A FEM for the Square Root of the Laplace Operator

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Outline of Topics

The continuos problem

The problem we shall be concerned with reads as follows: Given a smooth enough function f, find u such that

$$\begin{cases} (-\Delta)^{1/2} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$, with $d = 1, 2$ is a bounded domain with a smooth boundary $\partial\Omega$ and $(-\Delta)^{1/2}$ denotes the square root of the Laplace operator $-\Delta$ in Ω with zero boundary values on $\partial\Omega$.

Applications

Concerning applications, nonlocal operators are of importance in a wide range of applications:

- Finance.
- Image Processing.
- Quasi-geostrophic flow models.
- Modeling hydraulic fractures and the evolution of a viscous liquid thin film.

The development of efficient computational solution techniques for this problem is fundamental.

Definition of the square root of the Laplacian

Spectral theory of the Laplacian $-\Delta$ in a smooth bounded domain Ω with zero Dirichlet boundary values. There exists a sequence of eigenvalues

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and, there exists an orthonormal basis $\{\varphi_k\}$ of $L^2(\Omega)$, where $\varphi_k \in H_0^1(\Omega)$ is an eigenfunction corresponding to λ_k:

$$\left\{ \begin{array}{rcl} -\Delta\varphi_k & = & \lambda_k\varphi_k \quad \text{in } \Omega \\ \varphi_k & = & 0 \quad \text{on } \partial\Omega, \end{array} \right. \tag{1}$$

for $k = 1, 2, \cdots$. Regularity theory $\implies$ $\varphi_k \in C^\infty(\bar{\Omega})$ for $k = 1, 2, \cdots$.

Definition of the square root of the Laplacian

The square root of the Dirichlet Laplacian, for a smooth function u, is given by

$$u = \sum_{k=1}^{\infty} c_k \varphi_k \mapsto (-\Delta)^{1/2} u = \sum_{k=1}^{\infty} c_k \lambda_k^{1/2} \varphi_k .$$

Density results $\implies$ $(-\Delta)^{1/2} : H_0^1(\Omega) \to L^2(\Omega)$,

$$H_0^1(\Omega) = \{ u = \sum_{k=1}^{\infty} c_k \varphi_k | \sum_{k=1}^{\infty} \lambda_k c_k^2 < \infty \}.$$

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Then if $f \in L^2(\Omega)$, we have

$$f = \sum_{k=1}^{\infty} f_k \varphi_k \implies c_k = f_k \lambda_k^{-1/2}$$

Numerical disadvantages: We need to find a sufficiently large number of eigenfunctions to obtain an accurate approximation.

Definition of the square root of the Laplacian

On the other hand, this operator can be seen as a singular integral

$$(-\Delta)^{1/2}u(x) = C_d \int_\Omega \frac{u(x)-u(y)}{|x-y|^{d+1}}\,dy,$$

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where C_d is some normalization constant.

Numerical disadvantages: the integrand is singular and the matrix obtained is dense. These inconveniences complicate the numerical computation.

Dirichlet - Neumann Operator

Let u be a bounded and continuos function in $\mathbb{R}^n$. Then, there exists a unique harmonic extension v of u in $\mathbb{R}^{n+1}_+$:

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ v = u & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases}$$

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Consider $T : u \mapsto -\partial_y v(\cdot, 0)$. Then,

$$\begin{aligned} (T \circ T)(u) = T(-\partial_y v(\cdot, 0)) = \partial_{yy} v|_{y=0} &= -\Delta_x v|_{y=0} \\ &= -\Delta u \ \text{ in } \mathbb{R}^n. \end{aligned}$$

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T is positive and self- adjoint, then $T = (-\Delta)^{1/2}$.

Dirichlet - Neumann Operator

The same idea holds in the cylinder $\mathcal{C} := \Omega \times (0, \infty)$.

$$\begin{cases} -\Delta v = 0 & \text{in } \mathcal{C} = \Omega \times (0,\infty), \\ v = 0 & \text{on } \partial_L \mathcal{C} := \partial\Omega \times [0,\infty), \\ v = u & \text{on } \Omega \times \{0\}. \end{cases}$$

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$$Tu = (-\Delta)^{1/2} u = -\partial_y v(\cdot, 0) = \left.\frac{\partial v}{\partial \nu}\right|_{\Omega \times \{0\}}.$$

Harmonic extension

Approach presented in the paper by X. Cabré and J. Tan, (2010): relation between the nonlocal operator $(-\Delta)^{1/2}$ and the harmonic extension.

Given u defined in Ω, we consider its harmonic extension v in the cylinder $\mathcal{C} := \Omega \times (0, \infty)$, with v vanishing on $\partial_L \mathcal{C} := \partial\Omega \times [0, \infty)$.

