estimating the error in terms of the cost functional for semilinear optimal control problems [10, section 6] and its extensions to problems with control constraints [20, 30]. Recently, the authors of [4] have studied a posteriori error estimates for a distributed semilinear elliptic optimal control problem. They have proposed a general framework that, on the basis of global reliability estimates for the state and adjoints equations and second order optimality conditions, yields a global reliability result for the proposed error estimator of the underlying optimal control problem. For a particular residual-type setting, the authors obtain, on the basis of bubble functions arguments, local efficiency estimates. Regarding a posteriori error estimates for optimal control problems involving (2) we mention references [8, 9]. For particular boundary control problems, the authors of [8, 9] invoke the approach of [10] and construct an upper bound for the error  $J(\bar{\mathbf{y}}, \bar{\mathbf{u}}) - J(\bar{\mathbf{y}}_{\mathscr{T}}, \bar{\mathbf{u}}_{\mathscr{T}})$ . An efficiency analysis is, however, not provided.

In this work we propose a residual-based a posteriori error estimator for the optimal control problem (1)-(3) that can be decomposed as the sum of following three individual contributions: a contribution related to the discretization of the state equations, a second one associated to the discretization of the adjoint equations, and a third one that accounts for the discretization of the control variable. We must

immediately mention that, as is usual in the a posteriori error analysis for the Navier– Stokes equations, we shall operate under a smallness assumption on data; see [3, 24, 29]. Under this assumption, in two and three dimensional Lipschitz polytopes, we obtain global reliability and local efficiency estimates. On the basis of the devised error estimator, we also design a simple adaptive strategy that exhibits, for the examples that we present, optimal experimental rates of convergence for all the optimal variables but with the exception of the control variable. Several remarks and comparisons with the existing literature are now in order:

- In contrast to [8, 9], we show that our error estimator is *equivalent*, up to oscillation terms, to the total approximation error; see Theorems 9 and 12.
- In contrast to the a priori theory developed in [13, 19], our a posteriori error analysis only requires that  $\Omega$  is a Lipschitz polytope,  $\mathbf{y}_{\Omega} \in \mathbf{L}^{2}(\Omega)$ , and that both the optimal state  $(\bar{\mathbf{y}}, \bar{p})$  and adjoint state  $(\bar{\mathbf{z}}, \bar{r})$  belong to  $\mathbf{H}^{1}(\Omega) \times L^{2}(\Omega)$ .
- As opposed to the case when the state equation is linear, where, in general, only first order optimality conditions are needed to obtain a posteriori error estimates, the strategy that we develop here relies on the use of a second order sufficient optimality condition and on the particular structure of the associated critical cone; see Theorems 8 and 9.

The rest of the paper is organized as follows. In section 2 we set notation and review some preliminaries for the Navier–Stokes equations. Basic results for the optimal control problem (1)–(3) as well as first and second order optimality conditions are reviewed in section 3. The core of our work are sections 4 and 5, where we design an a posteriori error estimator, for a suitable inf-sup stable finite element scheme, and show its global reliability and local efficiency. Finally, two and three dimensional numerical examples are presented in section 6. These examples illustrate the theory and reveal a competitive performance of the devised AFEM.

2. Notation and preliminaries. Let us set notation and recall some facts that will be useful later.

**2.1. Notation.** We shall use standard notation for Lebesgue and Sobolev spaces. Let  $d \in \{1, 2, 3\}$  and  $U \subset \mathbb{R}^d$  be an open and bounded domain. The space of functions in  $L^2(U)$  that have zero average is denoted by  $L_0^2(U)$ . By  $W^{m,t}(U)$ , we denote the Sobolev space of functions in  $L^t(U)$  with partial derivatives of order up to m in  $L^t(U)$ . Here,  $0 \leq m < \infty$  and  $1 \leq t \leq \infty$ . The closure with respect to the norm in  $W^{m,t}(U)$ of the space of  $C^\infty$  functions compactly supported in U is denoted by  $W_0^{m,t}(U)$ . When t = 2 and  $m \in [0, \infty)$ , we set  $H^m(U) := W^{m,2}(U)$  and  $H_0^m(U) := W_0^{m,2}(U)$ . We use bold letters to denote the vector-valued counterparts of the aforementioned spaces. In particular, we set

$$\mathbf{V}(U) := \{ \mathbf{v} \in \mathbf{H}_0^1(U) : \text{div } \mathbf{v} = 0 \}.$$

If  $\mathcal{X}$  and  $\mathcal{Y}$  are normed vector spaces, we write  $\mathcal{X} \hookrightarrow \mathcal{Y}$  to denote that  $\mathcal{X}$  is continuously embedded in  $\mathcal{Y}$ . The relation  $a \leq b$  indicates that  $a \leq Cb$ , with a positive constant that depends neither on a, b nor the discretization parameter. The value of C might change at each occurrence.

Finally, throughout this work,  $\Omega$  denotes an open and bounded polytopal domain in  $\mathbb{R}^d$   $(d \in \{2, 3\})$  with Lipschitz boundary  $\partial \Omega$ .

**2.2.** Preliminaries for the Navier–Stokes equations. Let us, for the sake of future reference, collect here some standard results involved in the analysis of (2).

In order to write a weak formulation for (2), we introduce the trilinear form

$$b(\mathbf{v}_1;\mathbf{v}_2,\mathbf{v}_3):=((\mathbf{v}_1\cdot\nabla)\mathbf{v}_2,\mathbf{v}_3)_{\mathbf{L}^2(\Omega)}.$$

The form b satisfies the following properties [18, Chapter IV, Lemma 2.2], [26, Chapter II, Lemma 1.3]: Let  $\mathbf{v}_1 \in \mathbf{V}(\Omega)$  and  $\mathbf{v}_2, \mathbf{v}_3 \in \mathbf{H}_0^1(\Omega)$ . Then, we have

(4) 
$$b(\mathbf{v}_1; \mathbf{v}_2, \mathbf{v}_3) = -b(\mathbf{v}_1; \mathbf{v}_3, \mathbf{v}_2), \quad b(\mathbf{v}_1; \mathbf{v}_2, \mathbf{v}_2) = 0.$$

The form b is well-defined and continuous on  $\mathbf{H}_{0}^{1}(\Omega)^{3}$  and

(5) 
$$|b(\mathbf{v}_1;\mathbf{v}_2,\mathbf{v}_3)| \le C_b \|\nabla \mathbf{v}_1\|_{\mathbf{L}^2(\Omega)} \|\nabla \mathbf{v}_2\|_{\mathbf{L}^2(\Omega)} \|\nabla \mathbf{v}_3\|_{\mathbf{L}^2(\Omega)},$$

where  $C_b > 0$ ; see [17, Lemma IX.1.1] and [26, Chapter II, Lemma 1.1].

On the other hand, the surjectivity of the divergence operator yields the existence of a constant  $\beta > 0$  such that [18, Chapter I, section 5.1], [16, Corollary B. 71]

(6) 
$$\sup_{\mathbf{v}\in\mathbf{H}_{0}^{1}(\Omega)}\frac{(q,\operatorname{div}\mathbf{v})_{L^{2}(\Omega)}}{\|\nabla\mathbf{v}\|_{\mathbf{L}^{2}(\Omega)}} \geq \beta \|q\|_{L^{2}(\Omega)} \quad \forall q \in L^{2}_{0}(\Omega)$$

With these ingredients at hand, we introduce the following weak formulation of problem (2): Given  $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ , find  $(\mathbf{y}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  such that

(7) 
$$\nu(\nabla \mathbf{y}, \nabla \mathbf{v})_{\mathbf{L}^{2}(\Omega)} + b(\mathbf{y}; \mathbf{y}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v})_{L^{2}(\Omega)} = \langle \mathbf{f}, \mathbf{v} \rangle \qquad \forall \mathbf{v} \in \mathbf{H}_{0}^{1}(\Omega), (q, \operatorname{div} \mathbf{y})_{L^{2}(\Omega)} = 0 \qquad \forall q \in L_{0}^{2}(\Omega).$$

Here,  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $\mathbf{H}^{-1}(\Omega)$  and  $\mathbf{H}_{0}^{1}(\Omega)$ .

Denote by  $C_2$  the constant in the standard Sobolev embedding  $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ . The following result states the existence and uniqueness of solutions for the Navier– Stokes equations for small data (see [18, Chapter IV, Theorem 2.2] and [26, Chapter II, Theorem 1.3]). Since it will be useful later, we restrict the discussion to  $\mathbf{f} \in \mathbf{L}^2(\Omega)$ .

THEOREM 1 (well-posedness). If  $\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} < C_2^{-1} \mathcal{C}_b^{-1} \nu^2$ , then there exists a unique solution  $(\mathbf{y}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  of problem (7). In addition, we have

(8) 
$$\|\nabla \mathbf{y}\|_{\mathbf{L}^{2}(\Omega)} \leq \theta \mathcal{C}_{b}^{-1} \nu, \quad \theta \in [0, 1).$$

**3. The optimal control problem.** In this section we present a weak formulation for problem (1)-(3). We review first and second order optimality conditions in sections 3.2 and 3.3, respectively, and introduce, in section 3.4, a finite element discretization for problem (1)-(3).

We consider the following weak version of our control problem (1)-(3): Find

(9) 
$$\min_{\mathbf{H}_{0}^{1}(\Omega) \times \mathbb{U}_{ad}} J(\mathbf{y}, \mathbf{u})$$

subject to the weak formulation of the stationary Navier–Stokes equations

(10) 
$$\nu(\nabla \mathbf{y}, \nabla \mathbf{v})_{\mathbf{L}^{2}(\Omega)} + b(\mathbf{y}; \mathbf{y}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v})_{L^{2}(\Omega)} = (\mathbf{u}, \mathbf{v})_{\mathbf{L}^{2}(\Omega)} \quad \forall \mathbf{v} \in \mathbf{H}_{0}^{1}(\Omega), (q, \operatorname{div} \mathbf{y})_{L^{2}(\Omega)} = 0 \qquad \forall q \in L_{0}^{2}(\Omega).$$

Assume that

(11) 
$$\frac{\mathcal{C}_b C_2}{\nu^2} \sup_{\mathbf{u} \in \mathbb{U}_{ad}} \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} < 1.$$

Owing to Theorem 1 we immediately conclude that, for each  $\mathbf{u} \in \mathbb{U}_{ad}$ , there exists a unique pair  $(\mathbf{y}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  that solves problem (10).

Notice that, due to de Rham's Theorem (see section 4.1.3 and Theorem B. 73 in [16]), (10) is equivalent to the following formulation: Find  $\mathbf{y} \in \mathbf{V}(\Omega)$  such that

(12) 
$$\nu(\nabla \mathbf{y}, \nabla \mathbf{v})_{\mathbf{L}^{2}(\Omega)} + b(\mathbf{y}; \mathbf{y}, \mathbf{v}) = (\mathbf{u}, \mathbf{v})_{\mathbf{L}^{2}(\Omega)} \quad \forall \mathbf{v} \in \mathbf{V}(\Omega)$$

**3.1. Local solutions.** Since the optimal control problem (9)–(10) is not convex, we discuss optimality conditions in the context of local solutions. A control  $\bar{\mathbf{u}} \in \mathbb{U}_{ad}$  is said to be locally optimal in  $\mathbf{L}^2(\Omega)$  for (9)–(10), if there exists  $\delta > 0$  such that

$$J(\bar{\mathbf{y}}, \bar{\mathbf{u}}) \le J(\mathbf{y}, \mathbf{u})$$

for all  $\mathbf{u} \in \mathbb{U}_{ad}$  such that  $\|\mathbf{u} - \bar{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)} \leq \delta$ . Here,  $\bar{\mathbf{y}}$  and  $\mathbf{y}$  denote the velocity fields associated to  $\bar{\mathbf{u}}$  and  $\mathbf{u}$ , respectively.

The existence of a local solution  $(\bar{\mathbf{y}}, \bar{\mathbf{u}}) \in \mathbf{H}_0^1(\Omega) \times \mathbb{U}_{ad}$  for problem (9)–(10) follows standard arguments; see [15, Theorem 3.1].

**3.2. First order optimality conditions.** We now turn our attention to discuss first and second order optimality conditions for problem (9)–(10). We begin such a discussion by introducing the so-called control-to-state map  $S : \mathbf{L}^2(\Omega) \to \mathbf{V}(\Omega)$  which, given a control  $\mathbf{u} \in \mathbb{U}_{ad}$ , associates to it the unique velocity field  $\mathbf{y} \in \mathbf{V}(\Omega)$  that solves (12) under the smallness assumption (11). With this operator at hand, we introduce the reduced cost functional

$$j(\mathbf{u}) := J(\mathcal{S}\mathbf{u}, \mathbf{u}) = \frac{1}{2} \|\mathcal{S}\mathbf{u} - \mathbf{y}_{\Omega}\|_{\mathbf{L}^{2}(\Omega)}^{2} + \frac{\alpha}{2} \|\mathbf{u}\|_{\mathbf{L}^{2}(\Omega)}^{2}.$$

Under the smallness assumption (11), the control-to-state map S is Fréchet differentiable from  $\mathbf{L}^2(\Omega)$  to  $\mathbf{V}(\Omega)$ ; see [28, Lemma 3.8]. As a consequence, if  $\bar{\mathbf{u}}$  denotes a local optimal control for problem (9)–(10),  $\bar{\mathbf{u}}$  satisfies the variational inequality

(13) 
$$(\bar{\mathbf{z}} + \alpha \bar{\mathbf{u}}, \mathbf{u} - \bar{\mathbf{u}})_{\mathbf{L}^2(\Omega)} \ge 0 \quad \forall \mathbf{u} \in \mathbb{U}_{ad},$$

where  $(\bar{\mathbf{z}}, \bar{r}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  is the unique solution to the adjoint equations

(14) 
$$\nu(\nabla \mathbf{w}, \nabla \bar{\mathbf{z}})_{\mathbf{L}^{2}(\Omega)} + b(\bar{\mathbf{y}}; \mathbf{w}, \bar{\mathbf{z}}) + b(\mathbf{w}; \bar{\mathbf{y}}, \bar{\mathbf{z}}) - (\bar{r}, \operatorname{div} \mathbf{w})_{L^{2}(\Omega)} = (\bar{\mathbf{y}} - \mathbf{y}_{\Omega}, \mathbf{w})_{\mathbf{L}^{2}(\Omega)} = (s, \operatorname{div} \bar{\mathbf{z}})_{L^{2}(\Omega)} = 0,$$

for all  $(\mathbf{w}, s) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ . For details, we refer the reader to [25, Theorem 2.2], [28, Theorem 3.10], and [13, Theorem 3.2].

