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Maximum–norm a posteriori error estimates for an optimal control problem

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Abstract We analyze a reliable and efficient max-norm a posteriori error estimator for a control-constrained, linear–quadratic optimal control problem. The estimator yields optimal experimental rates of convergence within an adaptive loop.

Keywords Linear–quadratic optimal control problem · Finite element methods · A posteriori error analysis · Maximum–norm

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1 Introduction

Let Ω be an open and bounded polytope in $\mathbb{R}^d$, $d \in \{2,3\}$, with Lipschitz boundary $\partial\Omega$. Given $\mathsf{y}_\Omega \in L^2(\Omega)$ and $\lambda > 0$ we define the cost functional

$$J(\mathsf{y},\mathsf{u}) = \frac{1}{2}\|\mathsf{y}-\mathsf{y}_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\mathsf{u}\|^2_{L^2(\Omega)}. \tag{1}$$

In this article we devise max-norm a posteriori error estimators for the following optimal control problem: Find

$$\min J(\mathsf{y},\mathsf{u}) \tag{2}$$

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subject to, for a given $\mathsf{f} \in L^2(\Omega)$, the linear elliptic partial differential equation (PDE)

$$-\Delta \mathsf{y} = \mathsf{f} + \mathsf{u} \quad \text{in } \Omega, \qquad \mathsf{y} = 0 \quad \text{on } \partial\Omega, \tag{3}$$

and, for $\mathsf{a}, \mathsf{b} \in \mathbb{R}$ with $\mathsf{a} \leq \mathsf{b}$, the control constraints

$$\mathsf{u} \in \mathbb{U}_{\text{ad}}, \qquad \mathbb{U}_{\text{ad}} := \{\mathsf{v} \in L^2(\Omega) : \ \mathsf{a} \leq \mathsf{v}(x) \leq \mathsf{b} \text{ for almost every } x \text{ in } \Omega\}. \tag{4}$$

The pioneering work [7] presented and analyzed, in two dimensions, the first max-norm a posteriori error estimator for the state equation (3). These results were later extended to more dimensions and both nonlinear and geometric problems [3,4,8]. To our knowledge, max-norm a posteriori error estimation for the optimal control problem (2)–(4) has not been considered previously in the literature. This is the novelty of our contribution.

2 Notation

Let $\mathscr{T} = \{T\}$ be a conforming simplicial mesh of $\bar{\Omega}$ [5], $h_T = \text{diam}(T)$ and

$$\ell_{\mathscr{T}} = |\log(\max_{T \in \mathscr{T}} h_T^{-1})|. \tag{5}$$

We assume that $\mathscr{T}$ is a member of a shape regular family of meshes. Define

$$\mathbb{V}(\mathscr{T}) := \{w \in C^0(\bar{\Omega}) : \ w_{|T} \in \mathbb{P}_1(T) \ \forall T \in \mathscr{T} \text{ and } w_{|\partial\Omega} = 0\}, \qquad \mathbb{U}_{\text{ad}}(\mathscr{T}) = \mathbb{U}_{\text{ad}} \cap \mathbb{V}(\mathscr{T}).$$

We denote by $\mathscr{S} = \{S\}$ the set of internal $(d-1)$-dimensional interelement boundaries of $\mathscr{T}$ and $h_S = \text{diam}(S)$. If $T \in \mathscr{T}$, $\mathscr{S}_T \subset \mathscr{S}$ is the set of sides of T. For $S \in \mathscr{S}$ we set $\mathscr{N}_S = \{T^+, T^-\}$ such that $S = T^+ \cap T^-$. For $T \in \mathscr{T}$, we define

$$\mathscr{N}_T := \left\{T' \in \mathscr{T} : \mathscr{S}_T \cap \mathscr{S}_{T'} \neq \emptyset\right\}. \tag{6}$$

For $w_{\mathscr{T}} \in \mathbb{V}(\mathscr{T})$ and $S \in \mathscr{S}$ with $\mathscr{N}_S = \{T^+, T^-\}$, the jump or interelement residual is $[\![\nabla w_{\mathscr{T}} \cdot \nu]\!] = \nu^+ \cdot \nabla w_{\mathscr{T}|T^+} + \nu^- \cdot \nabla w_{\mathscr{T}|T^-}$, where ν^+, ν^- are the unit normals to S pointing towards T^+, $T^- \in \mathscr{T}$. The $L^2(\Omega)$ inner product is $(\cdot,\cdot)$. By $A \lesssim B$ we mean that $A \leq cB$ for a nonessential constant c that might change at each occurrence.