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where ν is the unit outer normal to $\mathcal{C}$ at $\Omega \times \{0\}$. Then,

$$u = \mathrm{tr}_\Omega v := v(\cdot, 0).$$

Spaces for v and u

Space for v:

$$H_0^1(\mathcal{C}) := \{v \in H^1(\mathcal{C}) | v = 0 \text{ a.e. on } \partial_L \mathcal{C} = \partial\Omega \times [0, \infty)\}.$$

Space for u:

$$\begin{aligned} \mathcal{V}_0(\Omega) = H_{00}^{1/2}(\Omega) &= \left[H_0^1(\Omega), L^2(\Omega)\right]_{1/2,2} \\ &= \left\{ u \in H^{1/2}(\Omega) \middle| \int_\Omega \frac{u^2(x)}{d(x)} dx < +\infty \right\}, \end{aligned}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$.

Proposition (Cabré - Tan, 2010)

Let $\mathcal{V}_0(\Omega)$ be the space of all traces on $\Omega \times \{0\}$ of functions in $H_0^1(\mathcal{C})$. Then, we have

$$
\begin{aligned}
\mathcal{V}_0(\Omega) &:= \{u = \mathrm{tr}_\Omega\, v \mid v \in H_0^1(\mathcal{C})\} \\
&= \left\{ u \in H^{1/2}(\Omega) \middle| \int_\Omega \frac{u^2(x)}{d(x)} < \infty \right\} \\
&= \left\{ u \in L^2(\Omega) \middle| u = \sum_{k=1}^{\infty} c_k \varphi_k \text{ s.t } \sum_{k=1}^{\infty} c_k^2 \lambda_k^{1/2} < \infty \right\},
\end{aligned}
$$

Moreover,

$$
\|u\|_{\mathcal{V}_0(\Omega)}^2 = \|u\|_{H^{1/2}(\Omega)}^2 + \int_\Omega \frac{u^2}{d}.
$$

Proposition (Cabré - Tan, 2010)

If $u \in \mathcal{V}_0(\Omega)$, then there exists a unique extension v of u s.t. $v \in H_0^1(\mathcal{C})$. In particular, if $u = \sum c_k \varphi_k$, then

$$v(x,y) = \sum_{k=1}^{\infty} c_k \varphi_k(x) e^{-\sqrt{\lambda_k} y}, \quad \forall (x,y) \in \mathcal{C}.$$

The operator $(-\Delta)^{1/2} : \mathcal{V}_0(\Omega) \mapsto \mathcal{V}_0(\Omega)^*$ is given by

$$(-\Delta)^{1/2} u = \left. \frac{\partial v}{\partial \nu} \right|_{\Omega \times \{0\}}.$$

Truncated problem

Numerically, it cannot be solved because $\mathcal{C}$ is an infinite domain $\implies$ We need to consider a suitable truncated problem.

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Why can we truncate the problem?

Lemma

For any $M > 0$, the harmonic extension v satisfies the following estimate

$$\|\nabla v\|_{L^2(\Omega\times(M,\infty))} < e^{-\sqrt{\lambda_1}M}\|f\|_{\mathcal{V}_0(\Omega)^*}.$$

In fact,

$$M > \frac{\ln\|f\|_{\mathcal{V}_0(\Omega)^*} + \ln(1/\epsilon)}{\sqrt{\lambda_1}} \Longrightarrow \|v\|^2_{L^2(\Omega\times\{M\})} \leq C\epsilon.$$

Truncated problem

Consider M adequately large and define v^M in a bounded domain $\mathcal{C}_M := \Omega \times (0, M)$, imposing a zero Dirichlet condition on $\Omega \times \{M\}$:

$$\begin{cases} -\Delta v^M & = & 0 & \text{in } \mathcal{C}_M = \Omega \times (0, M), \\ v^M & = & 0 & \text{on } \partial_L \mathcal{C}_M := \partial\Omega \times [0, M], \\ v^M & = & 0 & \text{on } \Omega \times \{M\}, \\ \dfrac{\partial v^M}{\partial \nu} & = & f & \text{on } \Omega \times \{0\}. \end{cases}$$

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Find $v^M \in H_0^1(\mathcal{C}_M)$ such that

$$\int_{\mathcal{C}_M} \nabla v^M \cdot \nabla \psi = \langle f, \mathrm{tr}_\Omega \psi \rangle_{\mathcal{V}_0(\Omega)^*, \mathcal{V}_0(\Omega)}, \quad \text{for all } \psi \in H_0^1(\mathcal{C}_M).$$

$$H_0^1(\mathcal{C}_M) := \{ v \in H^1(\mathcal{C}_M) | v = 0 \text{ a.e. on } \partial_L \mathcal{C}_M, \\ \text{and } v = 0 \text{ a.e. on } \Omega \times \{M\} \}.$$

Weak formulation of the truncated problem

How good is this truncated problem?

