

# A FEM FOR THE SQUARE ROOT OF THE LAPLACE OPERATOR:

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**Abstract.** We consider the square root of the Laplace operator  $(-\Delta)^{1/2}$  in a bounded domain. The square root of the Laplacian can be realized as the Dirichlet to Neumann operator of an extension problem posed on a semi-infinite cylinder. This extension problem involves a mixed boundary value problem, which we analyze in the framework of Sobolev spaces. For numerical approximation we propose a suitable truncated problem, which can be justified by the rapid decay of the solution to the extension problem. A finite element approximation is considered for the truncation. A priori error estimates are obtained for conforming and shape regular meshes and numerical experiments are presented illustrating the theory.

**1. Introduction.** Singular integrals and nonlocal operators have been a standard topic in different branches of mathematics such as operator theory and harmonic analysis (see [21]). They are now becoming increasingly important because of their strong connection with real-world problems. Concerning applications, nonlocal operators arise in: boundary control problems [12], finance [11], electromagnetic fluids [18], image processing [15], material science [5], turbulence [2] and others.

In this work, we shall be interested in a specific nonlocal positive operator: the square root of the Laplace operator in a bounded domain with zero Dirichlet boundary conditions and its respective numerical approximation via the finite element method.

The problem we shall be concerned with reads as follows: Given a smooth enough function  $f$ , find  $u$  such that

$$\begin{cases} (-\Delta)^{1/2}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^n$  with  $n \geq 1$  is a bounded domain with a smooth boundary  $\partial\Omega$ , and  $(-\Delta)^{1/2}$  denotes the square root of the Laplace operator  $-\Delta$  supplemented with homogeneous Dirichlet values on  $\partial\Omega$ .

The study of elliptic equations involving the square root of the Laplacian is important in many physical applications in which long-range or anomalous diffusion is considered. For instance in probability,  $(-\Delta)^{1/2}$  is the infinitesimal generator of a stable Lévy process (see [6]).

Problem (1.1) involves the square root of the Laplace operator, which is a nonlocal operator (see [16, 21, 8, 9]). It is well known that problem (1.1) can be realized in a local manner as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem in the semi-infinite cylinder  $\mathcal{C} := \Omega \times (0, \infty)$ . This extension problem corresponds to a mixed boundary value problem for the Laplace operator  $-\Delta$ :

$$\begin{cases} -\Delta \mathbf{u} = 0 & \text{in } \mathcal{C}, \\ \mathbf{u} = 0 & \text{on } \partial_L \mathcal{C}, \\ \frac{\partial \mathbf{u}}{\partial \nu} = f & \text{on } \Omega \times \{0\}, \end{cases} \quad (1.2)$$

where  $\partial_L \mathcal{C} := \partial\Omega \times [0, \infty)$  denotes the lateral boundary of  $\mathcal{C}$ , and  $\nu$  the unit outer normal to  $\mathcal{C}$  at  $\Omega \times \{0\}$ .

Then, the square root of the Laplace operator  $-\Delta$  in a bounded domain  $\Omega$  is related to the Dirichlet to Neumann operator for problem (1.2) in the following way:

$$(-\Delta)^{1/2}u = \frac{\partial \mathbf{u}}{\partial \nu} \text{ in } \Omega.$$

Using the idea developed above, our strategy to solve the nonlocal problem (1.1) is as follows: given a sufficiently smooth function  $f$  we solve problem (1.2), obtaining a function  $\mathbf{u}$ . Hence, taking the trace of  $\mathbf{u}$  at  $\Omega \times \{0\}$  we obtain the function  $u$  that solves (1.1).

The outline of this paper is as follows. In section 2 we consider a suitable function space designed to study problems (1.1) and (1.2), and we define the square root of the Laplace operator in  $\mathbb{R}^n$  and bounded domains. In order to approximate numerically  $u$ , in section 3 we study a suitable truncated problem of (1.2). In section 4, a finite element approximation is considered for the truncation and a priori error estimates for both  $u$  and  $\mathbf{u}$  are obtained. We conclude with numerical experiments, which illustrate the developed theory.

## 2. Preliminaries: Function spaces and the fractional Laplace operator.

**2.1. Function spaces.** In order to define the square root of the Laplace operator, we start by recalling some appropriate function spaces, see for instance ([17, 19, 22]). The fractional Sobolev space  $H^{1/2}(\Omega)$  can be defined via the so called Gagliardo-Slobodeckii seminorm (see [19, 22]). Let  $\Omega$  be an open non-empty subset of  $\mathbb{R}^n$ , we define the Gagliardo-Slobodeckii seminorm  $|\cdot|_{H^{1/2}(\Omega)}$  as follows:

$$|u|_{H^{1/2}(\Omega)} := \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+1}} dx dy \right)^{1/2}.$$

Then, the fractional Sobolev space  $H^{1/2}(\Omega)$  is defined by

$$H^{1/2}(\Omega) := \{u \in L^2(\Omega) : |u|_{H^{1/2}(\Omega)} < +\infty\}, \quad (2.1)$$

which, equipped with the norm

$$\|u\|_{H^{1/2}(\Omega)} := \left( \|u\|_{L^2(\Omega)}^2 + |u|_{H^{1/2}(\Omega)}^2 \right)^{1/2}$$

is a Banach space. Now, we can define the so-called *Lions-Magenes* space as follows:

$$H_{00}^{1/2}(\Omega) := \left\{ u \in H^{1/2}(\Omega) : \int_{\Omega} \frac{u^2(x)}{d(x)} dx < +\infty \right\},$$

where  $d(x) = \text{dist}(x, \partial\Omega)$ . This space equipped with the norm

$$\|u\|_{H_{00}^{1/2}(\Omega)} := \left\{ \|u\|_{H^{1/2}(\Omega)}^2 + \int_{\Omega} \frac{u^2}{d} \right\}^{1/2}$$

is a Banach space.