3 Optimal control problem

The necessary and sufficient optimality conditions of (2)–(4) read: Find $(\bar{\mathsf{y}}, \bar{\mathsf{p}}, \bar{\mathsf{u}}) \in H_0^1(\Omega) \times H_0^1(\Omega) \times \mathbb{U}_{\text{ad}}$ such that

$$\begin{cases} (\nabla \bar{\mathsf{y}}, \nabla \mathsf{v})_{L^2(\Omega)} = (\mathsf{f} + \bar{\mathsf{u}}, \mathsf{v})_{L^2(\Omega)} \ \forall \ \mathsf{v} \in H_0^1(\Omega), \\ (\nabla \mathsf{w}, \nabla \bar{\mathsf{p}})_{L^2(\Omega)} = (\bar{\mathsf{y}} - \mathsf{y}_\Omega, \mathsf{w})_{L^2(\Omega)} \ \forall \ \mathsf{w} \in H_0^1(\Omega), \\ (\bar{\mathsf{p}} + \lambda \bar{\mathsf{u}}, \mathsf{u} - \bar{\mathsf{u}})_{L^2(\Omega)} \geq 0 \ \forall \ \mathsf{u} \in \mathbb{U}_{ad}. \end{cases} \tag{7}$$

Following [6], there exists $r > d$ such that $\bar{\mathsf{y}}, \bar{\mathsf{p}} \in W^{1,r}(\Omega) \hookrightarrow C(\bar{\Omega})$. Hence, $\bar{\mathsf{u}} \in C(\bar{\Omega}) \cap H_0^1(\Omega)$ since

$$\bar{\mathsf{u}} = \Pi\left(-\lambda^{-1}\bar{\mathsf{p}}\right), \qquad \Pi(w)(x) := \min\{\mathsf{b}, \max\{\mathsf{a}, w(x)\}\} \text{ for all } x \text{ in } \bar{\Omega}. \tag{8}$$

For $\mathscr{G} \subset \Omega$ this operator is nonexpansive in $L^\infty(\mathscr{G})$, i.e.,

$$\|\Pi(w_1) - \Pi(w_2)\|_{L^\infty(\mathscr{G})} \leq \|w_1 - w_2\|_{L^\infty(\mathscr{G})} \quad \forall w_1, w_2 \in C(\bar{\Omega}). \tag{9}$$

We approximate the solution of (7) by finding $(\bar{\mathsf{y}}_{\mathscr{T}}, \bar{\mathsf{p}}_{\mathscr{T}}, \bar{\mathsf{u}}_{\mathscr{T}}) \in \mathbb{V}(\mathscr{T}) \times \mathbb{V}(\mathscr{T}) \times \mathbb{U}_{\text{ad}}(\mathscr{T})$ such that