Weak formulation of the truncated problem

How good is this truncated problem? We need two key steps. First, we have an orthogonality property,

$$\int_{\mathcal{C}_M} (\nabla v - \nabla v^M) \cdot \nabla \psi = 0, \quad \forall \psi \in H_0^1(\mathcal{C}_M),$$

which implies,

$$\|v - v^M\|_{H_0^1(\mathcal{C}_M)} = \inf_{\psi \in H_0^1(\mathcal{C}_M)} \|v - \psi\|_{H_0^1(\mathcal{C}_M)}.$$

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Then, we construct a function ψ_0 s.t.

$$\|v - \psi_0\|_{H_0^1(\mathcal{C}_M)} \le C(M, \lambda_1) e^{-\sqrt{\lambda_1} M} \|f\|_{\mathcal{V}_0(\Omega)^*}.$$

Weak formulation of the truncated problem

Then, we have the following result.

Lemma

For any $\epsilon > 0$, there exists a positive number M_0 s.t. for any $M > M_0$ the following estimate holds.

$$\|v - v^M\|_{H_0^1(\mathcal{C}_M)} \leq \epsilon \|f\|_{\mathcal{V}_0(\Omega)^*},$$

where

$$M_0 = \max\left(\sqrt{\frac{2}{\lambda_1}}, \frac{1}{\sqrt{\lambda_1}}\left(\ln(3) + \ln\left(\frac{1}{\epsilon}\right)\right)\right).$$

Galerkin approximation

Given a family of partitions $\mathcal{T}_k$ of the domain $\mathcal{C}_M$ into quadrilateral elements, we define for $n \geq 1$

$$\mathbb{V}^{n,0} := \{v \in C^0(\overline{\mathcal{C}_M}) : v|_T \in \mathcal{Q}_n(T)\ \forall T \in \mathcal{T}_k\} \cap H_0^1(\mathcal{C}_M).$$

Galerkin approximation for v^M: Find $v_h^M \in \mathbb{V}^{n,0}$ such that

$$\int_{\mathcal{C}_M} \nabla v_h^M \cdot \nabla w_h = \langle f, \mathrm{tr}_\Omega w_h \rangle_{\mathcal{V}_0(\Omega)^*, \mathcal{V}_0(\Omega)}, \qquad \text{for all } w_h \in \mathbb{V}^{n,0}.$$

Error estimates

We need to estimate the difference $v - v_h^M$ in the $H_0^1(\mathcal{C}) - norm$.

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\begin{aligned}
\|v - v_h^M\|_{H_0^1(\mathcal{C})} &\leq \|v - v^M\|_{H_0^1(\mathcal{C})} + \|v^M - v_h^M\|_{H_0^1(\mathcal{C}_M)} \\
&\leq \|v\|_{H_0^1(\mathcal{C} \setminus \mathcal{C}_M)} + \|v - v^M\|_{H_0^1(\mathcal{C}_M)} \\
&\quad + \|v^M - v_h^M\|_{H_0^1(\mathcal{C}_M)}.
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Lemma

For any $\epsilon > 0$, there exists a positive number M_0 s.t. for any $M > M_0$ the following estimate holds.

$$\|v - v_h^M\|_{H_0^1(\mathcal{C}_M)} \leq C \left(\epsilon \|f\|_{\mathcal{V}_0(\Omega)^*} + h\|v^M\|_{H^2(\mathcal{C}_M)}\right),$$

where $h = \max_{T\in\mathcal{T}} h_T$.

Error estimates

What about u?

Error estimates

What about u? Trace result implies an estimate for u

$$
\begin{aligned}
\|u - u_h^M\|_{H_{00}^{1/2}(\Omega)} &\leq \|v - v_h^M\|_{H_0^1(\mathcal{C})} \\
&\leq C\left(\epsilon\|f\|_{\mathcal{V}_0(\Omega)^*} + h\|f\|_{H^{1/2}(\Omega)}\right), \quad \epsilon = \epsilon(M).
\end{aligned}
$$

However, notice that this estimate is not optimal! Optimal estimate

$$
\|u - u_h^M\|_{H_{00}^{1/2}(\Omega)} \leq C\left(\epsilon\|f\|_{\mathcal{V}_0(\Omega)^*} + h^{3/2}\|f\|_{H^1(\Omega)}\right), \quad \epsilon = \epsilon(M).
$$

This optimal estimate needs $f \in H^1(\Omega)$.