When  $\Omega$  is an open and bounded subset of  $\mathbb{R}^n$  with a smooth boundary  $\partial\Omega$ , an equivalent approach to define the *Lions-Magenes* space is given by interpolation (see, for instance, [17, Ch. 1]). Let us use the standard convention  $H^0(\Omega) = L^2(\Omega)$ . Then, the *Lions-Magenes* space is equivalently defined as the interpolation space of index  $1/2$  for the pair  $[H_0^1(\Omega), L^2(\Omega)]$ :

$$H_{00}^{1/2}(\Omega) := [H_0^1(\Omega), L^2(\Omega)]_{1/2},$$

(see [17, Theorem 11.7]). In the case where  $\Omega$  has a Lipschitz boundary, the equivalence is still true (see [1, Ch. 7] for details).

We define  $H_0^{1/2}(\Omega)$  as the closure of  $C_0^\infty(\Omega)$  with respect to the norm  $\|\cdot\|_{H^{1/2}(\Omega)}$ , i.e.,

$$H_0^{1/2}(\Omega) := \overline{C_0^\infty(\Omega)}^{H^{1/2}(\Omega)}. \quad (2.2)$$

We have the strict inclusion

$$H_{00}^{1/2}(\Omega) \subsetneq H_0^{1/2}(\Omega);$$

for instance,  $1 \in H_0^{1/2}(\Omega)$  but  $1 \notin H_{00}^{1/2}(\Omega)$ . On the other hand, it is known that if  $\Omega$  is a bounded domain with a smooth boundary  $\partial\Omega$ , the space  $C_0^\infty(\Omega)$  is dense in  $H^{1/2}(\Omega)$  (see [17, Theorem 11.1]). Then, we have the following relationship among the spaces defined above

$$H_{00}^{1/2}(\Omega) \subsetneq H_0^{1/2}(\Omega) = H^{1/2}(\Omega).$$

To treat the nonlocal problem (1.1), we will study a corresponding extension problem in one more dimension, for which we need to consider an appropriate setting. Let us denote the upper half-space in  $\mathbb{R}^{n+1}$  by

$$\mathbb{R}_+^{n+1} = \{(x, y) = (x_1, x_2, \dots, x_n, y) \in \mathbb{R}^n \times \mathbb{R} : y > 0\},$$

Let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^n$ . Denote the half-cylinder with base  $\Omega$  by

$$\mathcal{C} = \Omega \times (0, \infty), \quad (2.3)$$

and its lateral boundary by

$$\partial_L \mathcal{C} = \partial\Omega \times [0, \infty). \quad (2.4)$$

Then, we define

$$\mathring{H}_L^1(\mathcal{C}) = \{v \in H^1(\mathcal{C}) : v = 0 \text{ a.e. on } \partial_L \mathcal{C}\}, \quad (2.5)$$

which as a consequence of the Poincaré inequality (see [14, 13]), can be equipped with the norm  $|v|_{H^1(\mathcal{C})} := \|\nabla v\|_{L^2(\mathcal{C})}$ .

Finally, we denote by  $tr_\Omega$  the trace operator on  $\Omega \times \{0\}$  for functions in  $H^1(\mathcal{C})$ . Notice that given  $v \in H^1(\mathcal{C})$ ,  $tr_\Omega v \in H^{1/2}(\Omega)$ , since traces of  $H^1$  functions are  $H^{1/2}$  functions on the boundary (see [1, 22]).

**2.2. The square root of the Laplace operator.** The square root of the Laplace operator of a smooth function with compact support  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  is defined via Fourier transform as

$$\mathcal{F}((-\Delta)^{1/2}u)(\xi) := |\xi|\mathcal{F}(u)(\xi). \quad (2.6)$$

This definition can be extended to any function belonging to the fractional Sobolev space  $H^{1/2}(\mathbb{R}^n)$ . It can also be expressed by the pointwise formula

$$(-\Delta)^{1/2}u(x) = C_n \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+1}} dy, \quad (2.7)$$

where P.V. stands for the Cauchy principal value and  $C_n$  is a normalization constant to guarantee that the symbol of the resulting operator is  $|\xi|$ ; see the references [9, 16] for more details. Observe from the pointwise formula (2.7) that the square root of the Laplace operator is a nonlocal operator.

Smooth and bounded functions are admissible for the definition above. In particular, the above integral is well defined if, for instance,  $u$  is bounded (which ensures the integrability at infinity) and  $u \in C_{\text{loc}}^2(\mathbb{R}^n)$  (which ensures the integrability at  $x = y$  in the principal value sense).

In order to define the square root of the Laplace operator in a bounded domain, we follow the approach presented in [8]. Recall the well known spectral theory of the Laplacian  $-\Delta$  in a bounded domain  $\Omega$  with zero Dirichlet boundary values (see, for instance [13, 14]). If  $\partial\Omega \in C^{0,1}$ , then regularity theory asserts that the operator  $(-\Delta)^{-1}$  is compact. Hence, there exists a sequence of eigenvalues of  $-\Delta$  repeated by their finite multiplicity

$$0 < \lambda_1 < \lambda_2 \leq \dots \lambda_k \leq \dots \rightarrow \infty, \quad \text{as } k \rightarrow \infty$$

and, there exists an orthonormal basis  $\{\varphi_k\}$  of  $L^2(\Omega)$ , such that  $\varphi_k \in H_0^1(\Omega)$  is an eigenfunction of  $-\Delta$  corresponding to  $\lambda_k$ :