$$\begin{cases} (\nabla \bar{\mathsf{y}}_{\mathscr{T}}, \nabla \mathsf{v}_{\mathscr{T}})_{L^2(\Omega)} = (\mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}, \mathsf{v}_{\mathscr{T}})_{L^2(\Omega)} \ \forall \mathsf{v}_{\mathscr{T}} \in \mathbb{V}(\mathscr{T}), \\ (\nabla \mathsf{w}_{\mathscr{T}}, \nabla \bar{\mathsf{p}}_{\mathscr{T}})_{L^2(\Omega)} = (\bar{\mathsf{y}}_{\mathscr{T}} - \mathsf{y}_\Omega, \mathsf{w}_{\mathscr{T}})_{L^2(\Omega)} \ \forall \mathsf{w}_{\mathscr{T}} \in \mathbb{V}(\mathscr{T}), \\ (\bar{\mathsf{p}}_{\mathscr{T}} + \lambda \bar{\mathsf{u}}_{\mathscr{T}}, \mathsf{u}_{\mathscr{T}} - \bar{\mathsf{u}}_{\mathscr{T}})_{L^2(\Omega)} \geq 0 \ \forall \mathsf{u}_{\mathscr{T}} \in \mathbb{U}_{ad}(\mathscr{T}). \end{cases} \tag{10}$$

4 A posteriori error analysis: reliability

We begin by defining the local error indicators

$$\mathscr{E}_{\mathsf{y}}(\bar{\mathsf{y}}_{\mathscr{T}}, \bar{\mathsf{u}}_{\mathscr{T}}; T) = h_T^{2-d/2} \|\mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(T)} + h_T \|[\![\nabla \bar{\mathsf{y}}_{\mathscr{T}} \cdot \nu]\!]\|_{L^\infty(\partial T \setminus \partial \Omega)}, \tag{11}$$

$$\mathscr{E}_{\mathsf{p}}(\bar{\mathsf{p}}_{\mathscr{T}}, \bar{\mathsf{y}}_{\mathscr{T}}; T) = h_T^{2-d/2} \|\bar{\mathsf{y}}_{\mathscr{T}} - \mathsf{y}_\Omega\|_{L^2(T)} + h_T \|[\![\nabla \bar{\mathsf{p}}_{\mathscr{T}} \cdot \nu]\!]\|_{L^\infty(\partial T \setminus \partial \Omega)}, \tag{12}$$

$$\mathscr{E}_{\mathsf{u}}(\bar{\mathsf{u}}_{\mathscr{T}}, \bar{\mathsf{p}}_{\mathscr{T}}; T) = \|\Pi(-\bar{\mathsf{p}}_{\mathscr{T}}/\lambda) - \bar{\mathsf{u}}_{\mathscr{T}}\|_{L^\infty(T)}, \tag{13}$$

$$\mathscr{E}(\bar{\mathsf{y}}_{\mathscr{T}}, \bar{\mathsf{p}}_{\mathscr{T}}, \bar{\mathsf{u}}_{\mathscr{T}}; T) = \left(\mathscr{E}_{\mathsf{y}}^2(\bar{\mathsf{y}}_{\mathscr{T}}, \bar{\mathsf{u}}_{\mathscr{T}}; T) + \mathscr{E}_{\mathsf{p}}^2(\bar{\mathsf{p}}_{\mathscr{T}}, \bar{\mathsf{y}}_{\mathscr{T}}; T) + \mathscr{E}_{\mathsf{u}}^2(\bar{\mathsf{u}}_{\mathscr{T}}, \bar{\mathsf{p}}_{\mathscr{T}}; T) \right)^{1/2}, \tag{14}$$

and the global a posteriori error estimators

$$\mathscr{E}_{\mathsf{y}}(\bar{\mathsf{y}}_{\mathscr{T}}, \bar{\mathsf{u}}_{\mathscr{T}}; \mathscr{T}) = \max_{T \in \mathscr{T}} \mathscr{E}_{\mathsf{y}}(\bar{\mathsf{y}}_{\mathscr{T}}, \bar{\mathsf{u}}_{\mathscr{T}}; T), \tag{15}$$

$$\mathscr{E}_{\mathsf{p}}(\bar{\mathsf{p}}_{\mathscr{T}}, \bar{\mathsf{y}}_{\mathscr{T}}; \mathscr{T}) = \max_{T \in \mathscr{T}} \mathscr{E}_{\mathsf{p}}(\bar{\mathsf{p}}_{\mathscr{T}}, \bar{\mathsf{y}}_{\mathscr{T}}; T), \tag{16}$$