Numerical example

We consider $\Omega = (0,1)$ and $f(x) = \pi sin(\pi x)$, then $\mathcal{C}_M = (0,1) \times (0,M)$
$u(x) = sin(\pi x)$ and $v(x,y) = sin(\pi x)e^{-\pi y}$.

$$\|v - v_h^M\|_{H_0^1(\mathcal{C})} \leq C\left(\epsilon\|f\|_{\mathcal{V}_0(\Omega)^*} + h\|v\|_{H^2(\mathcal{C}_M)}\right), \quad \epsilon = \epsilon(M)$$

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$$M = -\frac{2}{\pi} \ln\left(\frac{h}{\sqrt{2}}\right).$$

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In this case,

$$\|v - v_h^M\|_{H_0^1(\mathcal{C})} \leq Ch\|f\|_{\mathcal{V}_0(\Omega)}.$$

Some global meshes

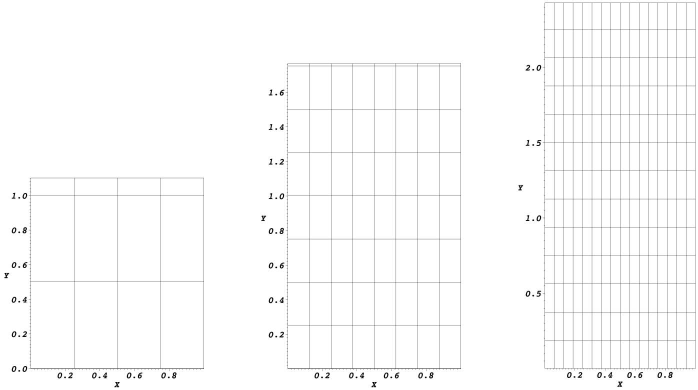
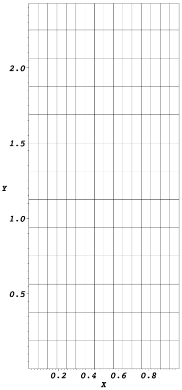


Figure: Degrees of freedom: 20, 81, 238 respectively.

Results with global refinement



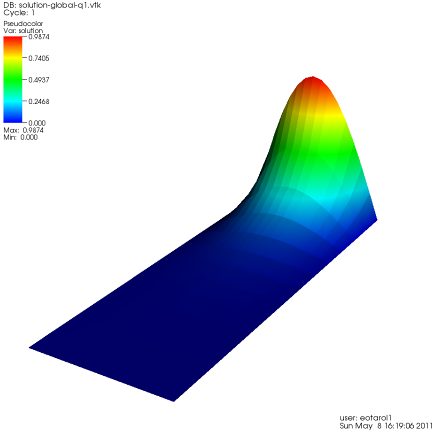


Figure: v_h^M with 238 degrees of freedom.

Results with global refinement

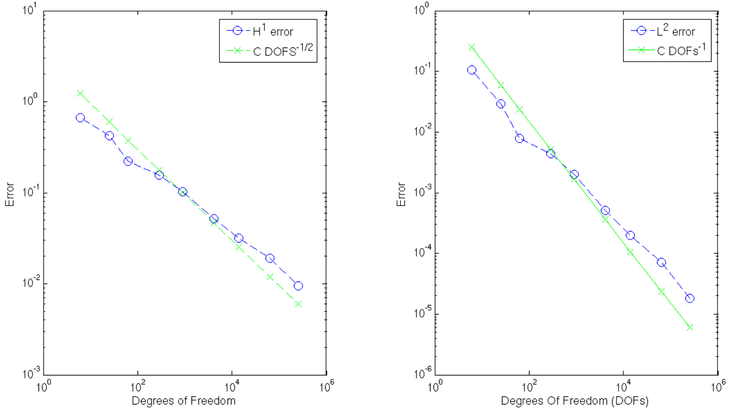


Figure: Decay of the L^2 and H^1 norms of the error for v.

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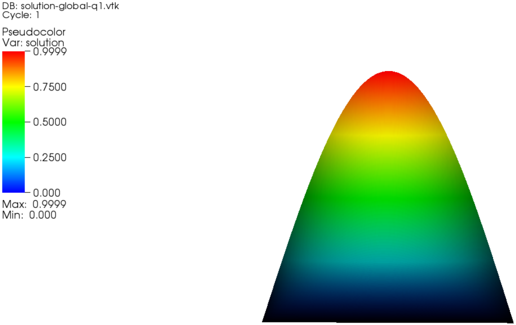

Figure: u_h^M with 141075 degrees of freedom.