*Remark* 2 (well–posedness of adjoint equations). Define  $\mathcal{B} : \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$  by

$$\mathcal{B}(\mathbf{w}, \mathbf{v}) := \nu(\nabla \mathbf{w}, \nabla \mathbf{v})_{\mathbf{L}^{2}(\Omega)} + b(\bar{\mathbf{y}}; \mathbf{w}, \mathbf{v}) + b(\mathbf{w}; \bar{\mathbf{y}}, \mathbf{v}).$$

Assume that (11) holds. Theorem 1 thus yields  $\|\nabla \bar{\mathbf{y}}\|_{\mathbf{L}^2(\Omega)} \leq \theta \mathcal{C}_b^{-1} \nu$ . Consequently,

(15) 
$$\mathcal{B}(\mathbf{w},\mathbf{w}) \ge \nu \|\nabla \mathbf{w}\|_{\mathbf{L}^{2}(\Omega)}^{2} - \mathcal{C}_{b} \|\nabla \mathbf{w}\|_{\mathbf{L}^{2}(\Omega)}^{2} \|\nabla \bar{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)} \ge \nu(1-\theta) \|\nabla \mathbf{w}\|_{\mathbf{L}^{2}(\Omega)}^{2},$$

where, we recall  $\theta < 1$ . We have thus proved that  $\mathcal{B}$  is coercive on  $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$ . The standard inf-sup theory for saddle point problems [16, Theorem 2.34] yields, on the basis of (6), the existence and uniqueness of a solution  $(\bar{\mathbf{z}}, \bar{r}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  to (14). In addition, set  $\mathbf{w} = \bar{\mathbf{z}}$  and invoke (4), (5), and (11) to arrive at the bound

(16) 
$$\|\nabla \bar{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)} \leq \frac{C_{2}}{\nu(1-\theta)} \|\bar{\mathbf{y}} - \mathbf{y}_{\Omega}\|_{\mathbf{L}^{2}(\Omega)}.$$

The local optimal control  $\bar{\mathbf{u}}$  satisfies (13) if and only if [25, equation (2.10)], [13, equation (3.9)]

(17) 
$$\bar{\mathbf{u}}(x) = \Pi_{[\mathbf{a},\mathbf{b}]} \left( -\alpha^{-1} \bar{\mathbf{z}}(x) \right) \text{ a.e. } x \in \Omega,$$

where the projection operator  $\Pi_{[\mathbf{a},\mathbf{b}]}: \mathbf{L}^1(\Omega) \to \mathbb{U}_{ad}$  is defined as

(18) 
$$\Pi_{[\mathbf{a},\mathbf{b}]}(\mathbf{v}) := \min\{\mathbf{b}, \max\{\mathbf{v},\mathbf{a}\}\}.$$

**3.3. Second order optimality conditions.** We now follow [13, section 3.2] and present necessary and sufficient second order optimality conditions. To introduce them, we define  $\mathbf{\bar{d}} := \mathbf{\bar{z}} + \alpha \mathbf{\bar{u}}$  and the cone of critical directions

$$\mathbf{C}_{\bar{\mathbf{u}}} := \{ \mathbf{v} \in \mathbf{L}^2(\Omega) \text{ that satisfies (19) and } \mathbf{v}_i(x) = 0 \text{ if } \bar{\mathbf{d}}_i(x) \neq 0, i = 1, \dots, d \}.$$

Here,  $\mathbf{d}_i$  corresponds to the *i*-th component of the vector  $\mathbf{d}$  and condition (19) reads

(19) 
$$\mathbf{v}(x) \begin{cases} \ge \mathbf{0} \text{ a.e. } x \in \Omega \text{ if } \bar{\mathbf{u}}(x) = \mathbf{a} \text{ and } \bar{\mathbf{d}}(x) = \mathbf{0}, \\ \le \mathbf{0} \text{ a.e. } x \in \Omega \text{ if } \bar{\mathbf{u}}(x) = \mathbf{b} \text{ and } \bar{\mathbf{d}}(x) = \mathbf{0}. \end{cases}$$

We are now in position to present second order necessary and sufficient optimality conditions; see [13, Theorems 3.6 and 3.8 and Corollary 3.9].

THEOREM 3 (second order optimality conditions). Assume that (11) holds. If  $\bar{\mathbf{u}} \in \mathbb{U}_{ad}$  is a local minimum for problem (9)–(10), then

$$j''(\bar{\mathbf{u}})\mathbf{v}^2 \ge 0 \quad \forall \mathbf{v} \in \mathbf{C}_{\bar{\mathbf{u}}}$$

Conversely, if  $(\bar{\mathbf{y}}, \bar{p}, \bar{\mathbf{z}}, \bar{r}, \bar{\mathbf{u}}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbb{U}_{ad}$  satisfies the first order optimality conditions (10), (13), and (14), and

(20) 
$$j''(\bar{\mathbf{u}})\mathbf{v}^2 > 0 \quad \forall \mathbf{v} \in \mathbf{C}_{\bar{\mathbf{u}}} \setminus \{\mathbf{0}\},$$

then, there exist  $\mu > 0$  and  $\varepsilon > 0$  such that

$$j(\mathbf{u}) \ge j(\bar{\mathbf{u}}) + \frac{\mu}{2} \left( \|\mathbf{u} - \bar{\mathbf{u}}\|_{\mathbf{L}^{2}(\Omega)}^{2} + \|\mathbf{y} - \bar{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)}^{2} \right),$$

for every pair  $(\mathbf{u}, \mathbf{y})$  that satisfies (2),  $\mathbf{u} \in \mathbb{U}_{ad}$ , and  $\|\mathbf{u} - \bar{\mathbf{u}}\|_{\mathbf{L}^{2}(\Omega)}^{2} + \|\mathbf{y} - \bar{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)}^{2} \leq \varepsilon$ .

We now introduce, given  $\tau > 0$ , the cone

(21) 
$$\mathbf{C}^{\tau}_{\bar{\mathbf{u}}} := \{ \mathbf{v} \in \mathbf{L}^2(\Omega) \text{ that satisfies (22)} \},$$

where condition (22) reads as follows:

(22) 
$$\mathbf{v}_{i}(x) \begin{cases} = 0 \text{ if } |\bar{\mathbf{d}}_{i}(x)| > \tau, \\ \geq 0 \text{ a.e. } x \in \Omega \text{ if } \bar{\mathbf{u}}_{i}(x) = \mathbf{a}_{i} \text{ and } |\bar{\mathbf{d}}_{i}(x)| \leq \tau, \\ \leq 0 \text{ a.e. } x \in \Omega \text{ if } \bar{\mathbf{u}}_{i}(x) = \mathbf{b}_{i} \text{ and } |\bar{\mathbf{d}}_{i}(x)| \leq \tau, \end{cases}$$

with  $i \in \{1, ..., d\}$ .

The next result will be of importance for deriving a posteriori error estimates for the numerical discretizations of (9)-(10) that we will propose; see [13, Corollary 3.11].

THEOREM 4 (equivalent second order optimality condition). Assume that (11) holds. Let  $(\bar{\mathbf{y}}, \bar{p}, \bar{\mathbf{z}}, \bar{\tau}, \bar{\mathbf{u}}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbb{U}_{ad}$  be a local solution to (9)–(10) satisfying the first order optimality conditions (10), (13), and (14). Then, (20) is equivalent to the existence of  $\mu > 0$  and  $\tau > 0$  such that

(23) 
$$j''(\bar{\mathbf{u}})\mathbf{v}^2 \ge \mu \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2 \quad \forall \mathbf{v} \in \mathbf{C}_{\bar{\mathbf{u}}}^\tau$$

We close this section with the next result.

LEMMA 5 (property of j''). Let  $\mathbf{u}, \mathbf{h}, \mathbf{v} \in \mathbf{L}^{\infty}(\Omega)$  and  $\mathsf{M} > 0$  be such that  $\max\{\|\mathbf{u}\|_{\mathbf{L}^{\infty}(\Omega)}, \|\mathbf{h}\|_{\mathbf{L}^{\infty}(\Omega)}\} \leq \mathsf{M}$ . Then, there exists  $C_{\mathsf{M}} > 0$  such that

(24) 
$$|j''(\mathbf{u} + \mathbf{h})\mathbf{v}^2 - j''(\mathbf{u})\mathbf{v}^2| \le C_{\mathsf{M}} \|\mathbf{h}\|_{\mathbf{L}^{\infty}(\Omega)} \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2$$

**3.4. Finite element approximation.** We now introduce the discrete setting in which we will operate. We consider  $\mathscr{T} = \{T\}$  to be a conforming partition of  $\overline{\Omega}$  into closed simplices T with size  $h_T = \operatorname{diam}(T)$ . Define  $h_{\mathscr{T}} := \max_{T \in \mathscr{T}} h_T$ . We denote by  $\mathbb{T}$  the collection of conforming and shape regular meshes that are refinements of an initial mesh  $\mathscr{T}_0$ . Let  $\mathscr{S}$  be the set of internal (d-1)-dimensional interelement boundaries S of  $\mathscr{T}$ . For  $T \in \mathscr{T}$ , let  $\mathscr{S}_T$  denote the subset of  $\mathscr{S}$  which contains the sides in  $\mathscr{S}$  which are sides of T. We denote by  $\mathcal{N}_S$  the subset of  $\mathscr{T}$  that contains the two elements that have S as a side, i.e.,  $\mathcal{N}_S = \{T^+, T^-\}$ , where  $T^+, T^- \in \mathscr{T}$  are such that  $S = T^+ \cap T^-$ . For  $T \in \mathscr{T}$ , we define the *star* associated with the element T as

(25) 
$$\mathcal{N}_T := \{ T' \in \mathscr{T} : \mathscr{S}_T \cap \mathscr{S}_{T'} \neq \emptyset \}.$$

In an abuse of notation, in what follows, by  $\mathcal{N}_T$  we will indistinctively denote either this set or the union of the triangles that comprise it.

For a discrete tensor valued function  $\mathbf{W}_{\mathscr{T}}$ , we define the jump or interelement residual on the internal side  $S \in \mathscr{S}$ , shared by the distinct elements  $T^+, T^- \in \mathcal{N}_S$ , by  $[\![\mathbf{W}_{\mathscr{T}} \cdot \mathbf{n}]\!] = \mathbf{W}_{\mathscr{T}}|_{T^+} \cdot \mathbf{n}^+ + \mathbf{W}_{\mathscr{T}}|_{T^-} \cdot \mathbf{n}^-$ . Here,  $\mathbf{n}^+$  and  $\mathbf{n}^-$  are unit normal on Spointing towards  $T^+$  and  $T^-$ , respectively.

We now introduce the inf-sup stable finite element spaces that we will consider in our work. Given a mesh  $\mathscr{T} \in \mathbb{T}$ , we denote by  $\mathbf{V}(\mathscr{T})$  and  $\mathcal{P}(\mathscr{T})$  the finite element spaces that approximate the velocity field and the pressure, respectively, based on the classical Taylor-Hood elements [16, section 4.2.5]:

(26) 
$$\mathbf{V}(\mathscr{T}) = \{\mathbf{v}_{\mathscr{T}} \in \mathbf{C}(\overline{\Omega}) : \mathbf{v}_{\mathscr{T}}|_{T} \in [\mathbb{P}_{2}(T)]^{d} \ \forall T \in \mathscr{T}\} \cap \mathbf{H}_{0}^{1}(\Omega),$$

(27) 
$$\mathcal{P}(\mathscr{T}) = \{q_{\mathscr{T}} \in C(\overline{\Omega}) : q_{\mathscr{T}}|_{T} \in \mathbb{P}_{1}(T) \ \forall T \in \mathscr{T}\} \cap L^{2}_{0}(\Omega)$$

To approximate local optimal controls, we consider piecewise quadratic functions, that is:  $\mathbf{\bar{u}}_{\mathscr{T}} \in \mathbb{U}_{ad}(\mathscr{T})$ , where

(28) 
$$\mathbb{U}_{ad}(\mathscr{T}) = \mathbb{U}(\mathscr{T}) \cap \mathbb{U}_{ad}, \quad \mathbb{U}(\mathscr{T}) = \{\mathbf{v}_{\mathscr{T}} \in \mathbf{L}^{\infty}(\Omega) : \mathbf{v}_{\mathscr{T}}|_{T} \in [\mathbb{P}_{2}(T)]^{d} \ \forall T \in \mathscr{T}\}.$$

The discrete counterpart of (9)–(10) thus reads as follows: Find min  $J(\mathbf{y}_{\mathscr{T}}, \mathbf{u}_{\mathscr{T}})$  subject to the discrete state equation

(29) 
$$\nu(\nabla \mathbf{y}_{\mathscr{T}}, \nabla \mathbf{v}_{\mathscr{T}})_{\mathbf{L}^{2}(\Omega)} + b(\mathbf{y}_{\mathscr{T}}; \mathbf{y}_{\mathscr{T}}, \mathbf{v}_{\mathscr{T}}) - (p_{\mathscr{T}}, \operatorname{div} \mathbf{v}_{\mathscr{T}})_{L^{2}(\Omega)} = (\mathbf{u}_{\mathscr{T}}, \mathbf{v}_{\mathscr{T}})_{\mathbf{L}^{2}(\Omega)}, (q_{\mathscr{T}}, \operatorname{div} \mathbf{y}_{\mathscr{T}})_{L^{2}(\Omega)} = 0,$$

for all  $(\mathbf{v}_{\mathscr{T}}, q_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ , and the discrete constraints  $\mathbf{u}_{\mathscr{T}} \in \mathbb{U}_{ad}(\mathscr{T})$ . Assume that (11) holds. Under the assumptions that the mesh  $\mathscr{T}$  is sufficiently refined and

that  $\mathbf{u}_{\mathscr{T}} \in \mathbb{U}_{ad}(\mathscr{T})$  is close enough to  $\bar{\mathbf{u}}$ , the discrete equations (29) admit a unique solution which lies in a neighborhood of  $(\bar{\mathbf{y}}, \bar{p})$ ; see [13, Theorem 4.8]. In addition, we have that our discrete optimal control problem admits at least one solution; see [13, Theorem 4.11].

If  $\bar{\mathbf{u}}_{\mathscr{T}}$  denotes a local solution, we have

(30) 
$$(\bar{\mathbf{z}}_{\mathscr{T}} + \alpha \bar{\mathbf{u}}_{\mathscr{T}}, \mathbf{u}_{\mathscr{T}} - \bar{\mathbf{u}}_{\mathscr{T}})_{\mathbf{L}^{2}(\Omega)} \geq 0 \quad \forall \mathbf{u}_{\mathscr{T}} \in \mathbb{U}_{ad}(\mathscr{T}),$$

where the pair  $(\bar{\mathbf{z}}_{\mathscr{T}}, \bar{r}_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$  solves

(31)  

$$\begin{aligned}
\nu(\nabla \mathbf{w}\,\mathcal{T},\nabla \bar{\mathbf{z}}\,\mathcal{T})_{\mathbf{L}^{2}(\Omega)} + b(\bar{\mathbf{y}}\,\mathcal{T};\mathbf{w}\,\mathcal{T},\bar{\mathbf{z}}\,\mathcal{T}) \\
+ b(\mathbf{w}\,\mathcal{T};\bar{\mathbf{y}}\,\mathcal{T},\bar{\mathbf{z}}\,\mathcal{T}) - (\bar{r}\,\mathcal{T},\operatorname{div}\,\mathbf{w}\,\mathcal{T})_{L^{2}(\Omega)} &= (\bar{\mathbf{y}}\,\mathcal{T} - \mathbf{y}_{\Omega},\mathbf{w}\,\mathcal{T})_{\mathbf{L}^{2}(\Omega)}, \\
(s\,\mathcal{T},\operatorname{div}\,\bar{\mathbf{z}}\,\mathcal{T})_{L^{2}(\Omega)} &= 0,
\end{aligned}$$

for all  $(\mathbf{w}_{\mathscr{T}}, s_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$  [13, Lemma 4.14]. Under the assumption that

(32) 
$$2\|\nabla \bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(\Omega)} \leq \theta \nu \mathcal{C}_{b}^{-1}, \quad \theta < 1,$$

the discrete problem (31) admits a unique solution. In fact, define

(33) 
$$\mathcal{C}(\mathbf{w}, \mathbf{v}) := \nu(\nabla \mathbf{w}, \nabla \mathbf{v})_{\mathbf{L}^{2}(\Omega)} + b(\bar{\mathbf{y}}_{\mathscr{T}}; \mathbf{w}, \mathbf{v}) + b(\mathbf{w}; \bar{\mathbf{y}}_{\mathscr{T}}, \mathbf{v}).$$

With (32) at hand, similar arguments to ones used to derive (15) yield the coercivity of  $\mathcal{C}$  in  $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$ . Since the pair ( $\mathbf{V}(\mathscr{T}), \mathcal{P}(\mathscr{T})$ ) satisfy a discrete inf–sup condition [16, Lemma 4.24], an application of [16, Theorem 2.42] allows us to conclude.

4. Reliability Analysis. In this section we propose and analyze an a posteriori error estimator for the optimal control problem (9)-(10). This estimator can be decomposed as the sum of three contributions which are related to the discretization of the state equations, adjoint equations, and the control set. To obtain a reliability estimate, i.e., an upper bound for the error in terms of the devised a posteriori error estimator, we invoke upper bounds on the error between the solution to the discretization (29)-(31) and auxiliary variables that we define in the following sections.

In order to guarantee the existence of a local solution  $(\bar{\mathbf{y}}_{\mathscr{T}}, \bar{\mathbf{u}}_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathbb{U}_{ad}(\mathscr{T})$ to the discrete version of the optimal control problem (9)–(10), satisfying the discrete system (29)–(31), we shall assume, throughout the following sections, that  $\bar{\mathbf{u}}_{\mathscr{T}}$  is close enough to the locally optimal control  $\bar{\mathbf{u}}$  and that the underlying mesh is sufficiently refined.

**4.1.** A posteriori error analysis for the state equations. We present, inspired in [24] (see also [3, section 9.3]), a posteriori error estimates for the stationary Navier–Stokes equations (2).