$$\begin{cases} -\Delta\varphi_k &= \lambda_k\varphi_k & \text{in } \Omega, \\ \varphi_k &= 0 & \text{on } \partial\Omega, \end{cases} \quad (2.8)$$

for  $k = 1, 2, \dots$ . The square root of the Dirichlet Laplacian  $(-\Delta)^{1/2}$  can be defined for any function  $u \in C_0^\infty(\Omega)$  by

$$u = \sum_{k=1}^{\infty} b_k \varphi_k \mapsto (-\Delta)^{1/2}u = \sum_{k=1}^{\infty} b_k \lambda_k^{1/2} \varphi_k, \quad (2.9)$$

where the coefficients  $b_k$  are given by

$$b_k = \int_{\Omega} u \varphi_k dx.$$

By density this operator can be extended to the Hilbert space

$$H := \left\{ u \in L^2(\Omega) : \|u\|_H^2 = \sum_{k=1}^{\infty} \lambda_k^{1/2} |b_k|^2 < \infty \right\}. \quad (2.10)$$

Following the theory of Hilbert scale presented in [17, Ch. 1] we have

$$[H_0^1(\Omega), L^2(\Omega)]_{1/2} = \text{Dom}(-\Delta)^{1/2},$$

where  $\text{Dom}(-\Delta)^{1/2}$  stands for the domain of the operator  $(-\Delta)^{1/2}$ , and as a consequence, we conclude

$$H = [H_0^1(\Omega), L^2(\Omega)]_{1/2} = H_{00}^{1/2}(\Omega). \quad (2.11)$$

As we discussed in section 2.1, the traces of functions in  $\mathring{H}_L^1(\mathcal{C})$  should be characterized. In fact, such a characterization is given in [8, Proposition 1.8], which is complemented with (2.11) to get the following result.

**PROPOSITION 2.1.** *Let  $\Omega$  be a bounded and smooth domain in  $\mathbb{R}^n$ . Then, we have*

$$H = H_{00}^{1/2}(\Omega) = \{u = \text{tr}_{\Omega} v : v \in \mathring{H}_L^1(\mathcal{C})\}. \quad (2.12)$$

It is known that the square root of the Laplace operator in a bounded domain  $\Omega$  defined as above, can be determined as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem (see [8, 10, 9]).

Let us consider a smooth function  $u$  defined in  $\Omega$ . Let  $\mathbf{u}$  be the unique solution of the Laplace equation in the cylinder  $\mathcal{C}$ , with  $\mathbf{u}$  vanishing on the lateral boundary  $\partial_L \mathcal{C}$  and  $u$  as the boundary condition on  $\Omega \times \{0\}$ :

$$\begin{cases} -\Delta \mathbf{u} = 0 & \text{in } \mathcal{C} = \Omega \times (0, \infty), \\ \mathbf{u} = 0 & \text{on } \partial_L \mathcal{C} = \partial \Omega \times [0, \infty), \\ \frac{\partial \mathbf{u}}{\partial \nu} = u & \text{on } \Omega \times \{0\}, \end{cases} \quad (2.13)$$

where  $\nu$  denotes the unit outer normal to the cylinder  $\mathcal{C}$  at  $\Omega \times \{0\}$ . Consider the operator

$$T : u \rightarrow -\partial_y \mathbf{u}(x, 0) = \frac{\partial \mathbf{u}}{\partial \nu} \Big|_{\Omega \times \{0\}}.$$

By a simple integration by parts argument, we see that

$$\begin{aligned} (Tu, u)_{L^2(\Omega)} &= - \int_{\Omega} \mathbf{u}(x, 0) \partial_y \mathbf{u}(x, 0) \\ &= \int_{\mathcal{C}} |\nabla \mathbf{u}|^2 \geq 0 \end{aligned}$$

Thus,  $T$  is a positive operator. Moreover, since  $-\partial_y \mathbf{u}(x, 0)$  is also a harmonic function, if we apply the operator  $T$  twice to the function  $u$ , we get

$$(T \circ T)u = T(-\partial_y \mathbf{u}(x, 0)) = \partial_{yy} \mathbf{u}(x, 0) = -\Delta_x \mathbf{u}(x, 0) = \Delta_x u$$

where  $\Delta_x$  denotes  $\partial_{x_1}^2 + \dots + \partial_{x_n}^2$ . Therefore, the operator  $T$  that maps the Dirichlet type condition  $u$  into the Neumann type condition  $-\partial_y \mathbf{u}(x, 0)$ , is actually the square root of the Laplace operator  $(-\Delta)^{1/2}$ . In this way, we transform the nonlocal problem (1.1) in a local one,

but in one more dimension. More precisely, we look for a function  $\mathbf{u}$  with  $\mathbf{u}(\cdot, 0) = u$  satisfying the following mixed boundary value problem in a half-cylinder:

$$\begin{cases} -\Delta \mathbf{u} &= 0 & \text{in } \mathcal{C} = \Omega \times (0, \infty), \\ \mathbf{u} &= 0 & \text{on } \partial_L \mathcal{C} = \partial\Omega \times [0, \infty), \\ \frac{\partial \mathbf{u}}{\partial \nu} &= f & \text{on } \Omega \times \{0\}. \end{cases} \quad (2.14)$$

Then, if  $\mathbf{u}$  satisfies problem (2.14), the trace  $u$  on  $\Omega \times \{0\}$  of  $\mathbf{u}$  is a solution of the problem (1.1).