Results with global refinement

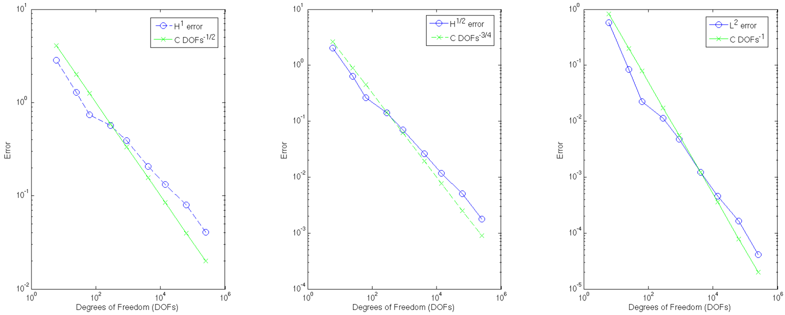

Figure: Decay of the L^2, $H^{1/2}$ and H^1 norms of the error for u.

Exponential refinement

We exploit the behavior of the real solution

$$v(x,y) = \sum c_k \varphi_k e^{-\sqrt{\lambda_k} y}, \quad \text{for all } (x,y) \in \mathcal{C},$$

to design an exponential mesh.

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$d = 2$: We do global refinement in x and exponential refinement in y. Using interpolation results we get

$$\begin{aligned}
\|v - v_h^M\|^2_{H_0^1(\mathcal{C}_M)} &\leq C \sum_{k=1}^{N_x} \sum_{l=1}^{N_y} \left((h_k^x)^2 + (h_l^y)^2 \right) |v|^2_{H^2(R_{kl})} \\
&\leq C \sum_{l=1}^{N_y} (h_l^y)^2 \, |v|^2_{H^2(\mathcal{C}_l)} \\
&\leq C \sum_{l=1}^{N_y} (h_l^y)^2 \, h_l^y \, e^{-\sqrt{\lambda_1} y_l}.
\end{aligned}$$

Exponential refinement

Finally, imposing

$$\|v - v_h^M\|^2_{H_0^1(\mathcal{C}_M)} \leq CN^{-1},$$

we get the following formula for the mesh on y:

$$y_{k+1} = y_k + \frac{1}{k} N^{-2/3} e^{\sqrt{\lambda_1}/3 y_k}.$$

Some exponential meshes

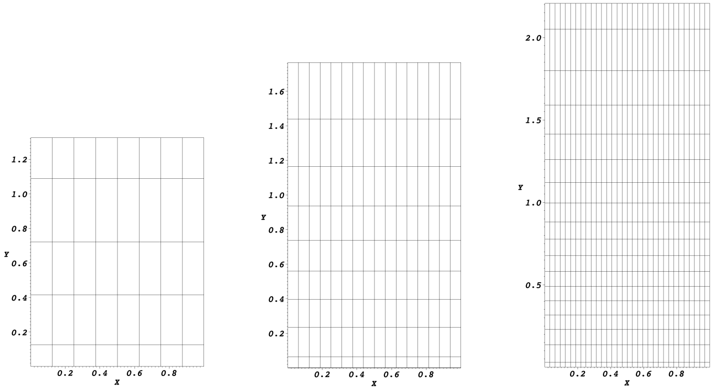

Figure: Degrees of freedom: 54, 170, 627 respectively.

Results with exponential refinement

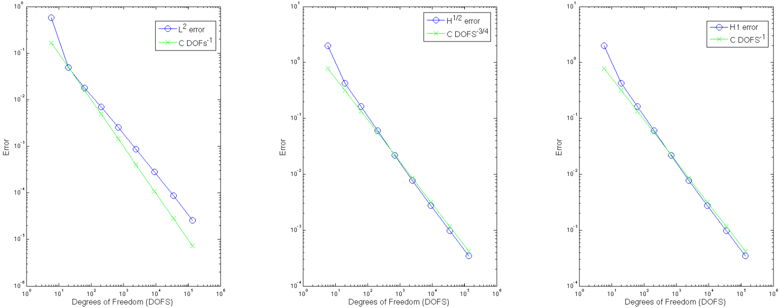

Figure: Decay of the L^2, $H^{1/2}$ and H^1 norms of the error for u.