We begin the discussion by introducing the following auxiliary variables. Assume that (11) holds. Let  $(\hat{\mathbf{y}}, \hat{p}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  be the solution to

(34) 
$$\nu(\nabla \hat{\mathbf{y}}, \nabla \mathbf{v})_{\mathbf{L}^{2}(\Omega)} + b(\hat{\mathbf{y}}; \hat{\mathbf{y}}, \mathbf{v}) - (\hat{p}, \operatorname{div} \mathbf{v})_{L^{2}(\Omega)} = (\bar{\mathbf{u}}_{\mathscr{T}}, \mathbf{v})_{\mathbf{L}^{2}(\Omega)} \quad \forall \mathbf{v} \in \mathbf{H}_{0}^{1}(\Omega), (q, \operatorname{div} \hat{\mathbf{y}})_{L^{2}(\Omega)} = 0 \qquad \forall q \in L_{0}^{2}(\Omega).$$

Theorem 1 guarantees the existence of a unique pair  $(\hat{\mathbf{y}}, \hat{p})$  solving problem (34) with

(35) 
$$\|\nabla \hat{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)} \leq \theta \mathcal{C}_{b}^{-1} \nu, \quad \theta < 1.$$

Notice that the pair  $(\bar{\mathbf{y}}_{\mathscr{T}}, \bar{p}_{\mathscr{T}})$ , which solves (29) with  $\mathbf{u}_{\mathscr{T}}$  replaced by  $\bar{\mathbf{u}}_{\mathscr{T}}$ , can be seen as the finite element approximation, within the space  $\mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ , of  $(\hat{\mathbf{y}}, \hat{p})$ . This observation motivates us to define the following a posteriori error estimator:

$$(36) \qquad \mathcal{E}_{st}^{2} := \sum_{T \in \mathscr{T}} \mathcal{E}_{st,T}^{2}, \quad \mathcal{E}_{st,T}^{2} := h_{T}^{2} \| \bar{\mathbf{u}}_{\mathscr{T}} + \nu \Delta \bar{\mathbf{y}}_{\mathscr{T}} - (\bar{\mathbf{y}}_{\mathscr{T}} \cdot \nabla) \bar{\mathbf{y}}_{\mathscr{T}} - \nabla \bar{p}_{\mathscr{T}} \|_{\mathbf{L}^{2}(T)}^{2} \\ + \| \operatorname{div} \bar{\mathbf{y}}_{\mathscr{T}} \|_{L^{2}(T)}^{2} + h_{T} \| \llbracket (\nu \nabla \bar{\mathbf{y}}_{\mathscr{T}} - \bar{p}_{\mathscr{T}} \mathbb{I}_{d}) \cdot \mathbf{n} \rrbracket \|_{\mathbf{L}^{2}(\partial T \setminus \partial \Omega)}^{2}.$$

We present the following global reliability result.

THEOREM 6 (global reliability of  $\mathcal{E}_{st}$ ). Assume that (11) holds. Let  $(\hat{\mathbf{y}}, \hat{p}) \in$  $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  be the unique solution to (34). Let  $(\bar{\mathbf{y}}_{\mathscr{T}}, \bar{p}_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$  be the solution to (29) with  $\mathbf{u}_{\mathscr{T}}$  replaced by  $\bar{\mathbf{u}}_{\mathscr{T}}$ . Assume that the estimate

(37) 
$$\|\nabla \bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(\Omega)} < \nu \mathcal{C}_{b}^{-1}$$

holds. Then, we have that

(38) 
$$\|\nabla(\hat{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}})\|_{\mathbf{L}^{2}(\Omega)}^{2} + \|\hat{p} - \bar{p}_{\mathscr{T}}\|_{L^{2}(\Omega)}^{2} \lesssim \mathcal{E}_{st}^{2}$$

with a hidden constant that is independent of  $(\hat{\mathbf{y}}, \hat{p})$  and  $(\bar{\mathbf{y}}_{\mathscr{T}}, \bar{p}_{\mathscr{T}})$ , the size of the elements in the mesh  $\mathscr{T}$ , and  $\#\mathscr{T}$ .

*Proof.* To perform a reliability analysis for the a posteriori error estimator (36), we introduce a Ritz projection  $(\varphi, \psi)$  of the residuals [2]. The pair  $(\varphi, \psi)$  is defined as the solution to the following problem: Find  $(\varphi, \psi) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  such that

(39) 
$$\begin{array}{l} (\nabla \boldsymbol{\varphi}, \nabla \mathbf{v})_{\mathbf{L}^{2}(\Omega)} = \nu(\nabla \hat{\mathbf{e}}_{\mathbf{y}}, \nabla \mathbf{v})_{\mathbf{L}^{2}(\Omega)} - (\hat{e}_{p}, \operatorname{div} \mathbf{v})_{L^{2}(\Omega)} + b(\hat{\mathbf{y}}; \hat{\mathbf{e}}_{\mathbf{y}}, \mathbf{v}) + b(\hat{\mathbf{e}}_{\mathbf{y}}; \bar{\mathbf{y}}_{\mathscr{T}}, \mathbf{v}), \\ (\psi, q)_{L^{2}(\Omega)} = (q, \operatorname{div} \hat{\mathbf{e}}_{\mathbf{y}})_{L^{2}(\Omega)}, \end{array}$$

for all  $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$  and  $q \in L_0^2(\Omega)$ , respectively. To shorten notation, we have introduced  $(\hat{\mathbf{e}}_{\mathbf{y}}, \hat{e}_p) := (\hat{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}}, \hat{p} - \bar{p}_{\mathscr{T}})$ . The existence and uniqueness of the pair  $(\varphi, \psi)$ follows from a simple application of the Lax–Milgram Lemma. On the other hand, under assumption (37), similar arguments to the ones developed in [24, Theorem 4] (see also [3, section 9.3]) yield the estimate

(40) 
$$\|\nabla \hat{\mathbf{e}}_{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)}^{2} + \|\hat{e}_{p}\|_{L^{2}(\Omega)}^{2} \lesssim \|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^{2}(\Omega)}^{2} + \|\psi\|_{L^{2}(\Omega)}^{2}$$

The rest of the proof is dedicated to bound the terms on the right-hand side of (40). Let  $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$  and set q = 0 in (39). This, combined with (34), yields

 $(\nabla \boldsymbol{\varphi}, \nabla \mathbf{v})_{\mathbf{L}^{2}(\Omega)} = (\bar{\mathbf{u}}_{\mathscr{T}}, \mathbf{v})_{\mathbf{L}^{2}(\Omega)} - \nu (\nabla \bar{\mathbf{y}}_{\mathscr{T}}, \nabla \mathbf{v})_{\mathbf{L}^{2}(\Omega)} - b(\bar{\mathbf{y}}_{\mathscr{T}}; \bar{\mathbf{y}}_{\mathscr{T}}, \mathbf{v}) + (\bar{p}_{\mathscr{T}}, \operatorname{div} \mathbf{v})_{L^{2}(\Omega)}.$ 

Denote by  $\mathcal{I}_{\mathscr{T}}: \mathbf{L}^{1}(\Omega) \to \mathbf{V}(\mathscr{T})$  the Clément interpolation operator [11, 14]. Invoke the previous relation, the discrete problem (29) with  $\mathbf{v}_{\mathscr{T}} = \mathcal{I}_{\mathscr{T}}\mathbf{v}$ , an elementwise integration by parts formula, standard approximation properties for  $\mathcal{I}_{\mathscr{T}}$ , and the finite overlapping property of stars, to conclude that

$$\begin{aligned} (\nabla \boldsymbol{\varphi}, \nabla \mathbf{v})_{\mathbf{L}^{2}(\Omega)} \lesssim \left( \sum_{T \in \mathscr{T}} h_{T}^{2} \| \bar{\mathbf{u}}_{\mathscr{T}} + \nu \Delta \bar{\mathbf{y}}_{\mathscr{T}} - (\bar{\mathbf{y}}_{\mathscr{T}} \cdot \nabla) \bar{\mathbf{y}}_{\mathscr{T}} - \nabla \bar{p}_{\mathscr{T}} \|_{\mathbf{L}^{2}(T)}^{2} \\ + h_{T} \| \llbracket (\nu \nabla \bar{\mathbf{y}}_{\mathscr{T}} - \bar{p}_{\mathscr{T}} \mathbb{I}_{d}) \cdot \mathbf{n} \rrbracket \|_{\mathbf{L}^{2}(\partial T \setminus \partial \Omega)}^{2} \right)^{\frac{1}{2}} \| \nabla \mathbf{v} \|_{\mathbf{L}^{2}(\Omega)}. \end{aligned}$$

Set  $\mathbf{v} = \boldsymbol{\varphi}$ . This yields the estimate  $\|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)} \lesssim \mathcal{E}_{st}$ .

Now, let  $q \in L^2_0(\Omega)$  and set  $\mathbf{v} = \mathbf{0}$  in (39). The Cauchy–Schwarz inequality yields

$$(\psi, q)_{L^2(\Omega)} \le \left(\sum_{T \in \mathscr{T}} \|\operatorname{div} \bar{\mathbf{y}}_{\mathscr{T}}\|_{L^2(T)}^2\right)^{\frac{1}{2}} \|q\|_{L^2(\Omega)}.$$

Consequently,  $\|\psi\|_{L^2(\Omega)} \leq \mathcal{E}_{st}$ .

A collection of the previous estimates yield  $\|\nabla \varphi\|_{\mathbf{L}^2(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 \lesssim \mathcal{E}_{st}^2$ . We thus invoke (40) to arrive at the desired estimate (38). This concludes the proof.

**4.2.** A posteriori error analysis for the adjoint equations. In this section we introduce an auxiliary problem related to the adjoint equations, devise an a posteriori error estimator for such a problem, and obtain a global reliability result. To the best of our knowledge, these results are not available in the literature.

Let  $(\hat{\mathbf{z}}, \hat{r}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  be the solution to

(41) 
$$\nu(\nabla \mathbf{w}, \nabla \hat{\mathbf{z}})_{\mathbf{L}^{2}(\Omega)} + b(\bar{\mathbf{y}}_{\mathscr{T}}; \mathbf{w}, \hat{\mathbf{z}}) + b(\mathbf{w}; \bar{\mathbf{y}}_{\mathscr{T}}, \hat{\mathbf{z}}) - (\hat{r}, \operatorname{div} \mathbf{w})_{L^{2}(\Omega)} = (\bar{\mathbf{y}}_{\mathscr{T}} - \mathbf{y}_{\Omega}, \mathbf{w})_{\mathbf{L}^{2}(\Omega)}, \\ (s, \operatorname{div} \hat{\mathbf{z}})_{L^{2}(\Omega)} = 0,$$

for all  $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$  and  $s \in L_0^2(\Omega)$ , respectively. Under assumption (32) problem (41) is well-posed. On the other hand, notice that  $(\bar{\mathbf{z}}_{\mathscr{T}}, \bar{r}_{\mathscr{T}})$ , the solution to (31), can be seen as the finite element approximation, within the space  $\mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ , of  $(\hat{\mathbf{z}}, \hat{r})$ . In view of this fact, we define, for  $T \in \mathscr{T}$ , the local error indicators

$$(42) \quad \mathcal{E}^{2}_{ad,T} := h_{T}^{2} \| \bar{\mathbf{y}}_{\mathscr{T}} - \mathbf{y}_{\Omega} + \nu \Delta \bar{\mathbf{z}}_{\mathscr{T}} - (\nabla \bar{\mathbf{y}}_{\mathscr{T}})^{\mathsf{T}} \bar{\mathbf{z}}_{\mathscr{T}} + (\bar{\mathbf{y}}_{\mathscr{T}} \cdot \nabla) \bar{\mathbf{z}}_{\mathscr{T}} - \nabla \bar{r}_{\mathscr{T}} \|_{\mathbf{L}^{2}(T)}^{2} + h_{T} \| \llbracket (\nu \nabla \bar{\mathbf{z}}_{\mathscr{T}} - \bar{r}_{\mathscr{T}} \mathbb{I}_{d}) \cdot \mathbf{n} \rrbracket \|_{\mathbf{L}^{2}(\partial T \setminus \partial \Omega)}^{2} + \| \operatorname{div} \bar{\mathbf{z}}_{\mathscr{T}} \|_{L^{2}(T)}^{2},$$

and the a posteriori error estimator

(43) 
$$\mathcal{E}_{ad}^2 := \sum_{T \in \mathscr{T}} \mathcal{E}_{ad,T}^2$$

The following result yields an upper bound for the error  $\|\nabla(\hat{\mathbf{z}} - \bar{\mathbf{z}}_{\mathscr{T}})\|_{\mathbf{L}^{2}(\Omega)} + \|\hat{r} - \bar{r}_{\mathscr{T}}\|_{L^{2}(\Omega)}$  in terms of the computable quantity  $\mathcal{E}_{ad}$ .

THEOREM 7 (global reliability of  $\mathcal{E}_{ad}$ ). Let  $(\hat{\mathbf{y}}, \hat{p})$  and  $(\bar{\mathbf{y}}_{\mathscr{T}}, \bar{p}_{\mathscr{T}})$  be as in the statement of Theorem 6. Let  $(\hat{\mathbf{z}}, \hat{r}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  and  $(\bar{\mathbf{z}}_{\mathscr{T}}, \bar{r}_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$  be the solutions to (41) and (31), respectively. Assume that the estimate (32) holds. Then, we have that

(44) 
$$\|\nabla(\hat{\mathbf{z}} - \bar{\mathbf{z}}_{\mathscr{T}})\|_{\mathbf{L}^{2}(\Omega)}^{2} + \|\hat{r} - \bar{r}_{\mathscr{T}}\|_{L^{2}(\Omega)}^{2} \lesssim \mathcal{E}_{ad}^{2},$$

with a hidden constant that is independent of  $(\hat{\mathbf{z}}, \hat{r})$ ,  $(\bar{\mathbf{z}}_{\mathscr{T}}, \bar{r}_{\mathscr{T}})$ , the size of the elements in the mesh  $\mathscr{T}$ , and  $\#\mathscr{T}$ .

*Proof.* We proceed on the basis of four steps.

Step 1. To simplify the presentation of the material, we define the pair  $(\hat{\mathbf{e}}_{\mathbf{z}}, \hat{e}_r) := (\hat{\mathbf{z}} - \bar{\mathbf{z}}_{\mathscr{T}}, \hat{r} - \bar{r}_{\mathscr{T}}).$ 

Define the Ritz projection  $(\boldsymbol{\eta}, \omega)$  of the residuals associated to the discretization (31) of (41) as the solution to the following problem: Find  $(\boldsymbol{\eta}, \omega) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  such that

(45) 
$$(\nabla \boldsymbol{\eta}, \nabla \mathbf{w})_{\mathbf{L}^{2}(\Omega)} = \mathcal{C}(\mathbf{w}, \hat{\mathbf{e}}_{\mathbf{z}}) - (\hat{e}_{r}, \operatorname{div} \mathbf{w})_{L^{2}(\Omega)} \qquad \forall \mathbf{w} \in \mathbf{H}_{0}^{1}(\Omega), \\ (\omega, s)_{L^{2}(\Omega)} = (s, \operatorname{div} \hat{\mathbf{e}}_{\mathbf{z}})_{L^{2}(\Omega)} \qquad \forall s \in L_{0}^{2}(\Omega),$$

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where  $\mathcal{C}$  is defined as in (33). The Lax–Milgram Lemma immediately yields the existence and uniqueness of  $(\boldsymbol{\eta}, \omega) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  solving (45).

The rest of the proof is dedicated to obtain the estimates

(46) 
$$\|\nabla \hat{\mathbf{e}}_{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)}^{2} + \|\hat{e}_{r}\|_{L^{2}(\Omega)}^{2} \lesssim \|\nabla \boldsymbol{\eta}\|_{\mathbf{L}^{2}(\Omega)}^{2} + \|\omega\|_{L^{2}(\Omega)}^{2} \lesssim \mathcal{E}_{ad}^{2}.$$

Step 2. The goal of this step is to prove the first estimate in (46). To accomplish this task, we first observe that the pair  $(\hat{\mathbf{e}}_{\mathbf{z}}, \hat{e}_r)$  satisfies the identities:

$$\mathcal{C}(\mathbf{w}, \hat{\mathbf{e}}_{\mathbf{z}}) - (\hat{e}_r, \operatorname{div} \mathbf{w})_{L^2(\Omega)} = (\nabla \boldsymbol{\eta}, \nabla \mathbf{w})_{\mathbf{L}^2(\Omega)} \qquad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega),$$
$$(s, \operatorname{div} \hat{\mathbf{e}}_{\mathbf{z}})_{L^2(\Omega)} = (\omega, s)_{L^2(\Omega)} \qquad \forall s \in L_0^2(\Omega).$$

In view of the fact that  $\bar{\mathbf{y}}_{\mathscr{T}} \in \mathbf{V}(\mathscr{T})$  satisfies assumption (32), similar arguments to those elaborated in the proof of (15) yield that  $\mathcal{C}$  is coercive in  $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$ . We can thus apply the inf-sup theory for saddle point problems given by Brezzi in [12] to conclude the stability estimate

(47) 
$$\|\nabla \hat{\mathbf{e}}_{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)}^{2} + \|\hat{e}_{r}\|_{L^{2}(\Omega)}^{2} \lesssim \|\nabla \boldsymbol{\eta}\|_{\mathbf{L}^{2}(\Omega)}^{2} + \|\omega\|_{L^{2}(\Omega)}^{2},$$

with a hidden constant that depends on  $\nu$ .