Now, let us consider the following Dirichlet to Neumann operator

$$\begin{aligned} \Gamma : H &\rightarrow H' \\ u &\mapsto \Gamma(u) = f := \frac{\partial \mathbf{u}}{\partial \nu}, \end{aligned}$$

where  $\mathbf{u}$  is the solution of problem (2.13) and  $H'$  denotes the dual space of  $H$ . We summarize the ideas explained above in the following result (see [8, 9]).

**THEOREM 2.2.** *For every  $u \in H$ , we have that*

$$(-\Delta)^{1/2} u = \Gamma(u) = \frac{\partial \mathbf{u}}{\partial \nu},$$

where the equality above holds in the distributional sense.

In other words, given  $f \in H'$ , a function  $u \in H$  solves the equation  $(-\Delta)^{1/2} u = f$  in  $\Omega$  if and only if its harmonic extension  $\mathbf{u}$  solves problem (2.14).

The associated weak formulation of problem (2.14) reads: Find  $\mathbf{u} \in \dot{H}_L^1(\mathcal{C})$  such that

$$\int_{\mathcal{C}} \nabla \mathbf{u} \cdot \nabla \phi = \langle f, \text{tr}_\Omega \phi \rangle, \quad \text{for all } \phi \in \dot{H}_L^1(\mathcal{C}). \quad (2.15)$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $H$  and  $H'$ , which is well defined because of the characterization of the space  $H$  given in Proposition 2.1.

We remark that via the Lax-Milgram Lemma, for every  $f \in H'$  problem (2.15) has a unique solution, and moreover, we have the following estimate (see [8]):

$$\|\mathbf{u}\|_{\dot{H}_L^1(\mathcal{C})} \lesssim \|u\|_H = \|f\|_{H'} \quad (2.16)$$

We conclude this section with a trace estimate, which will be important in order to obtain the error estimates derived in Section 4 (see estimate (2.17) in [8]):

$$\|u\|_H \lesssim \|\mathbf{u}\|_{\dot{H}_L^1(\mathcal{C})} \quad (2.17)$$

**3. A Truncated Problem.** The variational equation (2.15) is posed on the infinite domain  $\mathcal{C} = \Omega \times (0, \infty)$ . The rapid decay of the solution  $\mathbf{u}$  in the  $y$ -direction, suggests truncation to a bounded domain with a convenient Dirichlet condition. In fact, the next result shows that the energy of the solution  $\mathbf{u}$  outside of a truncated domain  $\Omega \times (0, \gamma)$  can be made as small as desired, by choosing properly the parameter  $\gamma$ .

**PROPOSITION 3.1.** *For any positive constant  $\gamma > 0$ , the solution  $\mathbf{u}$  of the problem (2.14) satisfies the following estimate*

$$\|\nabla \mathbf{u}\|_{L^2(\Omega \times (\gamma, \infty))} \lesssim e^{-\lambda_1^{1/2} \gamma} \|f\|_{H'}. \quad (3.1)$$

*Proof.* Let  $u \in H$ . Then  $u \in L^2(\Omega)$ , and we can consider its expansion in terms of the eigenfunctions  $\{\varphi_k\}_{k=1}^\infty$ :  $u(x) = \sum_{k=1}^\infty b_k \varphi_k(x)$ . Hence, the function

$$\mathbf{u}(x, y) = \sum_{k=1}^\infty b_k \varphi_k(x) e^{-\lambda_k^{1/2} y},$$

belongs to the Sobolev space  $\mathring{H}_L^1(\mathcal{C})$  and solves problem (2.14) (see [8, Lemma 2.10]). Using the expression above, and the fact that  $\{\varphi_k\}$  are eigenfunctions of  $-\Delta$  and orthonormal in  $L^2(\Omega)$ , we have

$$\begin{aligned} \int_{\mathcal{C}} \int_{\Omega} |\nabla \mathbf{u}|^2 &= \int_{\mathcal{C}} \int_{\Omega} |\nabla_x \mathbf{u}|^2 + |\partial_y \mathbf{u}|^2 \\ &= \sum_{k=1}^{\infty} b_k^2 \lambda_k^{1/2} e^{-2\lambda_k^{1/2} \mathcal{C}} \\ &\leq e^{-2\lambda_1^{1/2} \mathcal{C}} \|\mathbf{u}\|_{H'}^2. \end{aligned}$$

Finally, using the fact that  $\|\mathbf{u}\|_H = \|f\|_{H'}$ , we obtain the desired estimate (3.1).  $\square$

Then, due to the fast decay of the solution  $\mathbf{u}$ , given a tolerance  $\epsilon$  we may find an adequately large value of the parameter  $\mathcal{C}$  such that, the following estimate holds true:

$$\|\nabla \mathbf{u}\|_{L^2(\Omega \times (\mathcal{C}, \infty))} \leq \epsilon.$$

Hence, it turns out natural to define a truncated solution  $v$  in a bounded domain  $\mathcal{C}_{\mathcal{C}} := \Omega \times (0, \mathcal{C})$ , imposing a zero Dirichlet condition on the top of the bounded cylinder  $\mathcal{C}_{\mathcal{C}}$ . We define  $v$  to be the solution of the following mixed boundary value problem:

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{C}_{\mathcal{C}} = \Omega \times (0, \mathcal{C}), \\ v = 0 & \text{on } \partial_L \mathcal{C}_{\mathcal{C}} := \partial\Omega \times [0, \mathcal{C}], \\ v = 0 & \text{on } \Omega \times \{\mathcal{C}\}, \\ \frac{\partial v}{\partial \nu} = f & \text{on } \Omega \times \{0\}. \end{cases} \quad (3.2)$$