Adaptive refinement

The estimate

$$\|v - v_h^M\|_{H_0^1(\mathcal{C})} \leq C\left(\epsilon\|f\|_{\mathcal{V}_0(\Omega)^*} + h\|v\|_{H^2(\mathcal{C}_M)}\right),$$

is not computable and provides only asysmptotic information. We create a mesh adapted to the function v. Basic ingredient:

$$\|v - v_h^M\|_{H_0^1(\mathcal{C})} \leq C_1\mathcal{E}_{\mathcal{T}}(v_h^M) \leq C_2\left(\|v - v_h^M\|_{H_0^1(\mathcal{C})} + \mathrm{osc}_{\mathcal{T}}(v_h^M)\right)$$

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$$\|v - v_h^M\|_{H_0^1(\mathcal{C})} \leq C \left(\epsilon \|f\|_{\mathcal{V}_0(\Omega)^*} + h\|v\|_{H^2(\mathcal{C}_M)} \right),$$

is not computable and provides only asysmptotic information. We create a mesh adapted to the function v. Basic ingredient:

$$\|v - v_h^M\|_{H_0^1(\mathcal{C})} \leq C_1 \mathcal{E}_{\mathcal{T}}(v_h^M) \leq C_2 \left(\|v - v_h^M\|_{H_0^1(\mathcal{C})} + \mathrm{osc}_{\mathcal{T}}(v_h^M) \right)$$

Error estimator implemented:

$$\mathcal{E}_{\mathcal{T}}^2(v_h^M, T) = \frac{h_T}{24} \int_{\partial T} \left[\frac{\partial v_h^M}{\partial \nu} \right]$$

3D Numerical example

We consider $\Omega = (0,1)\times(0,1)$ and $f(x) = \sqrt{2}\pi\sin(\pi x)\sin(\pi y)$, then $u(x) = \sin(\pi x)\sin(\pi y)$ and $v(x,y) = sin(\pi x) sin(\pi y) e^{-\sqrt{2}\pi y}$.

We have optimal estimate for every refinement: adaptive, exponential and global. We show the results obtained using Adaptivity.

An adaptive mesh. $M = 4$

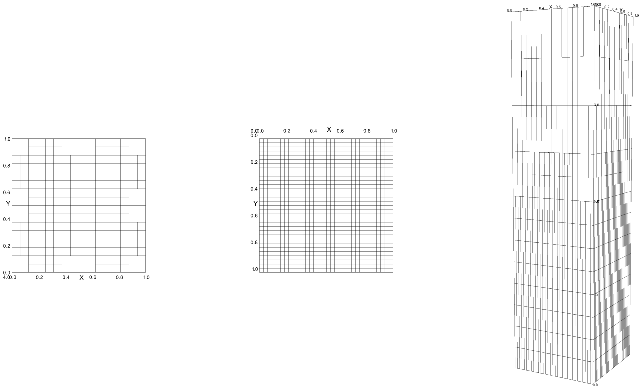


Figure: Degrees of freedom: 28314.

Convergence Table for v

n	cells	H^1-error		L^2-error	
0	4	4.016e-01	-	4.790e-02	-
1	32	6.419e-01	-0.68	7.156e-02	-0.58
2	228	6.252e-01	0.04	6.094e-02	0.23
3	1628	4.190e-01	0.58	2.845e-02	1.10
4	11400	2.312e-01	0.86	8.959e-03	1.67
5	79265	1.188e-01	0.96	2.394e-03	1.90
6	549238	5.983e-02	0.99	6.091e-04	1.97

Results with adaptive refinement

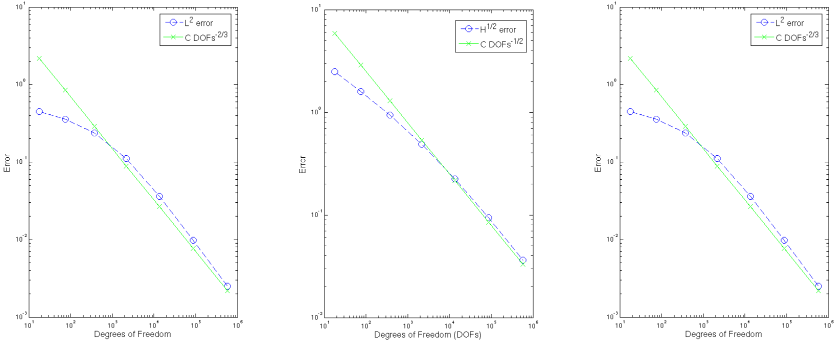
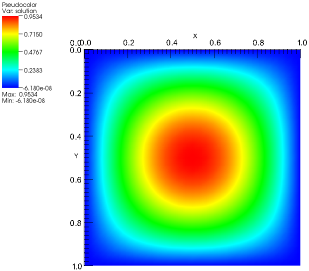


Figure: Decay of the L^2, $H^{1/2}$ and H^1 norms of the error for u.

Results with adaptive refinement



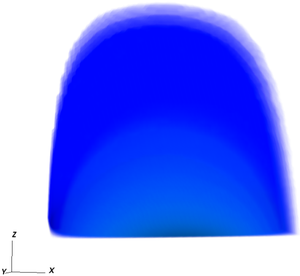


Figure: u_h^M and v_h^M with 13435 degrees of freedom.