Step 3. In this step we obtain the second estimate in (46). To accomplish this task, we invoke problems (45) and (41) to arrive at

(48)  

$$(\nabla \boldsymbol{\eta}, \nabla \mathbf{w})_{\mathbf{L}^{2}(\Omega)} = (\bar{\mathbf{y}}_{\mathscr{T}} - \mathbf{y}_{\Omega}, \mathbf{w})_{\mathbf{L}^{2}(\Omega)} - \nu(\nabla \mathbf{w}, \nabla \bar{\mathbf{z}}_{\mathscr{T}})_{\mathbf{L}^{2}(\Omega)} - b(\bar{\mathbf{y}}_{\mathscr{T}}; \mathbf{w}, \bar{\mathbf{z}}_{\mathscr{T}}) - b(\mathbf{w}; \bar{\mathbf{y}}_{\mathscr{T}}, \bar{\mathbf{z}}_{\mathscr{T}}) + (\bar{r}_{\mathscr{T}}, \operatorname{div} \mathbf{w})_{L^{2}(\Omega)},$$

$$(\omega, s)_{L^{2}(\Omega)} = (s, \operatorname{div} \hat{\mathbf{e}}_{\mathbf{z}})_{L^{2}(\Omega)},$$

for all  $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$  and  $s \in L_0^2(\Omega)$ , respectively. Let  $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$  and set s = 0 in (48). Invoke the discrete problem (31) with  $\mathbf{w}_{\mathscr{T}} = \mathcal{I}_{\mathscr{T}}\mathbf{w}$ , an elementwise integration by parts formula, standard approximation properties for  $\mathcal{I}_{\mathscr{T}}$ , and the finite overlapping property of stars, to conclude that

$$\begin{aligned} (\nabla \boldsymbol{\eta}, \nabla \mathbf{w})_{\mathbf{L}^{2}(\Omega)} \lesssim & \left( \sum_{T \in \mathscr{T}} h_{T}^{2} \| \bar{\mathbf{y}}_{\mathscr{T}} - \mathbf{y}_{\Omega} + \nu \Delta \bar{\mathbf{z}}_{\mathscr{T}} - (\nabla \bar{\mathbf{y}}_{\mathscr{T}})^{\mathsf{T}} \bar{\mathbf{z}}_{\mathscr{T}} + (\bar{\mathbf{y}}_{\mathscr{T}} \cdot \nabla) \bar{\mathbf{z}}_{\mathscr{T}} - \nabla \bar{r}_{\mathscr{T}} \|_{L^{2}(T)}^{2} \\ & + h_{T} \| \left[ (\nu \nabla \bar{\mathbf{z}}_{\mathscr{T}} - \bar{r}_{\mathscr{T}} \mathbb{I}_{d}) \cdot \mathbf{n} \right] \|_{L^{2}(\partial T \setminus \partial \Omega)}^{2} \right)^{\frac{1}{2}} \| \nabla \mathbf{w} \|_{\mathbf{L}^{2}(\Omega)}. \end{aligned}$$

Set  $\mathbf{w} = \boldsymbol{\eta}$ . This yields the estimate  $\|\nabla \boldsymbol{\eta}\|_{\mathbf{L}^2(\Omega)} \lesssim \mathcal{E}_{ad}$ .

Now, let  $s \in L_0^2(\Omega)$  and set  $\mathbf{w} = \mathbf{0}$  in (48). The Cauchy–Schwarz inequality, in view of the fact that  $\hat{\mathbf{z}} \in \mathbf{V}(\Omega)$ , yields

$$(\omega, s)_{L^2(\Omega)} \le \left(\sum_{T \in \mathscr{T}} \|\operatorname{div} \, \bar{\mathbf{z}}_{\mathscr{T}}\|_{L^2(T)}^2\right)^{\frac{1}{2}} \|s\|_{L^2(\Omega)},$$

which implies that  $\|\omega\|_{L^2(\Omega)} \leq \mathcal{E}_{ad}$ .

A collection of the previous estimates yield  $\|\nabla \eta\|_{\mathbf{L}^2(\Omega)}^2 + \|\omega\|_{L^2(\Omega)}^2 \lesssim \mathcal{E}_{ad}^2$ .

Step 4. Apply (47) and the bounds obtained in step 3 for  $\|\nabla \eta\|_{L^2(\Omega)}$  and  $\|\omega\|_{L^2(\Omega)}$  to arrive at the desired estimate (44).

4.3. Reliability analysis for the optimal control problem. In this section we design an a posteriori error estimator for problem (9)–(10) and provide a reliability analysis. The error estimator can be decomposed as the sum of three contributions: two contributions related to the discretization of the state and adjoint equations,  $\mathcal{E}_{st}$  and  $\mathcal{E}_{ad}$ , respectively (which have been already introduced in sections 4.1 and 4.2) and a contribution associated to the discretization of the optimal control variable.

To present the contribution associated to  $\bar{\mathbf{u}}$ , we define the auxiliary variable

(49) 
$$\tilde{\mathbf{u}} := \Pi_{[\mathbf{a},\mathbf{b}]} \left( -\alpha^{-1} \bar{\mathbf{z}}_{\mathscr{T}} \right)$$

A key property in favor of the definition of  $\tilde{\mathbf{u}} \in \mathbb{U}_{ad}$  is that  $\tilde{\mathbf{u}}$  satisfies the inequality

(50) 
$$(\bar{\mathbf{z}}_{\mathscr{T}} + \alpha \tilde{\mathbf{u}}, \mathbf{u} - \tilde{\mathbf{u}})_{\mathbf{L}^{2}(\Omega)} \ge 0 \quad \forall \mathbf{u} \in \mathbb{U}_{ad}.$$

With the variable  $\tilde{\mathbf{u}}$  at hand, we define the following error estimator and local error indicators associated to the discretization of the optimal control variable:

(51) 
$$\mathcal{E}_{ct}^2 := \sum_{T \in \mathscr{T}} \mathcal{E}_{ct,T}^2, \qquad \mathcal{E}_{ct,T} := \|\tilde{\mathbf{u}} - \bar{\mathbf{u}}_{\mathscr{T}}\|_{\mathbf{L}^2(T)}.$$

We define now the a posteriori error estimator for the control problem (9)-(10):

(52) 
$$\mathcal{E}_{ocp}^2 := \mathcal{E}_{st}^2 + \mathcal{E}_{ad}^2 + \mathcal{E}_{ct}^2.$$

The estimators  $\mathcal{E}_{st}$ ,  $\mathcal{E}_{ad}$ , and  $\mathcal{E}_{ct}$ , are defined as in (36), (43), and (51), respectively.

The next result is instrumental for our a posteriori error analysis.

THEOREM 8 (auxiliary control estimate). Assume that the smallness assumption (11) holds. Let  $(\bar{\mathbf{y}}, \bar{p}, \bar{\mathbf{z}}, \bar{r}, \bar{\mathbf{u}}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbb{U}_{ad}$  be a local solution of (9)–(10) that satisfies the sufficient second order optimality condition (20), or equivalently (23). Let M be a positive constant such that  $\max\{\|\bar{\mathbf{u}} + \theta_{\mathscr{T}}(\tilde{\mathbf{u}} - \bar{\mathbf{u}})\|_{\mathbf{L}^{\infty}(\Omega)}, \|\tilde{\mathbf{u}} - \bar{\mathbf{u}}\|_{\mathbf{L}^{\infty}(\Omega)}\} \leq \mathsf{M}$  with  $\theta_{\mathscr{T}} \in (0, 1)$ . Let  $\bar{\mathbf{z}}_{\mathscr{T}}$  be the unique solution to (31) and  $\mathscr{T}$  be a mesh such that

(53) 
$$\|\bar{\mathbf{z}} - \bar{\mathbf{z}}_{\mathscr{T}}\|_{\mathbf{L}^{\infty}(\Omega)} \le \min\{\alpha \mu (2C_{\mathsf{M}})^{-1}, \tau/2\}.$$

Then  $\tilde{\mathbf{u}} - \bar{\mathbf{u}} \in \mathbf{C}_{\bar{\mathbf{u}}}^{\tau}$  and

(54) 
$$\frac{\mu}{2} \|\bar{\mathbf{u}} - \tilde{\mathbf{u}}\|_{\mathbf{L}^{2}(\Omega)}^{2} \leq (j'(\tilde{\mathbf{u}}) - j'(\bar{\mathbf{u}}))(\tilde{\mathbf{u}} - \bar{\mathbf{u}}).$$

The constant  $C_{\mathsf{M}}$  is given by (24) while the auxiliary variable  $\tilde{\mathbf{u}}$  is defined in (49).

*Proof.* We proceed in two steps.

Step 1. We first prove that  $\tilde{\mathbf{u}} - \bar{\mathbf{u}} \in \mathbf{C}_{\bar{\mathbf{u}}}^{\tau}$ . Since  $\tilde{\mathbf{u}} \in \mathbb{U}_{ad}$ , we can immediately conclude that  $\mathbf{v} = \tilde{\mathbf{u}} - \bar{\mathbf{u}} \ge \mathbf{0}$  if  $\bar{\mathbf{u}} = \mathbf{a}$  and that  $\mathbf{v} = \tilde{\mathbf{u}} - \bar{\mathbf{u}} \le \mathbf{0}$  if  $\bar{\mathbf{u}} = \mathbf{b}$ . It thus suffices to verify the remaining condition in (22), i.e.,  $\mathbf{v}_i = (\tilde{\mathbf{u}} - \bar{\mathbf{u}})_i = 0$  if  $|\bar{\mathbf{d}}_i(x)| > \tau$ , with  $i \in \{1, \dots, d\}$ . To accomplish this task, we first use the triangle inequality and invoke the Lipschitz property of  $\Pi_{[\mathbf{a},\mathbf{b}]}$ , in conjunction with (53), to obtain

(55) 
$$\|\bar{\mathbf{z}} + \alpha \bar{\mathbf{u}} - (\bar{\mathbf{z}}_{\mathscr{T}} + \alpha \tilde{\mathbf{u}})\|_{\mathbf{L}^{\infty}(\Omega)} \le 2\|\bar{\mathbf{z}} - \bar{\mathbf{z}}_{\mathscr{T}}\|_{\mathbf{L}^{\infty}(\Omega)} \le \tau.$$

Now, let  $\xi \in \Omega$  and  $i \in \{1, \ldots, d\}$  be such that  $\bar{\mathbf{d}}_i(\xi) = (\bar{\mathbf{z}} + \alpha \bar{\mathbf{u}})_i(\xi) > \tau$ . Since  $\tau > 0$ , this implies that  $\bar{\mathbf{u}}_i(\xi) > -\alpha^{-1} \bar{\mathbf{z}}_i(\xi)$ . Therefore, from the projection formula

(17), we conclude that  $\bar{\mathbf{u}}_i(\xi) = \mathbf{a}_i$ . On the other hand, since  $\xi \in \Omega$  is such that  $(\bar{\mathbf{z}} + \alpha \bar{\mathbf{u}})_i(\xi) > \tau$ , from (55) we can conclude that

$$(\bar{\mathbf{z}}_{\mathscr{T}} + \alpha \tilde{\mathbf{u}})_i(\xi) > 0,$$

and thus that  $\tilde{\mathbf{u}}_i(\xi) > -\alpha^{-1}(\bar{\mathbf{z}}_{\mathscr{T}})_i(\xi)$ . This, on the basis of the definition of the auxiliary variable  $\tilde{\mathbf{u}}$ , given in (49), yields that  $\tilde{\mathbf{u}}_i(\xi) = \mathbf{a}_i$ . Consequently,  $\bar{\mathbf{u}}_i(\xi) = \tilde{\mathbf{u}}_i(\xi) = \mathbf{a}_i$ . Since *i* is arbitrary, we conclude that  $(\tilde{\mathbf{u}} - \bar{\mathbf{u}})(\xi) = \mathbf{0}$ . Similar arguments allow us to conclude that, if  $\bar{\mathbf{d}}_i(\xi) = (\bar{\mathbf{z}} + \alpha \bar{\mathbf{u}})_i(\xi) < -\tau$ , with  $i \in \{1, \ldots, d\}$ , then  $(\tilde{\mathbf{u}} - \bar{\mathbf{u}})(\xi) = \mathbf{0}$ .

Step 2. Since  $\tilde{\mathbf{u}} - \bar{\mathbf{u}} \in C_{\bar{\mathbf{u}}}^{\tau}$ , with  $C_{\bar{\mathbf{u}}}^{\tau}$  defined in (21), and  $\bar{\mathbf{u}}$  satisfies the sufficient second order optimality condition (23), we are allow to set  $\mathbf{v} = \tilde{\mathbf{u}} - \bar{\mathbf{u}}$  in (23). Thus

(56) 
$$\mu \| \tilde{\mathbf{u}} - \bar{\mathbf{u}} \|_{\mathbf{L}^2(\Omega)}^2 \le j''(\bar{\mathbf{u}}) (\tilde{\mathbf{u}} - \bar{\mathbf{u}})^2.$$

On the other hand, in view of the mean value theorem, we obtain, for some  $\theta_{\mathscr{T}} \in (0, 1)$ ,

$$(j'(\tilde{\mathbf{u}}) - j'(\bar{\mathbf{u}}))(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) = j''(\zeta)(\tilde{\mathbf{u}} - \bar{\mathbf{u}})^2$$

with  $\zeta = \bar{\mathbf{u}} + \theta_{\mathscr{T}}(\tilde{\mathbf{u}} - \bar{\mathbf{u}})$ . Thus, in view of (56), we arrive at

(57) 
$$\mu \|\tilde{\mathbf{u}} - \bar{\mathbf{u}}\|_{L^2(\Omega)}^2 \leq (j'(\tilde{\mathbf{u}}) - j'(\bar{\mathbf{u}}))(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) + (j''(\bar{\mathbf{u}}) - j''(\zeta))(\tilde{\mathbf{u}} - \bar{\mathbf{u}})^2.$$

Since  $\mathsf{M} > 0$  is such that  $\max\{\|\bar{\mathbf{u}} + \theta_{\mathscr{T}}(\tilde{\mathbf{u}} - \bar{\mathbf{u}})\|_{\mathbf{L}^{\infty}(\Omega)}, \|\tilde{\mathbf{u}} - \bar{\mathbf{u}}\|_{\mathbf{L}^{\infty}(\Omega)}\} \leq \mathsf{M} \text{ and } j \text{ is of class } C^2 \text{ in } \mathbf{L}^2(\Omega) \text{ [13, Theorem 3.3], we can thus apply (24) to derive$ 

$$(j''(\bar{\mathbf{u}}) - j''(\zeta))(\tilde{\mathbf{u}} - \bar{\mathbf{u}})^2 \le C_{\mathsf{M}} \|\tilde{\mathbf{u}} - \bar{\mathbf{u}}\|_{\mathbf{L}^{\infty}(\Omega)} \|\tilde{\mathbf{u}} - \bar{\mathbf{u}}\|_{\mathbf{L}^{2}(\Omega)}^2,$$

where we have also used that  $\theta_{\mathscr{T}} \in (0, 1)$ . Invoke (17) and (49), the Lipschitz property of the projection operator  $\Pi_{[a,b]}$ , and assumption (53), to arrive at

$$(j''(\bar{\mathbf{u}}) - j''(\zeta))(\tilde{\mathbf{u}} - \bar{\mathbf{u}})^2 \le C_{\mathsf{M}}\alpha^{-1} \|\bar{\mathbf{z}} - \bar{\mathbf{z}}_{\mathscr{T}}\|_{\mathbf{L}^{\infty}(\Omega)} \|\tilde{\mathbf{u}} - \bar{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}^2 \le \frac{\mu}{2} \|\tilde{\mathbf{u}} - \bar{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}^2.$$

Replacing this inequality into (57) allows us to conclude the desired inequality (54). This concludes the proof.