In order to look for a solution  $v$  of problem (3.2), we have to consider a suitable Sobolev space:

$$\mathring{H}_L^1(\mathcal{C}_{\mathcal{C}}) = \{v \in H^1(\mathcal{C}) : v = 0 \text{ on } \partial_L \mathcal{C}_{\mathcal{C}} \cup \Omega \times \{\mathcal{C}\}\}.$$

Then, the weak formulation of problem (3.2) reads: Find  $v \in \mathring{H}_L^1(\mathcal{C}_{\mathcal{C}})$  such that

$$\int_{\mathcal{C}_{\mathcal{C}}} \nabla v \cdot \nabla \phi = \langle f, \text{tr}_{\Omega} \phi \rangle, \quad \text{for all } \phi \in \mathring{H}_L^1(\mathcal{C}_{\mathcal{C}}). \quad (3.3)$$

Existence and uniqueness of the solution  $v$  in  $\mathring{H}_L^1(\mathcal{C}_{\mathcal{C}})$  follows from Lax-Milgram Lemma.

The next result shows that  $\mathbf{u}$ , the solution of the harmonic extension problem (2.14), can be approximated by the solution  $v$  of the truncated problem (3.2), if  $\mathcal{C}$  is chosen sufficiently large.

LEMMA 3.2. *For any positive  $\mathcal{C} > 0$ , the following estimate holds*

$$\|\nabla(\mathbf{u} - v)\|_{L^2(\mathcal{C}_{\mathcal{C}})} \lesssim e^{-\sqrt{\lambda_1} \mathcal{C}/2} \|f\|_{H'}. \quad (3.4)$$

*Proof.* Given  $\phi \in \mathring{H}_L^1(\mathcal{C}_{\mathcal{C}})$ , we can extend it by zero to  $\mathcal{C}$  to get  $\phi_e \in \mathring{H}_L^1(\mathcal{C})$ . Then, taking  $\phi_e$  and  $\phi$  as test functions in (2.15) and (3.3), respectively, and subtracting both expressions, we have

$$\int_{\mathcal{C}_{\mathcal{C}}} \nabla(\mathbf{u} - v) \cdot \nabla \phi = 0 \quad \text{for all } \phi \in \mathring{H}_L^1(\mathcal{C}_{\mathcal{C}}),$$

which means that  $v$  is the best approximation of  $\mathbf{u}$  in the space  $\mathring{H}_L^1(\mathcal{C}_{\mathcal{C}})$ , i.e.,

$$\|\nabla(\mathbf{u} - v)\|_{L^2(\mathcal{C}_{\mathcal{C}})} = \inf_{\phi \in \mathring{H}_L^1(\mathcal{C}_{\mathcal{C}})} \|\nabla(\mathbf{u} - \phi)\|_{L^2(\mathcal{C}_{\mathcal{C}})}. \quad (3.5)$$

Now we construct explicitly a function  $\phi_0$  such that the estimate (3.4) holds true with  $v = \phi_0$ . Let us consider the function  $\rho$  defined by

$$\rho(y) = \begin{cases} 1, & 0 \leq y \leq \mathcal{C}/2 \\ \frac{2}{\mathcal{C}}(\mathcal{C} - y), & \mathcal{C}/2 < y < \mathcal{C} \\ 0, & y \geq \mathcal{C} \end{cases} \quad (3.6)$$

Notice that  $\rho \in W^{1,\infty}(0, \infty)$  and satisfies the following bounds  $|\rho(y)| \leq 1$  and  $|\rho'(y)| \leq 2/\gamma$  for all  $y > 0$ . Using this function, we define  $\phi_0(x, y) := \mathbf{u}(x, y)\rho(y)$  for  $x \in \Omega$  and  $y > 0$ . A straightforward computation shows

$$\begin{aligned} |\nabla((1-\rho)\mathbf{u})|^2 &\leq 2((\rho')^2\mathbf{u}^2 + (1-\rho)^2|\nabla\mathbf{u}|^2) \\ &\leq 2\left(\frac{4}{\gamma^2}\mathbf{u}^2 + |\nabla\mathbf{u}|^2\right). \end{aligned}$$

Using the estimate above and the definition of the function  $\rho$ , we obtain

$$\|\nabla(\mathbf{u} - \phi_0)\|_{L^2(\mathcal{C}_\gamma)}^2 \leq 2\left(\frac{4}{\gamma^2} \int_{\gamma/2}^\gamma \int_\Omega |\mathbf{u}|^2 + \int_{\gamma/2}^\gamma \int_\Omega |\nabla\mathbf{u}|^2\right). \quad (3.7)$$

Now, we have to estimate both integrals appearing on the right hand side of inequality (3.7). In fact,

$$\begin{aligned} \int_{\gamma/2}^\gamma \int_\Omega |\mathbf{u}|^2 &= \sum_{k=1}^\infty b_k^2 \int_{\gamma/2}^\gamma e^{-2\lambda_k^{1/2}y} \\ &\leq \frac{1}{2}\lambda_1^{-1}e^{-\lambda_1^{1/2}\gamma} \sum_{k=1}^\infty b_k^2 \lambda_k^{1/2} (1 - e^{-\lambda_k^{1/2}\gamma}) \\ &\lesssim e^{-\lambda_1^{1/2}\gamma} \|f\|_{H'}^2. \end{aligned} \quad (3.8)$$

The second integral in (3.7) is estimated as follows:

$$\begin{aligned} \int_{\gamma/2}^\gamma \int_\Omega |\nabla\mathbf{u}|^2 &= \sum_{k=1}^\infty b_k^2 \lambda_k \int_{\gamma/2}^\gamma e^{-2\lambda_k^{1/2}y} \\ &\leq \frac{1}{2}e^{-\lambda_1^{1/2}\gamma} \sum_{k=1}^\infty b_k^2 \lambda_k^{1/2} (1 - e^{-\lambda_k^{1/2}\gamma}) \\ &\lesssim e^{-\lambda_1^{1/2}\gamma} \|f\|_{H'}. \end{aligned} \quad (3.9)$$