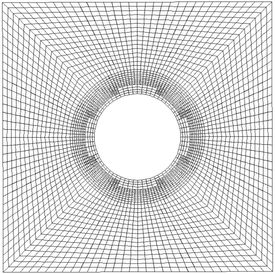
3D numerical example

We consider the following numerical example. Given a smooth function $f(x,y) = \sqrt{2}\pi \sin(\pi x) \sin(\pi y)$, find u such that

$$\begin{cases} (-\Delta)^{1/2} u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega = (-1,1)^2$ - $\text{disk}_{(0,0)}(0.5)$.

An adaptive mesh. $M = 4$



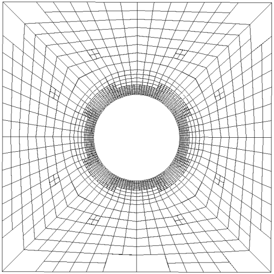


Figure: Meshes for $z = 0$ and $z = 4$.

An adaptive mesh. M = 4

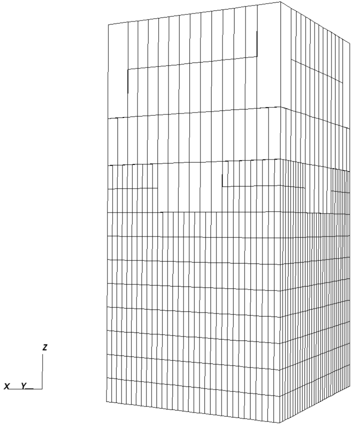


Figure: Degrees of freedom: 22492.

An adaptive mesh. M = 4

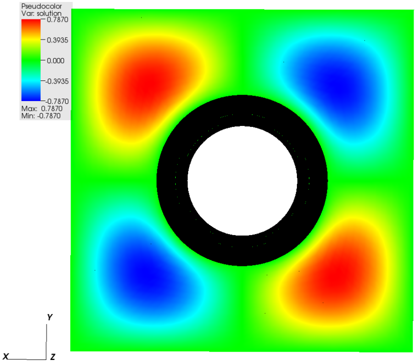


Figure: u_h^M computed with 22492 degrees of freedom.

A less regular example

We consider $\Omega = (0,1)$, and a function f s.t. the exact solution of the problem

$$\begin{cases} (-\Delta)^{1/2} u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

is given by

$$u(x) \begin{cases} 2x, & x \in (0,1/2), \\ 2(1-x), & x \in (1/2,1). \end{cases}$$

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$u \in H^s(\Omega)$ for $s < 3/2 \implies u \in H^{3/2-\epsilon}(\Omega)$, with $\epsilon > 0$. Then, we can not expect the optimal estimate:

$$\|u - u_h^M\|_{H_{00}^{1/2}(\Omega)} \leq C \left(\epsilon \|f\|_{\mathcal{V}_0(\Omega)^*} + h^{3/2} \|u\|_{H^2(\Omega)} \right), \quad \epsilon = \epsilon(M).$$

A less regular example

However, notice that $u \in H^{3/2-\epsilon}(\Omega) \implies v \in H^{2-\epsilon}(\Omega)$, and we have an almost optimal estimate for the function v,

$$\|v - v_h^M\|_{H_0^1(\mathcal{C})} \leq C\left(\epsilon\|f\|_{\mathcal{V}_0(\Omega)^*} + h^{1-\epsilon}\|v\|_{H^{2-\epsilon}(\mathcal{C}_M)}\right), \quad \epsilon = \epsilon(M),$$

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and, an almost optimal estimate for the function u,

$$\|u - u_h^M\|_{H_{00}^{1/2}(\Omega)} \leq C(\epsilon + h^{1-\epsilon})\|f\|_{\mathcal{V}_0(\Omega)}, \quad \epsilon = \epsilon(M).$$

Results with global refinement

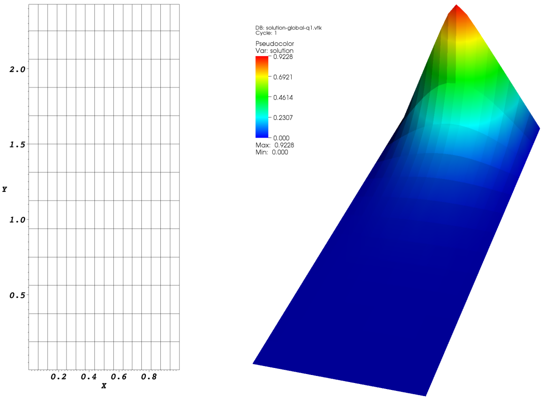

Figure: v_h^M with 238 degrees of freedom.