The following auxiliary variables are also of particular importance for our reliability analysis. Let  $(\tilde{\mathbf{y}}, \tilde{p}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  be the solution to

$$\nu(\nabla \tilde{\mathbf{y}}, \nabla \mathbf{v})_{\mathbf{L}^{2}(\Omega)} + b(\tilde{\mathbf{y}}; \tilde{\mathbf{y}}, \mathbf{v}) - (\tilde{p}, \operatorname{div} \mathbf{v})_{L^{2}(\Omega)} = (\tilde{\mathbf{u}}, \mathbf{v})_{\mathbf{L}^{2}(\Omega)} \quad \forall \mathbf{v} \in \mathbf{H}_{0}^{1}(\Omega),$$
$$(q, \operatorname{div} \tilde{\mathbf{y}})_{L^{2}(\Omega)} = 0 \qquad \forall q \in L_{0}^{2}(\Omega).$$

We also introduce the pair  $(\tilde{\mathbf{z}}, \tilde{r}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  as the solution to

$$\nu(\nabla \mathbf{w}, \nabla \tilde{\mathbf{z}})_{\mathbf{L}^{2}(\Omega)} + b(\tilde{\mathbf{y}}; \mathbf{w}, \tilde{\mathbf{z}}) + b(\mathbf{w}; \tilde{\mathbf{y}}, \tilde{\mathbf{z}}) - (\tilde{r}, \operatorname{div} \mathbf{w})_{L^{2}(\Omega)} = (\tilde{\mathbf{y}} - \mathbf{y}_{\Omega}, \mathbf{w})_{\mathbf{L}^{2}(\Omega)},$$
  
(s, div  $\tilde{\mathbf{z}})_{L^{2}(\Omega)} = 0,$ 

for all  $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$  and  $s \in L_0^2(\Omega)$ .

To present the following result, we define  $\mathbf{e}_{\mathbf{y}} := \bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}}, e_p := \bar{p} - \bar{p}_{\mathscr{T}}, \mathbf{e}_{\mathbf{z}} := \bar{\mathbf{z}} - \bar{\mathbf{z}}_{\mathscr{T}}, e_r := \bar{r} - \bar{r}_{\mathscr{T}}, \mathbf{e}_{\mathbf{u}} := \bar{\mathbf{u}} - \bar{\mathbf{u}}_{\mathscr{T}}, \text{ and the total error norm}$ 

(58) 
$$\|\mathbf{e}\|_{\Omega}^{2} := \|\nabla \mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)}^{2} + \|e_{p}\|_{L^{2}(\Omega)}^{2} + \|\nabla \mathbf{e}_{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)}^{2} + \|e_{r}\|_{L^{2}(\Omega)}^{2} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^{2}(\Omega)}^{2}.$$

We are now ready to state and prove the main result of this section.

THEOREM 9 (global reliability of  $\mathcal{E}_{ocp}$ ). Assume that the smallness assumptions (11) and (32) hold. Let  $(\bar{\mathbf{y}}, \bar{p}, \bar{\mathbf{z}}, \bar{r}, \bar{\mathbf{u}}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbb{U}_{ad}$  be a local solution of (9)–(10) that satisfies the sufficient second order optimality condition (20), or equivalently (23). Let  $\bar{\mathbf{u}}_{\mathcal{T}}$  be a local minimum of the associated discrete optimal control problem with  $(\bar{\mathbf{y}}_{\mathcal{T}}, \bar{p}_{\mathcal{T}})$  and  $(\bar{\mathbf{z}}_{\mathcal{T}}, \bar{r}_{\mathcal{T}})$  being the corresponding state and adjoint state discrete variables, respectively. Let  $\mathcal{T}$  be a mesh such that (53) holds. Then

(59) 
$$\|\mathbf{e}\|_{\Omega}^2 \lesssim \mathcal{E}_{ocr}^2$$

The hidden constant is independent of the continuous and discrete optimal variables, the size of the elements in the mesh  $\mathcal{T}$ , and  $\#\mathcal{T}$ .

*Proof.* We proceed in six steps.

Step 1. The goal of this step is to control  $\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^{2}(\Omega)}$  in (58). Invoke the auxiliary variable  $\tilde{\mathbf{u}}$ , defined in (49), and definition (51) to arrive at

(60) 
$$\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^{2}(\Omega)} \leq \|\bar{\mathbf{u}} - \tilde{\mathbf{u}}\|_{\mathbf{L}^{2}(\Omega)} + \mathcal{E}_{ct}.$$

It thus suffices to bound  $\|\bar{\mathbf{u}} - \tilde{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}$ . To accomplish this task, we set  $\mathbf{u} = \tilde{\mathbf{u}}$  in (13) and  $\mathbf{u} = \bar{\mathbf{u}}$  in (50) to obtain

$$j'(\bar{\mathbf{u}})(\tilde{\mathbf{u}}-\bar{\mathbf{u}}) = (\bar{\mathbf{z}}+\alpha\bar{\mathbf{u}},\tilde{\mathbf{u}}-\bar{\mathbf{u}})_{\mathbf{L}^{2}(\Omega)} \ge 0, \qquad -(\bar{\mathbf{z}}_{\mathscr{T}}+\alpha\tilde{\mathbf{u}},\tilde{\mathbf{u}}-\bar{\mathbf{u}})_{\mathbf{L}^{2}(\Omega)} \ge 0.$$

With these estimates at hand, we invoke inequality (54) to conclude

$$\begin{aligned} &\frac{\mu}{2} \| \bar{\mathbf{u}} - \tilde{\mathbf{u}} \|_{\mathbf{L}^{2}(\Omega)}^{2} \leq j'(\tilde{\mathbf{u}})(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) - j'(\bar{\mathbf{u}})(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) \leq j'(\tilde{\mathbf{u}})(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) \\ &\leq j'(\tilde{\mathbf{u}})(\tilde{\mathbf{u}} - \bar{\mathbf{u}}) - (\bar{\mathbf{z}}_{\mathscr{T}} + \alpha \tilde{\mathbf{u}}, \tilde{\mathbf{u}} - \bar{\mathbf{u}})_{\mathbf{L}^{2}(\Omega)} = (\tilde{\mathbf{z}} - \bar{\mathbf{z}}_{\mathscr{T}}, \tilde{\mathbf{u}} - \bar{\mathbf{u}})_{\mathbf{L}^{2}(\Omega)}. \end{aligned}$$

Adding and subtracting the auxiliary variable  $\hat{\mathbf{z}}$ , defined as the solution to (41), and utilizing the Cauchy–Schwarz and triangle inequalities we obtain

$$\|ar{\mathbf{u}}-ar{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}\lesssim \|ar{\mathbf{z}}-\hat{\mathbf{z}}\|_{\mathbf{L}^2(\Omega)}+\|ar{\mathbf{z}}-ar{\mathbf{z}}_\mathscr{T}\|_{\mathbf{L}^2(\Omega)}$$

A Poincaré inequality combined with the a posteriori error estimate (44) yield

(61) 
$$\|\bar{\mathbf{u}} - \tilde{\mathbf{u}}\|_{\mathbf{L}^{2}(\Omega)} \lesssim \|\nabla(\tilde{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)} + \mathcal{E}_{ad}$$

We now estimate the remaining term  $\|\nabla(\tilde{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^2(\Omega)}$ . To accomplish this task, we notice that the pair  $(\tilde{\mathbf{z}} - \hat{\mathbf{z}}, \tilde{r} - \hat{r}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  solves

$$\nu(\nabla \mathbf{w}, \nabla(\tilde{\mathbf{z}} - \hat{\mathbf{z}}))_{\mathbf{L}^{2}(\Omega)} + b(\tilde{\mathbf{y}}; \mathbf{w}, \tilde{\mathbf{z}}) - b(\bar{\mathbf{y}}_{\mathscr{T}}; \mathbf{w}, \hat{\mathbf{z}}) + b(\mathbf{w}; \tilde{\mathbf{y}}, \tilde{\mathbf{z}}) - b(\mathbf{w}; \bar{\mathbf{y}}_{\mathscr{T}}, \hat{\mathbf{z}}) - (\tilde{r} - \hat{r}, \operatorname{div} \mathbf{w})_{L^{2}(\Omega)} = (\tilde{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}}, \mathbf{w})_{\mathbf{L}^{2}(\Omega)}, (s, \operatorname{div} (\tilde{\mathbf{z}} - \hat{\mathbf{z}}))_{L^{2}(\Omega)} = 0,$$

for all  $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$  and  $s \in L_0^2(\Omega)$ , respectively. Set s = 0 and  $\mathbf{w} = \tilde{\mathbf{z}} - \hat{\mathbf{z}}$  to obtain

$$\begin{split} \nu \|\nabla(\tilde{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)}^{2} + b(\tilde{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}}; \tilde{\mathbf{z}} - \hat{\mathbf{z}}, \tilde{\mathbf{z}}) + b(\bar{\mathbf{y}}_{\mathscr{T}}; \tilde{\mathbf{z}} - \hat{\mathbf{z}}, \tilde{\mathbf{z}} - \hat{\mathbf{z}}) \\ + b(\tilde{\mathbf{z}} - \hat{\mathbf{z}}; \tilde{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}}, \tilde{\mathbf{z}}) + b(\tilde{\mathbf{z}} - \hat{\mathbf{z}}; \bar{\mathbf{y}}_{\mathscr{T}}, \tilde{\mathbf{z}} - \hat{\mathbf{z}}) = (\tilde{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}}, \tilde{\mathbf{z}} - \hat{\mathbf{z}})_{\mathbf{L}^{2}(\Omega)}. \end{split}$$

Invoke now estimate (5) and the Cauchy–Schwarz inequality to obtain

$$\begin{split} \nu \|\nabla(\tilde{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)}^{2} &\leq 2\mathcal{C}_{b} \|\nabla(\tilde{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)} \|\nabla(\tilde{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}})\|_{\mathbf{L}^{2}(\Omega)} \|\nabla\tilde{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)} \\ &+ 2\mathcal{C}_{b} \|\nabla(\tilde{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)}^{2} \|\nabla\bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(\Omega)} + \|\tilde{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(\Omega)} \|\tilde{\mathbf{z}} - \hat{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)}. \end{split}$$

Then, in view of assumption (32), it immediately follows that

$$\nu(1-\theta) \|\nabla(\tilde{\mathbf{z}}-\hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)}^{2} \leq \|\tilde{\mathbf{y}}-\bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(\Omega)} \|\tilde{\mathbf{z}}-\hat{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)} + 2C_{b} \|\nabla(\tilde{\mathbf{z}}-\hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)} \|\nabla(\tilde{\mathbf{y}}-\bar{\mathbf{y}}_{\mathscr{T}})\|_{\mathbf{L}^{2}(\Omega)} \|\nabla\tilde{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)}.$$

Applying a Poincaré inequality, adding and subtracting the auxiliary variable  $\hat{\mathbf{y}}$ , defined as the solution to (34), and using the triangle inequality, we arrive at

(62) 
$$\|\nabla(\tilde{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)} \lesssim \left(1 + \|\nabla\tilde{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)}\right) \left(\|\nabla(\tilde{\mathbf{y}} - \hat{\mathbf{y}})\|_{\mathbf{L}^{2}(\Omega)} + \|\nabla(\hat{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}})\|_{\mathbf{L}^{2}(\Omega)}\right).$$

Notice that the stability estimate for the problem that  $(\tilde{\mathbf{z}}, \tilde{r})$  solves yields

$$\|\nabla \tilde{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)} \leq \frac{C_{2}}{\nu(1-\theta)} \|\tilde{\mathbf{y}} - \mathbf{y}_{\Omega}\|_{\mathbf{L}^{2}(\Omega)} \leq \frac{C_{2}}{\nu(1-\theta)} \left( C_{2} \mathcal{C}_{b}^{-1} \theta \nu + \|\mathbf{y}_{\Omega}\|_{\mathbf{L}^{2}(\Omega)} \right),$$

where we have also used (8). Replacing this estimate into (62) and invoking the a posteriori error estimate (38) we obtain

(63) 
$$\|\nabla(\tilde{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)} \lesssim \|\nabla(\tilde{\mathbf{y}} - \hat{\mathbf{y}})\|_{\mathbf{L}^{2}(\Omega)} + \mathcal{E}_{st},$$

with a hidden constant that is independent of the continuous and discrete optimal variables, the size of the elements in the mesh  $\mathscr{T}$ , and  $\#\mathscr{T}$  but depends on the continuous problem data and  $C_2$ ,  $\mathcal{C}_b$ , and  $\theta$ .

The rest of this step is dedicated to bound the term  $\|\nabla(\tilde{\mathbf{y}} - \hat{\mathbf{y}})\|_{\mathbf{L}^2(\Omega)}$  in (63). To accomplish this task, we first notice that the pair  $(\tilde{\mathbf{y}} - \hat{\mathbf{y}}, \tilde{p} - \hat{p}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  solves the problem

$$\nu(\nabla(\tilde{\mathbf{y}}-\hat{\mathbf{y}}),\nabla\mathbf{v})_{\mathbf{L}^{2}(\Omega)}+b(\tilde{\mathbf{y}};\tilde{\mathbf{y}},\mathbf{v})-b(\hat{\mathbf{y}};\hat{\mathbf{y}},\mathbf{v})-(\tilde{p}-\hat{p},\operatorname{div}\mathbf{v})_{L^{2}(\Omega)}=(\tilde{\mathbf{u}}-\bar{\mathbf{u}}_{\mathscr{T}},\mathbf{v})_{\mathbf{L}^{2}(\Omega)},$$

$$(q,\operatorname{div}(\tilde{\mathbf{y}}-\hat{\mathbf{y}}))_{L^{2}(\Omega)}=0,$$

for all  $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$  and  $q \in L_0^2(\Omega)$ , respectively. Set  $\mathbf{v} = \tilde{\mathbf{y}} - \hat{\mathbf{y}}$  and q = 0, and invoke the second property for the form b stated in (4) to arrive at

$$\nu \|\nabla(\tilde{\mathbf{y}} - \hat{\mathbf{y}})\|_{\mathbf{L}^2(\Omega)}^2 + b(\tilde{\mathbf{y}} - \hat{\mathbf{y}}; \hat{\mathbf{y}}, \tilde{\mathbf{y}} - \hat{\mathbf{y}}) = (\tilde{\mathbf{u}} - \bar{\mathbf{u}}_{\mathscr{T}}, \tilde{\mathbf{y}} - \hat{\mathbf{y}})_{\mathbf{L}^2(\Omega)}$$

Estimates (5) and (35) thus yield

(64) 
$$\|\nabla(\tilde{\mathbf{y}} - \hat{\mathbf{y}})\|_{\mathbf{L}^{2}(\Omega)} \lesssim \|\tilde{\mathbf{u}} - \bar{\mathbf{u}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(\Omega)} = \mathcal{E}_{ct},$$

upon using definition (51). Replacing estimate (64) into (63), and the obtained one into (61), we obtain

(65) 
$$\|\bar{\mathbf{u}} - \tilde{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)} \lesssim \mathcal{E}_{ad} + \mathcal{E}_{st} + \mathcal{E}_{ct}.$$

On the basis of (65) and (60), we can thus obtain the a posteriori error estimate

(66) 
$$\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^{2}(\Omega)} \lesssim \mathcal{E}_{ad} + \mathcal{E}_{st} + \mathcal{E}_{ct}.$$

Step 2. The goal of this step is to bound  $\|\nabla \mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)}$  in (58). We begin with a simple application of the triangle inequality and (38):

(67) 
$$\|\nabla \mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)} \leq \|\nabla(\bar{\mathbf{y}} - \hat{\mathbf{y}})\|_{\mathbf{L}^{2}(\Omega)} + C\mathcal{E}_{st},$$

where C > 0. We now bound  $\|\nabla(\bar{\mathbf{y}} - \hat{\mathbf{y}})\|_{\mathbf{L}^2(\Omega)}$ . To accomplish this task, we first notice that the pair  $(\bar{\mathbf{y}} - \hat{\mathbf{y}}, \bar{p} - \hat{p}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  solves the problem