Replacing the estimates (3.8) and (3.9) into (3.7), we get

$$\|\nabla(\mathbf{u} - \phi_0)\|_{L^2(\mathcal{C}_\gamma)}^2 \lesssim e^{-\lambda_1^{1/2}\gamma} \|f\|_{H'}^2.$$

Finally, using the best approximation property (3.5), we obtain the desired estimate

$$\begin{aligned} \|\nabla(\mathbf{u} - v)\|_{L^2(\mathcal{C}_\gamma)}^2 &= \inf_{\phi \in \hat{H}_L^1(\mathcal{C}_\gamma)} \|\nabla(\mathbf{u} - \phi)\|_{L^2(\mathcal{C}_\gamma)}^2 \\ &\leq \|\nabla(\mathbf{u} - \phi_0)\|_{L^2(\mathcal{C}_\gamma)}^2 \\ &\lesssim e^{-\lambda_1^{1/2}\gamma} \|f\|_{H'}^2. \quad \square \end{aligned}$$

Notice that we have introduced a truncation parameter  $\gamma$  which determines the size of the truncated domain  $\mathcal{C}_\gamma = \Omega \times (0, \gamma)$ . Lemma 3.2 shows that the solution of the problem on the truncated domain (3.2) converges exponentially to that of the original problem (2.14) in the domain of interest.

To conclude this section, we present and estimate for the difference  $\nabla(\mathbf{u} - v)$ , in the whole domain  $\mathcal{C}$ , i.e., in the  $L^2(\mathcal{C})$ -norm. The key ingredients are Proposition 3.1 and Lemma 3.2.

**THEOREM 3.3.** *For any positive  $\gamma > 0$ , the following estimate holds*

$$\|\nabla(\mathbf{u} - v)\|_{L^2(\mathcal{C})} \lesssim e^{-\sqrt{\lambda_1}\gamma/2} \|f\|_{H'}. \quad (3.10)$$

*In particular, given a tolerance  $\epsilon > 0$ , there exists a positive number  $\gamma_0$  such that for any  $\gamma > \gamma_0$ , the following estimate holds*

$$\|\nabla(\mathbf{u} - v)\|_{L^2(\mathcal{C})} \leq \epsilon \|f\|_{H'}. \quad (3.11)$$

Proof. Notice that the function  $v$  can be extended by zero to  $\bar{\Omega} \times [\gamma, \infty)$ . Then, we have

$$\|\nabla(\mathbf{u} - v)\|_{L^2(\mathcal{C})}^2 = \|\nabla(\mathbf{u} - v)\|_{L^2(\mathcal{C})}^2 + \|\nabla \mathbf{u}\|_{L^2(\Omega \times (\gamma, \infty))}^2$$

Using Lemma 3.2 and Proposition 3.1, we obtain

$$\begin{aligned} \|\nabla(\mathbf{u} - v)\|_{L^2(\mathcal{C})}^2 &\leq C e^{-\lambda_1^{1/2} \gamma} \|f\|_H^2 \\ &\leq \epsilon^2 \|f\|_{H'}^2, \end{aligned} \quad (3.12)$$

for all  $\gamma > \gamma_0$ , where

$$\gamma_0 = \frac{1}{\sqrt{\lambda_1}} \left( \ln(C) + 2 \ln \left( \frac{1}{\epsilon} \right) \right),$$

with  $C$  denoting the constant in inequality (3.12).  $\square$

**4. A Priori Error Estimates.** Let  $\mathcal{C}_\gamma$  be a polyhedral domain in  $\mathbb{R}_+^{n+1}$ . A triangulation  $\mathcal{C}_\gamma$  (also called mesh or grid) of  $\mathcal{C}_\gamma$  is a partition of  $\bar{\mathcal{C}}_\gamma$  into a set of  $n+1$ -simplexes.

We impose two conditions on a triangulation  $\mathcal{T}$  which are important in finite element construction. First, a triangulation  $\mathcal{T}$  is called conforming or compatible if the intersection of any two simplexes  $T$  and  $T'$  in  $\mathcal{T}$  is either empty or a common lower dimensional simplex.

The second important condition is shape regularity. A set of triangulations  $\mathbb{T}$  is called shape regular if there exists a constant  $\sigma$  such that

$$\max_{T \in \mathcal{T}} \frac{\text{diam}(T)^{n+1}}{|T|} \leq \sigma, \quad \text{for all } \mathcal{T} \in \mathbb{T},$$

where  $\text{diam}(T)$  is the diameter of  $T$  and  $|T|$  is the measure of  $T$  in  $\mathbb{R}^{n+1}$ . For shape regular triangulations,  $\text{diam}(T) \approx h_T := |T|^{1/(n+1)}$  which will be used to represent the size of  $T$  and we define  $h_{\mathcal{T}} := \max_{T \in \mathcal{T}} h_T$ .