Results with global refinement

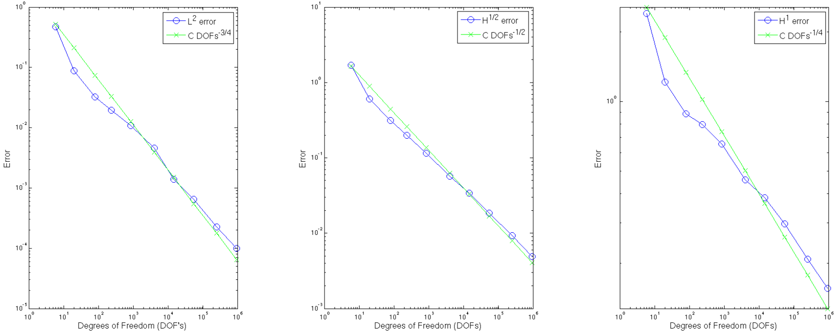

Figure: Decay of the L^2, $H^{1/2}$ and H^1 norms of the error for u.

Results with global refinement

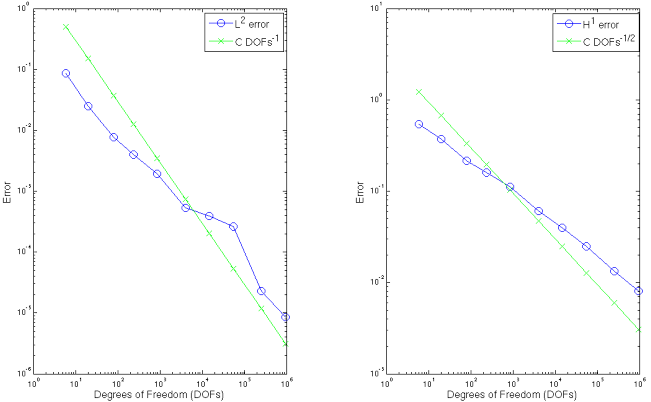

Figure: Decay of the L^2 and H^1 norms of the error for v.

Convergence Table for v: adaptive refinement.

n	cells	H^1-error		L^2-error	
0	2	5.419e-01	-	8.615e-02	-
1	8	6.595e-01	-0.28	9.719e-02	-0.17
2	32	5.297e-01	0.32	5.996e-02	0.70
3	110	3.357e-01	0.66	2.277e-02	1.40
4	386	1.945e-01	0.79	6.773e-03	1.75
5	1313	1.086e-01	0.84	1.963e-03	1.79
6	4445	5.950e-02	0.87	5.625e-04	1.80
7	14903	3.213e-02	0.89	3.096e-04	0.86
8	49838	1.715e-02	0.91	3.953e-05	2.97
9	166577	9.092e-03	0.92	1.120e-05	1.82

Results with adaptive refinement

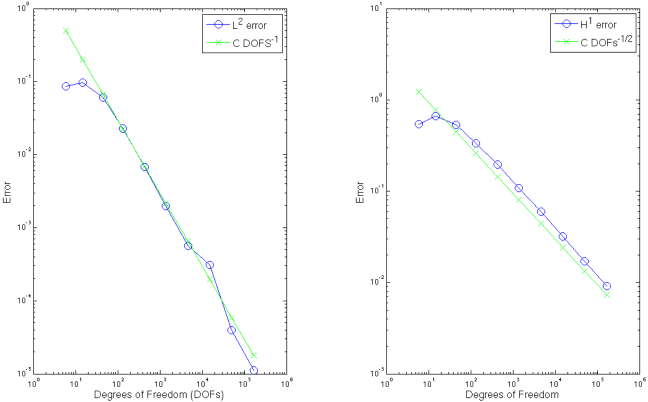

Figure: Decay of the L^2 and H^1 norms of the error for v.

Future work

We are interested in develop an efficient computational technique to solve the problem: Given a smooth enough function f, find u such that

$$\begin{cases} (-\Delta)^s u & = & f & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$, with $d = 1, 2$ is a bounded domain with a smooth boundary $\partial\Omega$.

Caffarelli - Silvestre (2007)

$$(-\Delta)^s u = -C_s \lim_{y \to 0^+} y^a \partial_y v,$$

where $a = 1 - 2s$, and C_s is a positive constant depending only on s.

Future work

v solves the following degenerate elliptic equation:

$$\begin{cases} div(y^a \nabla v) & = & 0 \quad \text{in } \mathcal{C} \\ v & = & 0 \quad \text{on } \partial_L \mathcal{C} \\ \lim_{y \to 0^+} y^a \partial_y v & = & C_s^{-1} f \quad \text{on } \Omega \times \{0\}. \end{cases}$$

$$u = \mathrm{tr}_\Omega v \text{ on } \Omega.$$