(68)  

$$(\nabla(\bar{\mathbf{y}} - \hat{\mathbf{y}}), \nabla \mathbf{v})_{\mathbf{L}^{2}(\Omega)} + b(\bar{\mathbf{y}}; \bar{\mathbf{y}} - \hat{\mathbf{y}}, \mathbf{v}) + b(\bar{\mathbf{y}} - \hat{\mathbf{y}}; \hat{\mathbf{y}}, \mathbf{v}) - (\bar{p} - \hat{p}, \operatorname{div} \mathbf{v})_{L^{2}(\Omega)} = (\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\mathscr{T}}, \mathbf{v})_{\mathbf{L}^{2}(\Omega)},$$

$$(q, \operatorname{div} (\bar{\mathbf{y}} - \hat{\mathbf{y}}))_{L^{2}(\Omega)} = 0,$$

for all  $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$  and  $q \in L_0^2(\Omega)$ , respectively. Set  $\mathbf{v} = \bar{\mathbf{y}} - \hat{\mathbf{y}}$  and q = 0, and invoke (4) and the fact that  $\bar{\mathbf{y}} - \hat{\mathbf{y}} \in \mathbf{V}(\Omega)$  to arrive at

$$\nu \|\nabla(\bar{\mathbf{y}} - \hat{\mathbf{y}})\|_{\mathbf{L}^{2}(\Omega)}^{2} + b(\bar{\mathbf{y}} - \hat{\mathbf{y}}; \hat{\mathbf{y}}, \bar{\mathbf{y}} - \hat{\mathbf{y}}) = (\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\mathscr{T}}, \bar{\mathbf{y}} - \hat{\mathbf{y}})_{\mathbf{L}^{2}(\Omega)}$$

We thus invoke (5) and the stability estimate (35) to obtain

(69) 
$$\|\nabla(\bar{\mathbf{y}} - \hat{\mathbf{y}})\|_{\mathbf{L}^{2}(\Omega)} \lesssim \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^{2}(\Omega)}.$$

We finally replace estimate (69) into (67) and invoke (66) to obtain the error estimate

(70) 
$$\|\nabla \mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)} \lesssim \mathcal{E}_{ad} + \mathcal{E}_{st} + \mathcal{E}_{ct}.$$

Step 3. We now estimate the term  $||e_p||_{L^2(\Omega)}$  in (58). A trivial application of the triangle inequality in conjunction with the a posteriori estimate (38) yield

(71) 
$$||e_p||_{L^2(\Omega)} \lesssim ||\bar{p} - \hat{p}||_{L^2(\Omega)} + \mathcal{E}_{st}.$$

It thus suffices to bound  $\|\bar{p} - \hat{p}\|_{L^2(\Omega)}$ . To do this, we utilize the inf-sup condition (6), the fact that  $(\bar{\mathbf{y}} - \hat{\mathbf{y}}, \bar{p} - \hat{p}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  solves (68) and estimate (5) to arrive at

(72) 
$$\|\bar{p} - \hat{p}\|_{L^{2}(\Omega)} \lesssim \sup_{\mathbf{v} \in \mathbf{H}_{0}^{1}(\Omega)} \frac{(\bar{p} - \hat{p}, \operatorname{div} \mathbf{v})_{L^{2}(\Omega)}}{\|\nabla \mathbf{v}\|_{\mathbf{L}^{2}(\Omega)}} \lesssim \|\nabla (\bar{\mathbf{y}} - \hat{\mathbf{y}})\|_{\mathbf{L}^{2}(\Omega)} + \|\nabla (\bar{\mathbf{y}} - \hat{\mathbf{y}})\|_{\mathbf{L}^{2}(\Omega)} (\|\nabla \bar{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)} + \|\nabla \hat{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)}) + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^{2}(\Omega)}.$$

Since the smallness assumption (11) holds, we immediately deduce the stability estimates (8) and (35). Thus,

$$\|\nabla \bar{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)} + \|\nabla \hat{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)} \le 2\theta \mathcal{C}_{b}^{-1}\nu, \quad \theta < 1.$$

Replace this estimate into (72) and invoke (69) and (66) to arrive at  $\|\bar{p} - \hat{p}\|_{L^2(\Omega)} \lesssim \mathcal{E}_{ad} + \mathcal{E}_{st} + \mathcal{E}_{ct}$ . This estimate, in view of (71), yields the a posteriori error estimate

(73) 
$$\|e_p\|_{L^2(\Omega)} \lesssim \mathcal{E}_{ad} + \mathcal{E}_{st} + \mathcal{E}_{ct}.$$

Step 4. We bound  $\|\nabla \mathbf{e}_{\mathbf{z}}\|_{\mathbf{L}^2(\Omega)}$ . To accomplish this task, we apply the triangle inequality and invoke the a posteriori estimate (44). These arguments yield

(74) 
$$\|\nabla \mathbf{e}_{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)} \lesssim \|\nabla(\bar{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)} + \mathcal{E}_{ad}.$$

To bound  $\|\nabla(\bar{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^2(\Omega)}$  we observe that  $(\bar{\mathbf{z}} - \hat{\mathbf{z}}, \bar{r} - \hat{r}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  solves

$$\nu(\nabla \mathbf{w}, \nabla(\bar{\mathbf{z}} - \hat{\mathbf{z}}))_{\mathbf{L}^{2}(\Omega)} + b(\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}}; \mathbf{w}, \bar{\mathbf{z}}) + b(\bar{\mathbf{y}}_{\mathscr{T}}; \mathbf{w}, \bar{\mathbf{z}} - \hat{\mathbf{z}})$$

$$(75) \qquad +b(\mathbf{w}; \bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}}, \bar{\mathbf{z}}) + b(\mathbf{w}; \bar{\mathbf{y}}_{\mathscr{T}}, \bar{\mathbf{z}} - \hat{\mathbf{z}}) - (\bar{r} - \hat{r}, \operatorname{div} \mathbf{w})_{L^{2}(\Omega)} = (\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}}, \mathbf{w})_{\mathbf{L}^{2}(\Omega)},$$

$$(s, \operatorname{div} (\bar{\mathbf{z}} - \hat{\mathbf{z}}))_{L^{2}(\Omega)} = 0,$$

for all  $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$  and  $s \in L_0^2(\Omega)$ , respectively. Set  $\mathbf{w} = \bar{\mathbf{z}} - \hat{\mathbf{z}}$  and s = 0, and invoke the estimate (5) to obtain

$$\begin{split} \nu \|\nabla(\bar{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)}^{2} &\leq 2\mathcal{C}_{b} \|\nabla(\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}})\|_{\mathbf{L}^{2}(\Omega)} \|\nabla\bar{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)} \|\nabla(\bar{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)}^{2} \\ &+ 2\mathcal{C}_{b} \|\nabla(\bar{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)}^{2} \|\nabla\bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(\Omega)} + \|\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(\Omega)} \|\bar{\mathbf{z}} - \hat{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)}. \end{split}$$

Utilize (32) and a Poincaré inequality to obtain

(76) 
$$\nu(1-\theta) \|\nabla(\bar{\mathbf{z}}-\hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)} \leq \left(2\mathcal{C}_{b} \|\nabla\bar{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)} + C_{2}^{2}\right) \|\nabla(\bar{\mathbf{y}}-\bar{\mathbf{y}}_{\mathscr{T}})\|_{\mathbf{L}^{2}(\Omega)}.$$

We thus invoke the stability estimate (16), the smallness assumption (11), and the results of Theorem 1 to obtain

(77) 
$$\|\nabla \bar{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)} \leq \frac{C_{2}}{\nu(1-\theta)} \left(C_{2}\theta \mathcal{C}_{b}^{-1}\nu + \|\mathbf{y}_{\Omega}\|_{\mathbf{L}^{2}(\Omega)}\right).$$

Replace this estimate into (76) to obtain

$$\|\nabla(\bar{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)} \lesssim \|\nabla \mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)},$$

with a hidden constant that is independent of the continuous and discrete optimal variables, the size of the elements in the mesh  $\mathscr{T}$ , and  $\#\mathscr{T}$  but depends on the continuous problem data and  $C_2$ ,  $\mathcal{C}_b$ , and  $\theta$ . We thus invoke (70) to obtain

(78) 
$$\|\nabla(\bar{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)} \lesssim \mathcal{E}_{ad} + \mathcal{E}_{st} + \mathcal{E}_{ct},$$

which, in view of (74), yields the a posteriori error estimate

(79) 
$$\|\nabla \mathbf{e}_{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)} \lesssim \mathcal{E}_{ad} + \mathcal{E}_{st} + \mathcal{E}_{ct}.$$

Step 5. We now control  $||e_r||_{L^2(\Omega)}$  in (58). We begin by applying (44) to derive

(80) 
$$||e_r||_{L^2(\Omega)} \lesssim ||\bar{r} - \hat{r}||_{L^2(\Omega)} + \mathcal{E}_{ad}.$$

To estimate  $\|\bar{r} - \hat{r}\|_{L^2(\Omega)}$  we utilize the inf-sup condition (6), problem (75), and the basic estimate (5):

$$\begin{aligned} \|\bar{r} - \hat{r}\|_{L^{2}(\Omega)} &\lesssim \sup_{\mathbf{w} \in \mathbf{H}_{0}^{1}(\Omega)} \frac{(\bar{r} - \hat{r}, \operatorname{div} \mathbf{w})_{L^{2}(\Omega)}}{\|\nabla \mathbf{w}\|_{\mathbf{L}^{2}(\Omega)}} \lesssim \|\nabla(\bar{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)} + \|\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(\Omega)} \\ &+ \|\nabla(\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mathscr{T}})\|_{\mathbf{L}^{2}(\Omega)} \|\nabla\bar{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)} + \|\nabla\bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(\Omega)} \|\nabla(\bar{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)} \end{aligned}$$

We thus invoke assumption (32) and estimate (77) to arrive at

$$\|\bar{r} - \hat{r}\|_{L^{2}(\Omega)} \lesssim \|\nabla(\bar{\mathbf{z}} - \hat{\mathbf{z}})\|_{\mathbf{L}^{2}(\Omega)} + \|\nabla\mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)},$$

with a hidden constant that is independent of the continuous and discrete optimal variables, the size of the elements in the mesh  $\mathscr{T}$ , and  $\#\mathscr{T}$  but depends on the continuous problem data and  $C_2$ ,  $\mathcal{C}_b$ , and  $\theta$ . The estimates (70) and (78) immediately yield  $\|\bar{r} - \hat{r}\|_{L^2(\Omega)} \lesssim \mathcal{E}_{ad} + \mathcal{E}_{st} + \mathcal{E}_{ct}$ . Finally, we replace this estimate into (80) to obtain the a posteriori error estimate

(81) 
$$\|e_r\|_{L^2(\Omega)} \lesssim \mathcal{E}_{ad} + \mathcal{E}_{st} + \mathcal{E}_{ct}.$$

Step 6. The desired estimate (59) follows from collecting the estimates (66), (70), (73), (79), and (81). This concludes the proof.  $\hfill \Box$ 

5. Local efficiency analysis. In this section we analyze the efficiency properties of the a posteriori error estimator  $\mathcal{E}_{ocp}$ , defined in (52), on the basis of standard bubble function arguments. Before proceeding with such an analysis, we introduce the following notation: For an edge, triangle, or tetrahedron G, let  $\mathcal{V}(G)$  be the set of vertices of G. With this notation at hand, we introduce, for  $T \in \mathscr{T}$  and  $S \in \mathscr{S}$ , the following standard element and edge bubble functions [29]:

$$\varphi_T = (d+1)^{(d+1)} \prod_{\mathbf{v} \in \mathcal{V}(T)} \lambda_{\mathbf{v}}, \qquad \varphi_S = d^d \prod_{\mathbf{v} \in \mathcal{V}(S)} \lambda_{\mathbf{v}}|_{T'} \text{ with } T' \subset \mathcal{N}_S.$$

In these formulas, by  $\lambda_{v}$  we denote the barycentric coordinate function associated to  $v \in \mathcal{V}(T)$ . We recall that  $\mathcal{N}_{S}$  corresponds to the patch composed of the two elements of  $\mathscr{T}$  sharing S.

We now proceed to derive local efficiency properties for the indicator  $\mathcal{E}_{st,T}$  defined in (36).

THEOREM 10 (local efficiency of  $\mathcal{E}_{st}$ ). Assume that the smallness assumption (11) holds. Let  $(\bar{\mathbf{y}}, \bar{p}, \bar{\mathbf{z}}, \bar{r}, \bar{\mathbf{u}}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbb{U}_{ad}$  be a local solution of (9)–(10). Let  $\bar{\mathbf{u}}_{\mathscr{T}}$  be a local minimum of the associated discrete optimal control problem with  $(\bar{\mathbf{y}}_{\mathscr{T}}, \bar{p}_{\mathscr{T}})$  and  $(\bar{\mathbf{z}}_{\mathscr{T}}, \bar{r}_{\mathscr{T}})$  being the corresponding state and adjoint state discrete variables, respectively. If assumption (32) holds, then, for  $T \in \mathscr{T}$ , the local error indicator  $\mathcal{E}_{st,T}$  satisfies

(82) 
$$\mathcal{E}_{st,T} \lesssim (1+h_T) \left( \| \mathbf{e}_{\mathbf{y}} \|_{\mathbf{H}^1(\mathcal{N}_T)} + \| e_p \|_{L^2(\mathcal{N}_T)} + \| \mathbf{e}_{\mathbf{u}} \|_{\mathbf{L}^2(\mathcal{N}_T)} \right),$$

where  $\mathcal{N}_T$  is defined as in (25). The hidden constant is independent of the continuous and discrete optimal variables, the size of the elements in the mesh  $\mathscr{T}$ , and  $\#\mathscr{T}$ .

*Proof.* We begin by noticing that, since  $(\bar{\mathbf{y}}, \bar{p}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  solves (10) with **u** replaced by  $\bar{\mathbf{u}}$ , an elementwise integration by parts formula allows us to derive the following identity:

(83) 
$$\nu(\nabla \mathbf{e}_{\mathbf{y}}, \nabla \mathbf{v})_{\mathbf{L}^{2}(\Omega)} + b(\bar{\mathbf{y}}; \mathbf{e}_{\mathbf{y}}, \mathbf{v}) + b(\mathbf{e}_{\mathbf{y}}; \bar{\mathbf{y}}_{\mathscr{T}}, \mathbf{v}) - (e_{p}, \operatorname{div} \mathbf{v})_{L^{2}(\Omega)} + (q, \operatorname{div} \mathbf{e}_{\mathbf{y}})_{L^{2}(\Omega)} - (\mathbf{e}_{\mathbf{u}}, \mathbf{v})_{\mathbf{L}^{2}(\Omega)} = \sum_{T \in \mathscr{T}} (\bar{\mathbf{u}}_{\mathscr{T}} + \nu \Delta \bar{\mathbf{y}}_{\mathscr{T}} - (\bar{\mathbf{y}}_{\mathscr{T}} \cdot \nabla) \bar{\mathbf{y}}_{\mathscr{T}} - \nabla \bar{p}_{\mathscr{T}}, \mathbf{v})_{\mathbf{L}^{2}(T)} + \sum_{S \in \mathscr{S}} (\llbracket (\nu \nabla \bar{\mathbf{y}}_{\mathscr{T}} - \bar{p}_{\mathscr{T}} \mathbb{I}_{d}) \cdot \mathbf{n} \rrbracket, \mathbf{v})_{\mathbf{L}^{2}(S)} - \sum_{T \in \mathscr{T}} (q, \operatorname{div} \bar{\mathbf{y}}_{\mathscr{T}})_{L^{2}(T)},$$

which holds for every  $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$  and  $q \in L_0^2(\Omega)$ . With the aid of this identity, in the following steps, we will estimate separately each of the individual terms that appear in the definition of the local error indicator  $\mathcal{E}_{st,T}$ .

We now proceed on the basis of four steps.