The standard finite element method is to approximate problem (3.3) within a piecewise polynomial finite dimensional subspace. For simplicity we consider the piecewise linear finite element space  $\mathbb{V}(\mathcal{T})$  over a shape-regular triangulation  $\mathcal{T}$  of  $\mathcal{C}_\gamma$ :

$$\mathbb{V}(\mathcal{T}) := \{V \in \mathcal{C}^0(\bar{\mathcal{C}}_\gamma) : V|_T \in \mathbb{P}_1(T) \forall T \in \mathcal{T} \text{ and } V|_{\Gamma_D} = 0\} \subset \mathring{H}_L^1(\mathcal{C}). \quad (4.1)$$

where  $\Gamma_D$  denotes the Dirichlet boundary given by the union of  $\partial_L \mathcal{C}_\gamma$  and  $\Omega \times \{\gamma\}$ . We now solve (3.3) in the finite element space  $\mathbb{V}(\mathcal{T})$ : find  $V_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$  such that

$$\int_{\mathcal{C}_\gamma} \nabla V_{\mathcal{T}} \cdot \nabla W = \langle f, \text{tr}_\Omega W \rangle, \quad (4.2)$$

for all  $W \in \mathbb{V}(\mathcal{T})$ . The existence and uniqueness of the solution to (4.2) follows again from Lax-Milgram Lemma since  $\mathbb{V}(\mathcal{T})$  is Hilbert.

The results of section 2.2 show that for  $f \in H'$ , if  $\mathbf{u}$  solves the harmonic extension problem (2.14), then  $u = \text{tr}_\Omega \mathbf{u}$  solves problem (1.1). Using this result, we define a finite element approximation of  $u$ . First, we define  $\mathbb{U}(\mathcal{T}) := \text{tr}_\Omega \mathbb{V}(\mathcal{T})$ , which corresponds to a space of piecewise polynomials of degree 1 over the mesh  $\mathcal{T}_\Omega := \text{tr}_\Omega \mathcal{T}$ . Now, we can define a finite element approximation of  $u \in H$ , by  $U_{\mathcal{T}} := \text{tr}_\Omega V_{\mathcal{T}} \in \mathbb{U}(\mathcal{T})$ .

We are now in position to derive a global error estimate. To this end, it is convenient to introduce the mesh-size function  $h \in L^\infty(\mathcal{C}_\gamma)$  given by  $h|_T = h_T$  for all  $T \in \mathcal{T}$ . Then, a combination of the Céa Lemma (see [7, Theorem 2.8.1]), interpolation estimates (see [7, Ch. 4]) and Theorem 3.3 yields the following error estimate.

**THEOREM 4.1.** *Assume that the exact solution  $\mathbf{u}$  of (1.1) satisfies  $v \in H^s(\mathcal{C}_\gamma)$  with  $1 \leq s \leq 2$ , and set  $r := s - 1$ . Then, the error of the finite element solution  $U_{\mathcal{T}} \in \mathbb{U}(\mathcal{T})$  of (4.2) satisfies the following global a priori upper bound*

$$\|u - U_{\mathcal{T}}\|_H \leq \epsilon \|f\|_{H'} + \|h^r D^s v\|_{L^2(\mathcal{C}_\gamma)}. \quad (4.3)$$

*Proof.* We start from the trace estimate (2.17) applied to  $u - U_{\mathcal{T}}$  and the triangle inequality:

$$\begin{aligned} \|u - U_{\mathcal{T}}\|_H &\lesssim \|u - V_{\mathcal{T}}\|_{\dot{H}_L^1(\mathcal{C})} \\ &\leq \|u - v\|_{\dot{H}_L^1(\mathcal{C})} + \|v - V_{\mathcal{T}}\|_{\dot{H}_L^1(\mathcal{C}_{\mathcal{Y}})}. \end{aligned} \quad (4.4)$$

Now, given a tolerance  $\epsilon$ , we know there exists a positive number  $\mathcal{Y}_0$  such that the estimate (3.11) holds, which reads

$$\|u - v\|_{\dot{H}_L^1(\mathcal{C})} \lesssim \epsilon \|f\|_{H'} \quad (4.5)$$

Standard finite element approximation (see [7, Ch. 4]) provides the following estimate

$$\|v - V_{\mathcal{T}}\|_{\dot{H}_L^1(\mathcal{C}_{\mathcal{Y}})} \lesssim \|h^r D^s v\|_{L^2(\Omega)} \quad (4.6)$$

Finally, replacing the estimates (4.5) and (4.6) in (4.4), we obtained the desired a priori upper bound (4.3).  $\square$

If  $s = 2$ , and so  $v$  has the maximal regularity  $v \in H^2(\mathcal{C}_{\mathcal{Y}})$ , we obtain the optimal convergence rate in a linear Sobolev scale

$$\|\nabla(v - V_{\mathcal{T}})\|_{L^2(\mathcal{C}_{\mathcal{Y}})} \lesssim h_{\mathcal{T}} \|v\|_{H^2(\mathcal{C}_{\mathcal{Y}})}. \quad (4.7)$$

The order 1 is dictated by the polynomial degree 1 and cannot be improved upon assuming either higher regularity  $H^2(\mathcal{C}_{\mathcal{Y}})$  or a graded mesh  $\mathcal{T}$ . Notice that, in this case, (4.7) does not provide an optimal estimate for the approximation of the solution of (1.1). In fact, the estimate obtained reads

$$\|u - U_{\mathcal{T}}\|_H \lesssim \epsilon \|f\|_{H'} + h_{\mathcal{T}} \|v\|_{H^2(\mathcal{C}_{\mathcal{Y}})}$$

If  $u$  has the maximal regularity  $u \in H^2(\Omega)$ , the optimal convergence rate in a linear Sobolev scale is

$$\|u - U_{\mathcal{T}}\|_H \lesssim h_{\mathcal{T}}^{3/2} \|u\|_{H^2(\Omega)}. \quad (4.8)$$

However, to have such an estimate, we need  $f \in H_0^1(\Omega)$  (see [8, Proposition 3.1]). Then, if we assume just  $f \in H$  the estimate for  $u$  would be

$$\|u - U_{\mathcal{T}}\|_H \lesssim h_{\mathcal{T}} \|f\|_H, \quad (4.9)$$

which is recovered by Theorem 4.1.