Step 1. Let  $T \in \mathscr{T}$ . Define

$$\mathbf{R}_T^{st} := (\bar{\mathbf{u}}_{\mathscr{T}} + \nu \Delta \bar{\mathbf{y}}_{\mathscr{T}} - (\bar{\mathbf{y}}_{\mathscr{T}} \cdot \nabla) \bar{\mathbf{y}}_{\mathscr{T}} - \nabla \bar{p}_{\mathscr{T}}) |_T$$

We bound  $h_T^2 \|\mathbf{R}_T^{st}\|_{\mathbf{L}^2(T)}^2$  in (36). To accomplish this task, we set  $\mathbf{v} = \varphi_T \mathbf{R}_T^{st}$  and q = 0 in (83), and utilize standard properties of the bubble function  $\varphi_T$ , a standard Sobolev embedding, and basic inequalities to obtain

$$\begin{aligned} \|\mathbf{R}_{T}^{st}\|_{\mathbf{L}^{2}(T)}^{2} \lesssim \left(\|\bar{\mathbf{y}}\|_{\mathbf{H}^{1}(T)}\|\nabla\mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(T)} + \|\mathbf{e}_{\mathbf{y}}\|_{\mathbf{H}^{1}(T)}\|\nabla\bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(T)}\right)\|\varphi_{T}\mathbf{R}_{T}^{st}\|_{\mathbf{H}^{1}(T)} \\ + \left(\|\nabla\mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(T)} + \|e_{p}\|_{L^{2}(T)}\right)\|\nabla(\varphi_{T}\mathbf{R}_{T}^{st})\|_{\mathbf{L}^{2}(T)} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^{2}(T)}\|\varphi_{T}\mathbf{R}_{T}^{st}\|_{\mathbf{L}^{2}(T)}.\end{aligned}$$

We thus apply standard inverse inequalities and bubble functions arguments to obtain

(84) 
$$\|\mathbf{R}_{T}^{st}\|_{\mathbf{L}^{2}(T)} \lesssim (1+h_{T}^{-2})^{\frac{1}{2}} \left(\|\bar{\mathbf{y}}\|_{\mathbf{H}^{1}(T)}\|\nabla\mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(T)} + \|\mathbf{e}_{\mathbf{y}}\|_{\mathbf{H}^{1}(T)}\|\nabla\bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(T)}\right) + h_{T}^{-1} \left(\|\nabla\mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(T)} + \|e_{p}\|_{L^{2}(T)}\right) + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^{2}(T)}.$$

On other hand, a Poincaré inequality combined with the stability estimate (8) yield

(85) 
$$\|\bar{\mathbf{y}}\|_{\mathbf{H}^{1}(T)} \leq \|\bar{\mathbf{y}}\|_{\mathbf{H}^{1}(\Omega)} \leq C \|\nabla \bar{\mathbf{y}}\|_{\mathbf{L}^{2}(\Omega)} \leq C \theta \mathcal{C}_{b}^{-1} \nu,$$

where C > 0. Similarly, in view of assumption (32), we have that

(86) 
$$\|\nabla \bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(T)} \leq \|\nabla \bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(\Omega)} < (\nu \mathcal{C}_{b}^{-1})/2.$$

Replacing this estimate and (85) into inequality (84) we obtain

(87) 
$$h_T^2 \|\mathbf{R}_T^{st}\|_{\mathbf{L}^2(T)}^2 \lesssim (1+h_T^2) \left( \|\mathbf{e}_{\mathbf{y}}\|_{\mathbf{H}^1(T)}^2 + \|e_p\|_{L^2(T)}^2 + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(T)}^2 \right),$$

with a hidden constant that is independent of the continuous and discrete optimal variables, the size of the elements in the mesh  $\mathscr{T}$ , and  $\#\mathscr{T}$  but depends on the continuous problem data and  $\nu$ ,  $\mathcal{C}_b$ , and  $\theta$ .

Step 2. Let  $T \in \mathscr{T}$  and  $S \in \mathscr{S}_T$ . Define  $\mathbf{J}_S^{st} := \llbracket (\nu \nabla \bar{\mathbf{y}}_{\mathscr{T}} - \bar{p}_{\mathscr{T}} \mathbb{I}_d) \cdot \mathbf{n} \rrbracket$ . We bound the jump term  $h_T \| \mathbf{J}_S^{st} \|_{\mathbf{L}^2(S)}^2$  in (36). To accomplish this task, we set  $\mathbf{v} = \varphi_S \mathbf{J}_S^{st}$  and q = 0 in (83) and proceed on the basis of similar arguments to the ones that lead to (84). These arguments yield

$$\begin{split} \|\mathbf{J}_{S}^{st}\|_{\mathbf{L}^{2}(S)}^{2} &\lesssim \sum_{T' \in \mathcal{N}_{S}} \left( \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^{2}(T')} + \|\mathbf{R}_{T'}^{st}\|_{\mathbf{L}^{2}(T')} + (1 + h_{T'}^{-2})^{\frac{1}{2}} \left( \|\bar{\mathbf{y}}\|_{\mathbf{H}^{1}(T')} \|\nabla \mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(T')} \\ &+ \|\mathbf{e}_{\mathbf{y}}\|_{\mathbf{H}^{1}(T')} \|\nabla \bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{L}^{2}(T')} \right) + h_{T'}^{-1} \left( \|\nabla \mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(T')} + \|e_{p}\|_{L^{2}(T')} \right) \right) h_{T}^{\frac{1}{2}} \|\mathbf{J}_{S}^{st}\|_{\mathbf{L}^{2}(S)}. \end{split}$$

Invoke (85), (86), and (87) to arrive at

(88) 
$$h_T \|\mathbf{J}_S^{st}\|_{\mathbf{L}^2(S)}^2 \lesssim \sum_{T' \in \mathcal{N}_S} (1 + h_{T'}^2) \left( \|\mathbf{e}_{\mathbf{y}}\|_{\mathbf{H}^1(T')}^2 + \|e_p\|_{L^2(T')}^2 + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(T')}^2 \right)$$

Step 3. Let  $T \in \mathscr{T}$ . The goal of this step is to control the term  $\|\operatorname{div} \bar{\mathbf{y}}_{\mathscr{T}}\|_{L^2(T)}^2$  in (36). From the incompressibility condition div  $\bar{\mathbf{y}} = 0$ , it immediately follows that

(89) 
$$\|\operatorname{div} \, \bar{\mathbf{y}}_{\mathscr{T}}\|_{L^{2}(T)} \leq \|\operatorname{div} \, \mathbf{e}_{\mathbf{y}}\|_{L^{2}(T)} \lesssim \|\nabla \mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(T)} \lesssim \|\mathbf{e}_{\mathbf{y}}\|_{\mathbf{H}^{1}(T)}.$$

Step 4. The proof concludes by gathering the estimates (87), (88), and (89).

We now investigate the local efficiency properties of the local indicator  $\mathcal{E}_{ad,T}$  defined in (42). To accomplish this task, for any  $\mathbf{g} \in \mathbf{L}^2(\Omega)$  and  $\mathcal{M} \subset \mathscr{T}$ , we define the oscillation term

(90) 
$$\operatorname{osc}_{\mathcal{M}}(\mathbf{g}) := \left(\sum_{T \in \mathcal{M}} h_T^2 \|\mathbf{g} - \Pi_T(\mathbf{g})\|_{\mathbf{L}^2(T)}^2\right)^{\frac{1}{2}},$$

where  $\Pi_T$  denotes the L<sup>2</sup>-projection onto piecewise constant functions over T.

THEOREM 11 (local efficiency of  $\mathcal{E}_{ad}$ ). Assume that the smallness assumption (11) holds. Let  $(\bar{\mathbf{y}}, \bar{p}, \bar{\mathbf{z}}, \bar{r}, \bar{\mathbf{u}}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbb{U}_{ad}$  be a local solution of (9)–(10). Let  $\bar{\mathbf{u}}_{\mathscr{T}}$  be a local minimum of the associated discrete optimal control problem with  $(\bar{\mathbf{y}}_{\mathscr{T}}, \bar{p}_{\mathscr{T}})$  and  $(\bar{\mathbf{z}}_{\mathscr{T}}, \bar{r}_{\mathscr{T}})$  being the corresponding state and adjoint state discrete variables, respectively. If assumption (32) holds, then, for  $T \in \mathscr{T}$ , the local error indicator  $\mathcal{E}_{ad,T}$  satisfies

(91) 
$$\mathcal{E}_{ad,T} \lesssim (1+h_T) \left( \|\mathbf{e}_{\mathbf{z}}\|_{\mathbf{H}^1(\mathcal{N}_T)} + \|\mathbf{e}_{\mathbf{y}}\|_{\mathbf{H}^1(\mathcal{N}_T)} + \|e_r\|_{L^2(\mathcal{N}_T)} \right) + \operatorname{osc}_{\mathcal{N}_T}(\mathbf{y}_{\Omega}),$$

where  $\mathcal{N}_T$  and  $\operatorname{osc}_{\mathcal{N}_T}(\mathbf{y}_{\Omega})$  are defined as in (25) and (90), respectively. The hidden constant is independent of the continuous and discrete optimal variables, the size of the elements in the mesh  $\mathscr{T}$ , and  $\#\mathscr{T}$ .

*Proof.* Since the pair  $(\bar{\mathbf{z}}, \bar{r}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  solves (14), an elementwise integration by parts formula yields the identity

$$(92) \quad \nu(\nabla \mathbf{w}, \nabla \mathbf{e}_{\mathbf{z}})_{\mathbf{L}^{2}(\Omega)} + b(\mathbf{e}_{\mathbf{y}}; \mathbf{w}, \bar{\mathbf{z}}) + b(\bar{\mathbf{y}}_{\mathscr{T}}; \mathbf{w}, \mathbf{e}_{\mathbf{z}}) + b(\mathbf{w}; \mathbf{e}_{\mathbf{y}}, \bar{\mathbf{z}}) + b(\mathbf{w}; \bar{\mathbf{y}}_{\mathscr{T}}, \mathbf{e}_{\mathbf{z}}) - (e_{r}, \operatorname{div} \mathbf{w})_{L^{2}(\Omega)} + (s, \operatorname{div} \mathbf{e}_{\mathbf{z}})_{L^{2}(\Omega)} - (\mathbf{e}_{\mathbf{y}}, \mathbf{w})_{\mathbf{L}^{2}(\Omega)} = \sum_{T \in \mathscr{T}} \left( (\Pi_{T}(\mathbf{y}_{\Omega}) - \mathbf{y}_{\Omega}, \mathbf{w})_{\mathbf{L}^{2}(T)} \right) + (\bar{\mathbf{y}}_{\mathscr{T}} - \Pi_{T}(\mathbf{y}_{\Omega}) + \nu \Delta \bar{\mathbf{z}}_{\mathscr{T}} - (\nabla \bar{\mathbf{y}}_{\mathscr{T}})^{\mathsf{T}} \bar{\mathbf{z}}_{\mathscr{T}} + (\bar{\mathbf{y}}_{\mathscr{T}} \cdot \nabla) \bar{\mathbf{z}}_{\mathscr{T}} - \nabla \bar{r}_{\mathscr{T}}, \mathbf{w})_{\mathbf{L}^{2}(T)} - (s, \operatorname{div} \bar{\mathbf{z}}_{\mathscr{T}})_{L^{2}(T)} \right) + \sum_{S \in \mathscr{S}} (\llbracket (\nu \nabla \bar{\mathbf{z}}_{\mathscr{T}} - \bar{r}_{\mathscr{T}} \mathbb{I}_{d}) \cdot \mathbf{n}], \mathbf{w})_{\mathbf{L}^{2}(S)},$$

which holds for every  $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$  and  $s \in L_0^2(\Omega)$ . With equation (92) at hand, in the following steps, we will estimate separately each of the individual terms that appear in the definition of  $\mathcal{E}_{ad,T}$ .

We now proceed on the basis of four steps.

Step 1. Let  $T \in \mathscr{T}$ . Define

$$\begin{aligned} \mathbf{R}_{T}^{ad} &:= (\bar{\mathbf{y}}_{\mathscr{T}} - \mathbf{y}_{\Omega} + \nu \Delta \bar{\mathbf{z}}_{\mathscr{T}} - (\nabla \bar{\mathbf{y}}_{\mathscr{T}})^{\mathsf{T}} \bar{\mathbf{z}}_{\mathscr{T}} + (\bar{\mathbf{y}}_{\mathscr{T}} \cdot \nabla) \bar{\mathbf{z}}_{\mathscr{T}} - \nabla \bar{r}_{\mathscr{T}}) \mid_{T}, \\ \hat{\mathbf{R}}_{T}^{ad} &:= (\bar{\mathbf{y}}_{\mathscr{T}} - \Pi_{T}(\mathbf{y}_{\Omega}) + \nu \Delta \bar{\mathbf{z}}_{\mathscr{T}} - (\nabla \bar{\mathbf{y}}_{\mathscr{T}})^{\mathsf{T}} \bar{\mathbf{z}}_{\mathscr{T}} + (\bar{\mathbf{y}}_{\mathscr{T}} \cdot \nabla) \bar{\mathbf{z}}_{\mathscr{T}} - \nabla \bar{r}_{\mathscr{T}}) \mid_{T}, \end{aligned}$$

We estimate the residual term  $h_T^2 || \mathbf{R}_T^{ad} ||_{\mathbf{L}^2(T)}^2$  in (42). We begin with a simple application of the triangle inequality to obtain

(93) 
$$h_T \|\mathbf{R}_T^{ad}\|_{\mathbf{L}^2(T)} \le h_T \|\hat{\mathbf{R}}_T^{ad}\|_{\mathbf{L}^2(T)} + \operatorname{osc}_T(\mathbf{y}_{\Omega}).$$

To bound  $h_T \| \hat{\mathbf{R}}_T^{ad} \|_{\mathbf{L}^2(T)}$  we set  $\mathbf{w} = \varphi_T \hat{\mathbf{R}}_T^{ad}$  and s = 0 in identity (92), utilize standard properties of the bubble function  $\varphi_T$ , and basic estimates to arrive at

(94) 
$$\|\hat{\mathbf{R}}_{T}^{ad}\|_{\mathbf{L}^{2}(T)} \lesssim (1+h_{T}^{-2})^{\frac{1}{2}} \left( \|\mathbf{e}_{\mathbf{y}}\|_{\mathbf{H}^{1}(T)} \|\bar{\mathbf{z}}\|_{\mathbf{H}^{1}(T)} + \|\bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{H}^{1}(T)} \|\mathbf{e}_{\mathbf{z}}\|_{\mathbf{H}^{1}(T)} \right) + h_{T}^{-1} \left( \|\nabla \mathbf{e}_{\mathbf{z}}\|_{\mathbf{L}^{2}(T)} + \|e_{r}\|_{L^{2}(T)} \right) + \|\mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(T)} + \|\Pi_{T}(\mathbf{y}_{\Omega}) - \mathbf{y}_{\Omega}\|_{\mathbf{L}^{2}(T)}.$$

On the other hand, notice that the stability estimate (77) yields

(95) 
$$\|\bar{\mathbf{z}}\|_{\mathbf{H}^{1}(T)} \leq \|\bar{\mathbf{z}}\|_{\mathbf{H}^{1}(\Omega)} \leq C \|\nabla\bar{\mathbf{z}}\|_{\mathbf{L}^{2}(\Omega)} \leq \frac{CC_{2}}{\nu(1-\theta)} \left(C_{2}\theta \mathcal{C}_{b}^{-1}\nu + \|\mathbf{y}_{\Omega}\|_{\mathbf{L}^{2}(\Omega)}\right),$$

where C > 0. Replacing this estimate and (86) into (94), we conclude

(96) 
$$h_T^2 \| \hat{\mathbf{R}}_T^{ad} \|_{\mathbf{L}^2(T)}^2 \lesssim (1 + h_T^2) \left( \| \mathbf{e}_{\mathbf{z}} \|_{\mathbf{H}^1(T)}^2 + \| \mathbf{e}_{\mathbf{y}} \|_{\mathbf{H}^1(T)}^2 + \| e_r \|_{L^2(T)}^2 \right) + \operatorname{osc}_T^2(\mathbf{y}_{\Omega}).$$

We conclude the desired estimate by gathering the inequalities (93) and (96). Step 2. Let  $T \in \mathscr{T}$  and  $S \in \mathscr{S}_T$ . We bound  $h_T \| [\![ (\nu \nabla \bar{\mathbf{z}}_{\mathscr{T}} - \bar{r}_{\mathscr{T}} \mathbb{I}_d) \cdot \mathbf{n} ]\!] \|_{\mathbf{L}^2(S)}^2$  in (42). To simplify the presentation of the material, we define

$$\mathbf{J}_{S}^{ad} := \llbracket (\nu \nabla \bar{\mathbf{z}}_{\mathscr{T}} - \bar{r}_{\mathscr{T}} \mathbb{I}_{d}) \cdot \mathbf{n} \rrbracket.$$

Set  $\mathbf{w} = \varphi_S \mathbf{J}_S^{ad}$  and s = 0 in (92) and proceed on the basis of similar arguments to the ones used to derive (94). These arguments yield

$$\begin{split} \|\mathbf{J}_{S}^{ad}\|_{\mathbf{L}^{2}(S)}^{2} \lesssim \sum_{T' \in \mathcal{N}_{S}} \left( (1 + h_{T'}^{-2})^{\frac{1}{2}} \left( \|\mathbf{e}_{\mathbf{y}}\|_{\mathbf{H}^{1}(T')} \|\bar{\mathbf{z}}\|_{\mathbf{H}^{1}(T')} + \|\bar{\mathbf{y}}_{\mathscr{T}}\|_{\mathbf{H}^{1}(T')} \|\mathbf{e}_{\mathbf{z}}\|_{\mathbf{H}^{1}(T')} \right) \\ &+ h_{T'}^{-1} \left( \|\nabla \mathbf{e}_{\mathbf{z}}\|_{\mathbf{L}^{2}(T')} + \|e_{r}\|_{L^{2}(T')} \right) + \|\mathbf{e}_{\mathbf{y}}\|_{\mathbf{L}^{2}(T')} \\ &+ \|\hat{\mathbf{R}}_{T'}^{ad}\|_{\mathbf{L}^{2}(T')} + \|\Pi_{T'}(\mathbf{y}_{\Omega}) - \mathbf{y}_{\Omega}\|_{\mathbf{L}^{2}(T')} \right) h_{T}^{\frac{1}{2}} \|\mathbf{J}_{S}^{ad}\|_{\mathbf{L}^{2}(S)}. \end{split}$$

Invoke estimates (86), (95), and (96) to conclude and obtain

(97) 
$$h_T \|\mathbf{J}_S^{ad}\|_{\mathbf{L}^2(S)}^2 \lesssim \sum_{T' \in \mathcal{N}_S} (1 + h_{T'}^2) \left( \|\mathbf{e}_{\mathbf{z}}\|_{\mathbf{H}^1(T')}^2 + \|\mathbf{e}_{\mathbf{y}}\|_{\mathbf{H}^1(T')}^2 + \|\mathbf{e}_{\mathbf{y}}\|_{\mathbf{H}^1(T')}^2 + \operatorname{osc}_{T'}^2(\mathbf{y}_{\Omega}) \right).$$

Step 3. Let  $T \in \mathscr{T}$ . Since div  $\bar{\mathbf{z}} = 0$ , we immediately obtain that

(98) 
$$\|\operatorname{div} \bar{\mathbf{z}}_{\mathscr{T}}\|_{L^2(T)} \le \|\operatorname{div} \mathbf{e}_{\mathbf{z}}\|_{L^2(T)} \lesssim \|\mathbf{e}_{\mathbf{z}}\|_{\mathbf{H}^1(T)}.$$

Step 4. The proof concludes by gathering (93), (96), (97), and (98).

The results obtained in Theorems 10 and 11 yield the local efficiency of

(99) 
$$\mathcal{E}_{ocp,T}^2 := \mathcal{E}_{ad,T}^2 + \mathcal{E}_{st,T}^2 + \mathcal{E}_{ct,T}^2$$

THEOREM 12 (local efficiency of  $\mathcal{E}_{ocp,T}$ ). Assume that the smallness assumption (11) holds. Let  $(\bar{\mathbf{y}}, \bar{p}, \bar{\mathbf{z}}, \bar{r}, \bar{\mathbf{u}}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbb{U}_{ad}$  be a local solution of (9)–(10). Let  $\bar{\mathbf{u}}_{\mathscr{T}}$  be a local minimum of the associated discrete optimal control problem with  $(\bar{\mathbf{y}}_{\mathscr{T}}, \bar{p}_{\mathscr{T}})$  and  $(\bar{\mathbf{z}}_{\mathscr{T}}, \bar{r}_{\mathscr{T}})$  being the corresponding state and adjoint state discrete variables, respectively. If assumption (32) holds, then, for  $T \in \mathscr{T}$ , we have that

$$\mathcal{E}_{ocp,T} \lesssim (1+h_T) \left( \| \mathbf{e}_{\mathbf{y}} \|_{\mathbf{H}^1(\mathcal{N}_T)} + \| \mathbf{e}_{\mathbf{u}} \|_{\mathbf{L}^2(\mathcal{N}_T)} + \| \mathbf{e}_{\mathbf{z}} \|_{\mathbf{H}^1(\mathcal{N}_T)} \\ \| e_p \|_{L^2(\mathcal{N}_T)} + \| e_r \|_{L^2(\mathcal{N}_T)} \right) + \operatorname{osc}_{\mathcal{N}_T}(\mathbf{y}_{\Omega}),$$

where  $\mathcal{N}_T$  and  $\operatorname{osc}_{\mathcal{N}_T}(\mathbf{y}_{\Omega})$  are defined as in (25) and (90), respectively. The hidden constant is independent of the continuous and discrete optimal variables, the size of the elements in the mesh  $\mathscr{T}$ , and  $\#\mathscr{T}$ .

*Proof.* Let  $T \in \mathscr{T}$ . In view of the local efficiency estimates (82) and (91), it suffices to bound  $\mathcal{E}_{ct,T}$ . Invoke (51) and an application of the triangle inequality to obtain

$$\begin{aligned} \mathcal{E}_{ct,T} &\leq \|\tilde{\mathbf{u}} - \bar{\mathbf{u}}\|_{\mathbf{L}^{2}(T)} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^{2}(T)} \\ &= \|\Pi_{[\mathbf{a},\mathbf{b}]}(-\alpha^{-1}\bar{\mathbf{z}}_{\mathscr{T}}) - \Pi_{[\mathbf{a},\mathbf{b}]}(-\alpha^{-1}\bar{\mathbf{z}})\|_{\mathbf{L}^{2}(T)} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^{2}(T)}. \end{aligned}$$

Invoke the Lipschitz property of  $\Pi_{[\mathbf{a},\mathbf{b}]}$ , introduced in (18), to obtain

(100) 
$$\mathcal{E}_{ct,T} \le \alpha^{-1} \| \mathbf{e}_{\mathbf{z}} \|_{\mathbf{L}^{2}(T)} + \| \mathbf{e}_{\mathbf{u}} \|_{\mathbf{L}^{2}(T)} \le \alpha^{-1} \| \mathbf{e}_{\mathbf{z}} \|_{\mathbf{H}^{1}(T)} + \| \mathbf{e}_{\mathbf{u}} \|_{\mathbf{L}^{2}(T)}.$$

The proof concludes by collecting estimates (82), (91), and (100).

Π

6. Numerical examples. In this section we conduct a series of numerical examples that illustrate the performance of the devised a posteriori error estimator  $\mathcal{E}_{ocp}$ defined in (52).

**6.1.** Implementation. The presented numerical examples have been carried out with the help of a code that we implemented using C++. All matrices have been assembled exactly and global linear systems were solved using the multifrontal massively parallel sparse direct solver (MUMPS) [5, 6]. The right hand sides, the approximation errors, and the error estimators are computed by a quadrature formula which is exact for polynomials of degree nineteen (19) for two dimensional domains and degree fourteen (14) for three dimensional domains.

For a given partition  $\mathscr{T}$  we seek  $(\bar{\mathbf{y}}_{\mathscr{T}}, \bar{p}_{\mathscr{T}}, \bar{\mathbf{z}}_{\mathscr{T}}, \bar{\mathbf{r}}_{\mathscr{T}}, \bar{\mathbf{u}}_{\mathscr{T}}) \in \mathbf{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T}) \times \mathbf{V}(\mathscr{T}) \times$  $\mathcal{P}(\mathscr{T}) \times \mathbb{U}_{ad}(\mathscr{T})$  that solves the discrete optimality conditions (29)–(31). This system is solved by using a primal-dual active set strategy [27, section 2.12.4] combined with a fixed point strategy: for each active set iteration, the ensuing nonlinear system is solved by using a fixed point method. Once the discrete solution is obtained, we compute, for  $T \in \mathscr{T}$ , the error indicator  $\mathcal{E}_{ocp,T}$ , defined in (99), to drive the adaptive mesh refinement procedure described in Algorithm 1. A sequence of adaptively refined meshes is thus generated from the initial meshes shown in Figure 1. To visualize finite element approximations we have used the open-source application ParaView [1, 7].

## Algorithm 1 Adaptive algorithm.

**Input:** Initial mesh  $\mathscr{T}_0$ , fluid viscosity  $\nu$ , desired state  $\mathbf{y}_{\Omega}$ , external source  $\mathbf{f}$ , constraints **a** and **b**, and regularization parameter  $\alpha$ ;

**Set:** i = 0.

## Active set strategy:

1: Choose an initial discrete guess  $(\mathbf{y}_{\mathcal{J}_i}^0, p_{\mathcal{J}_i}^0, \mathbf{z}_{\mathcal{J}_i}^0, \mathbf{r}_{\mathcal{J}_i}^0, \mathbf{u}_{\mathcal{J}_i}^0) \in \mathbf{V}(\mathcal{J}_i) \times \mathcal{P}(\mathcal{J}_i) \times \mathbf{V}(\mathcal{J}_i) \times \mathbf{V}(\mathcal{J}_i)$  $\mathcal{P}(\mathscr{T}_i) \times \mathbb{U}(\mathscr{T}_i);$ 

**2**: Compute  $[\bar{\mathbf{y}}_{\mathscr{T}_i}, \bar{p}_{\mathscr{T}_i}, \bar{\mathbf{z}}_{\mathscr{T}_i}, \bar{\mathbf{x}}_{\mathscr{T}_i}, \bar{\mathbf{u}}_{\mathscr{T}_i}] = \mathbf{Active-Set}[\mathscr{T}_i, \nu, \mathbf{y}_{\Omega}, \mathbf{f}, \mathbf{a}, \mathbf{b}, \alpha, \mathbf{y}_{\mathscr{T}_i}^0, p_{\mathscr{T}_i}^0, \mathbf{z}_{\mathscr{T}_i}^0,$  $r_{\mathcal{J}_i}^0, \mathbf{u}_{\mathcal{J}_i}^0$ ]. Active-Set implements the active set strategy of [27, section 2.12.4]; for each active set iteration, the ensuing nonlinear system is solved by using a fixed point method:

#### Adaptive loop:

3: For each  $T \in \mathscr{T}_i$  compute the local error indicator  $\mathcal{E}_{ocp,T}$  defined in (99); 4: Mark an element  $T \in \mathscr{T}_i$  for refinement if  $\mathcal{E}^2_{ocp,T} > \frac{1}{2} \max_{T' \in \mathscr{T}_i} \mathcal{E}^2_{ocp,T'}$ ; 5: From step 4, construct a new mesh  $\mathscr{T}_{i+1}$ , using a longest edge bisection algorithm. Set  $i \leftarrow i + 1$  and go to step 1.



FIG. 1. The initial meshes used when the domain  $\Omega$  is a two-dimensional L-shape (Example 1) and a cube (Example 2).

The total number of degrees of freedom when solving (29)-(31) corresponds to  $\mathsf{Ndof} = 2[\dim(\mathbf{V}(\mathscr{T})) + \dim(\mathscr{P}(\mathscr{T}))] + \dim(\mathbb{U}(\mathscr{T})).$  We recall that the discrete spaces  $\mathbf{V}(\mathscr{T}), \mathscr{P}(\mathscr{T}), \text{ and } \mathbb{U}(\mathscr{T}) \text{ are defined by (26), (27), and (28), respectively. The error$ is measured in the norm  $\|\mathbf{e}\|_{\Omega}$ , which is defined in (58).

To simplify the construction of exact solutions, we incorporate an extra source term  $\mathbf{f} \in \mathbf{L}^{\infty}(\Omega)$  in the right hand side of the momentum equation of (10). With such a modification, the right hand side of the first equation in (10) now reads  $(\mathbf{f}+\mathbf{u},\mathbf{v})_{\mathbf{L}^{2}(\Omega)}$ .

We now provide two numerical experiments. We first consider a problem in two dimensions where we go beyond the presented analysis and perform a numerical experiment where we violate the assumption of homogeneous Dirichlet boundary conditions. Second, we consider a problem with homogeneous Dirichlet boundary conditions in three dimensions. We mention that in both numerical examples the exact solutions are known.

**6.2. Example 1 (two dimensional non-convex domain).** We set  $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$ ,  $\mathbf{a} = (-2,-2)$ ,  $\mathbf{b} = (2,2)$ ,  $\alpha = 10^{-4}$ , and  $\nu = 1$ . The optimal state and adjoint state are given, in polar coordinates  $(\rho, \vartheta)$ , by

$$\bar{\mathbf{y}}(\rho,\vartheta) = \bar{\mathbf{z}}(\rho,\vartheta) = 10^{-2}\rho^{\sigma} \begin{pmatrix} (1+\sigma)\sin(\vartheta)\psi(\vartheta) + \cos(\vartheta)\psi'(\vartheta) \\ -(1+\sigma)\cos(\vartheta)\psi(\vartheta) + \sin(\vartheta)\psi'(\vartheta) \end{pmatrix},$$
$$\bar{p}(\rho,\vartheta) = \bar{r}(\rho,\vartheta) = \frac{1}{1-\sigma}\rho^{\sigma-1} \left( (1+\sigma)^2)\psi'(\vartheta) + \psi'''(\vartheta) \right),$$
$$\psi(\vartheta) = \left(\frac{\sin((1+\sigma)\vartheta)}{1+\sigma} + \frac{\sin((\sigma-1)\vartheta)}{\sigma-1}\right)\cos(\gamma\sigma) - \cos((1+\sigma)\vartheta) + \cos((\sigma-1)\vartheta),$$

where  $\vartheta \in [0, 3\pi/2]$ ,  $\sigma = 856399/1572864$ , and  $\gamma = 3\pi/2$ .

In Figure 2 we present the results obtained for Example 1. Subfigures (A.1) and (A.2) show that our AFEM outperforms uniform refinement. Subfigures (A.2) and (A.3) show that our devised AFEM exhibits optimal experimental rates of convergence for all the individual contributions of  $\|\mathbf{e}\|_{\Omega}$  and  $\mathcal{E}_{ocp}$ , respectively, but with the exception of the ones related to the control variable. For each adaptive iteration, the effectivity index  $\mathcal{I} := \mathcal{E}_{ocp}/\|\mathbf{e}\|_{\Omega}$  is presented in subfigure (A.4). The final value is stabilized around the value of 2 and shows the accuracy of the proposed error estimator when is used in our adaptive loop. The mesh obtained after 45 iterations of our AFEM is presented in subfigure (B.1); the mesh contains 3914 elements and 2031 nodes. We observe that most of the adaptive refinement occurs near to the interface of the control variable and the geometric singularity. This attests to the efficiency of the devised estimator. Finally, in subfigures (B.2), (B.3), and (B.4), we present the numerical approximations of  $\bar{r}_{\mathcal{T}}$ , the first component of  $\bar{\mathbf{u}}_{\mathcal{T}}$ , and the first component of  $\bar{\mathbf{z}}_{\mathcal{T}}$ , respectively.

6.3. Example 2 (three dimensional convex domain). We consider  $\Omega = (0,1)^3$ ,  $\mathbf{a} = 10^{-3}(-7,-7,-7)$ ,  $\mathbf{b} = 10^{-3}(7,7,7)$ ,  $\alpha = 10^{-1}$ , and  $\nu = 10^{-2}$ . The exact optimal state and adjoint state are given by

$$\bar{\mathbf{y}}(x_1, x_2, x_3) = 10^{-3} \mathbf{curl} \left( \left( x_2 x_3 (1 - x_2) (1 - x_3) \right)^2 \left( 1 - x_1 - \frac{e^{-x_1/\nu} - e^{-1/\nu}}{1 - e^{-1/\nu}} \right) \right), \\ \bar{\mathbf{z}}(x_1, x_2, x_3) = \mathbf{curl} \left( \left( x_1 x_2 x_3 (1 - x_1) (1 - x_2) (1 - x_3) \right)^2 \right), \\ \bar{p}(x_1, x_2) = \bar{r}(x_1, x_2) = (x_1 x_2 x_3 - 1/8).$$

In Figure 3 we present the results obtained for Example 2. Subfigures (C.1) and (C.2) show that, as in the two dimensional example, our AFEM outperforms uniform refinement; it also exhibits optimal experimental rates of convergence for all the individual contributions of  $\|\mathbf{e}\|_{\Omega}$  and  $\mathcal{E}_{ocp}$ , respectively. Finally, for each adaptive

iteration, the effectivity index  $\mathcal{I}$  is presented in subfigure (C.4). The final value is stabilized around the value of 1.



FIG. 2. Example 1. Experimental rates of convergence for the individual contributions of  $\|\mathbf{e}\|_{\Omega}$  for uniform (A.1) and adaptive refinement (A.2); experimental rates of convergence for the individual contributions of  $\mathcal{E}_{ocp}$  (A.3); effectivity index  $\mathcal{I}$  (A.4); adaptively refined mesh obtained after 45 iterations of our adaptive loop (3914 elements and 2031 nodes) (B.1); finite element approximation of  $\bar{r}_{\mathcal{T}}$  (B.2), and the first component of  $\bar{\mathbf{u}}_{\mathcal{T}}$  (B.3) and  $\bar{\mathbf{z}}_{\mathcal{T}}$  (B.4).



FIG. 3. Example 2. Experimental rates of convergence for the individual contributions of  $\|\mathbf{e}\|_{\Omega}$  for uniform (C.1) and adaptive refinement (C.2); experimental rates of convergence for the individual contributions of  $\mathcal{E}_{ocp}$  (C.3) and effectivity index  $\mathcal{I}$  (C.4).

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