In order to have the optimal estimate (4.7), we need the function  $v \in H^2(\mathcal{C}_{\mathcal{Y}})$ . It is well known that in general the solution of a mixed boundary value problem is not smooth, even for  $C^\infty$ -data. This singular behavior occurs *near* the points of interseccion between the Dirichlet and Neumann boundary. For instance, let us consider an example. The solution of the following problem:  $\Delta w(x, y) = 0$  in  $\{y > 0\}$ ,  $w_y = 0$  for  $\{x \leq 0\}$  and  $\{y = 0\}$  and  $w = r^{1/2} \sin(\theta/2)$  otherwise, does not belong to  $H^2$ . In order to recover more regular solutions, one has to impose some compatibility conditions between the data, the operator and the boundary.

For a mixed boundary value problem with an elliptic second order operator, optimal regularity results under weak assumptions on the data are obtained in [20]. For instance, for  $\theta \in (-1/2, 1/2)$ ,  $f \in H^{-1/2+\theta}$  implies  $v \in H^{1+\theta}$ . In our particular problem, given that  $\mathcal{C}_{\mathcal{Y}} = \Omega \times (0, \mathcal{Y})$ , using the spectral theory, it is possible to prove that if  $f \in H$  then  $v \in H^2(\mathcal{C}_{\mathcal{Y}})$ .

**5. A numerical example.** In the following numerical example we choose a function  $f$  such that the solution of (1.1) does not have  $H^2$ -regularity. In fact, we consider  $\Omega = (0, 1)$  and a function  $f$  such that  $f(0) = f(1) = 0$  and the exact solution of problem (1.1) is given by

$$u(x) = \begin{cases} 2x, & x \in (0, 1/2), \\ 2(1-x), & x \in (1/2, 1). \end{cases}$$

This function is such that  $u \in H^s(\Omega)$  for every  $s < 3/2$ , then we have the estimate

$$\|u - U_{\mathcal{T}}\|_H \lesssim \epsilon \|f\|_H + h_{\mathcal{T}}^{s-1/2} \|v\|_{H^2(\mathcal{C}_{\mathcal{Y}})}, \quad \epsilon = \epsilon(\mathcal{Y}). \quad (5.1)$$

The implementation of this numerical experiment has been carried out with the help of the `deal.II` library (see [3, 4]). We implemented the truncated problem (3.3) over the domain  $(0, 1) \times (0, \gamma)$ , where  $\gamma$  denotes the truncation parameter defined in section 3. For this type of elliptic problems, it is well known (see for instance [7, 3]) that the discretization via the finite element method of problem (3.3) is reduced to solve a linear system of equations. In our case the obtained linear system is solved using the conjugate gradient method preconditioned with SSOR. The stopping criterion is as follows: stop after 1000 iterations and stop if the norm of the residual is below  $10^{-12}$ .

In order to compare the theoretical and experimental orders of convergence (EOC), we consider a sequence of 10 meshes  $\{\mathcal{T}_k\}_{k=1}^{10}$ , and over each mesh  $\mathcal{T}_k$  we choose the parameter  $\gamma$  such that  $\epsilon \approx h_{\mathcal{T}}$ . In this way, the estimate (5.1) becomes

$$\|u - U_{\mathcal{T}}\|_H \lesssim h_{\mathcal{T}}^{s-1/2} \|v\|_{H^2(\mathcal{C}_\gamma)}. \quad (5.2)$$

In figure 5.1 we show the numerical approximation  $V_{\mathcal{T}}$  obtained in a uniform mesh with  $\#\mathcal{T} = 1024$  degrees of freedom, and the EOC for the  $H^{1/2}$ -norm. Notice that this estimate is the one predicted by Theorem 4.1.

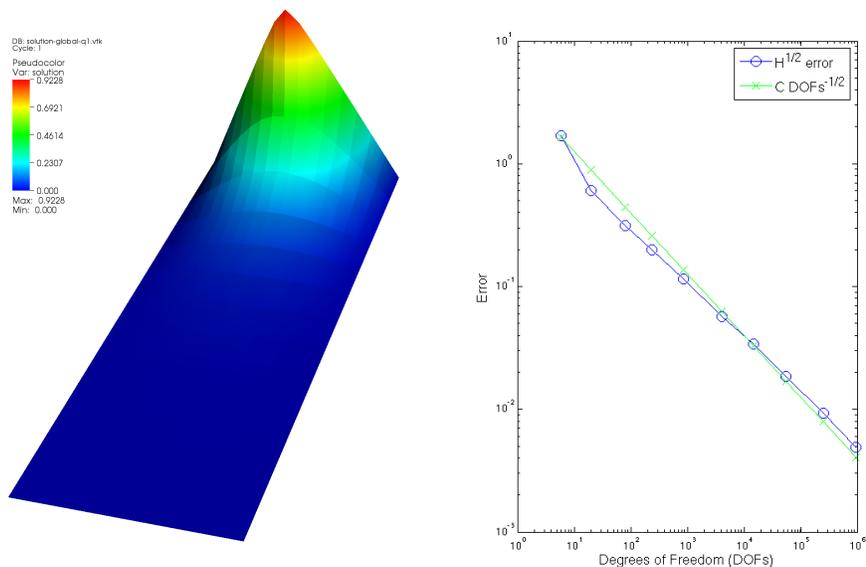


Fig. 5.1: Numerical approximation  $V_{\mathcal{T}}$  with  $\#\mathcal{T} = 1024$  and EOC for the  $H^{1/2}$ -norm.

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