$$\mathscr{E}_{\mathsf{u}}(\bar{\mathsf{u}}_{\mathscr{T}}, \bar{\mathsf{p}}_{\mathscr{T}}; \mathscr{T}) = \max_{T \in \mathscr{T}} \mathscr{E}_{\mathsf{u}}(\bar{\mathsf{u}}_{\mathscr{T}}, \bar{\mathsf{p}}_{\mathscr{T}}; T), \tag{17}$$

$$\mathscr{E}(\bar{\mathsf{y}}_{\mathscr{T}}, \bar{\mathsf{p}}_{\mathscr{T}}, \bar{\mathsf{u}}_{\mathscr{T}}; \mathscr{T}) = \left(\mathscr{E}_{\mathsf{y}}^2(\bar{\mathsf{y}}_{\mathscr{T}}, \bar{\mathsf{u}}_{\mathscr{T}}; \mathscr{T}) + \mathscr{E}_{\mathsf{p}}^2(\bar{\mathsf{p}}_{\mathscr{T}}, \bar{\mathsf{y}}_{\mathscr{T}}; \mathscr{T}) + \mathscr{E}_{\mathsf{u}}^2(\bar{\mathsf{u}}_{\mathscr{T}}, \bar{\mathsf{p}}_{\mathscr{T}}; \mathscr{T}) \right)^{1/2}. \tag{18}$$

We also introduce two auxilliary variables: $\hat{\mathsf{y}}, \hat{\mathsf{p}} \in H_0^1(\Omega)$ which solve, respectively,

$$(\nabla \hat{\mathsf{y}}, \nabla \mathsf{v}) = (\mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}, \mathsf{v}) \quad \forall\, \mathsf{v} \in H_0^1(\Omega), \quad (\nabla \mathsf{w}, \nabla \hat{\mathsf{p}}) = (\bar{\mathsf{y}}_{\mathscr{T}} - \mathsf{y}_\Omega, \mathsf{w}) \quad \forall\, \mathsf{w} \in H_0^1(\Omega). \tag{19}$$

We note that $\hat{\mathsf{y}}, \hat{\mathsf{p}} \in H_0^1(\Omega) \cap C(\bar{\Omega})$ and thus we can invoke [2, Lemma 4.2] to conclude that

$$\|\hat{\mathsf{y}} - \bar{\mathsf{y}}_{\mathscr{T}}\|_{L^\infty(\Omega)} \lesssim \ell_{\mathscr{T}} \mathscr{E}_{\mathsf{y}}(\bar{\mathsf{y}}_{\mathscr{T}}, \bar{\mathsf{u}}_{\mathscr{T}}; \mathscr{T}), \qquad \|\hat{\mathsf{p}} - \bar{\mathsf{p}}_{\mathscr{T}}\|_{L^\infty(\Omega)} \lesssim \ell_{\mathscr{T}} \mathscr{E}_{\mathsf{p}}(\bar{\mathsf{p}}_{\mathscr{T}}, \bar{\mathsf{y}}_{\mathscr{T}}; \mathscr{T}). \tag{20}$$

Finally, for $e_{\bar{\mathsf{y}}} = \bar{\mathsf{y}} - \bar{\mathsf{y}}_{\mathscr{T}}$, $e_{\bar{\mathsf{p}}} = \bar{\mathsf{p}} - \bar{\mathsf{p}}_{\mathscr{T}}$ and $e_{\bar{\mathsf{u}}} = \bar{\mathsf{u}} - \bar{\mathsf{u}}_{\mathscr{T}}$ we define

$$\|(e_{\bar{\mathsf{y}}}, e_{\bar{\mathsf{p}}}, e_{\bar{\mathsf{u}}})\|_\Omega^2 := \|e_{\bar{\mathsf{y}}}\|_{L^\infty(\Omega)}^2 + \|e_{\bar{\mathsf{p}}}\|_{L^\infty(\Omega)}^2 + \|e_{\bar{\mathsf{u}}}\|_{L^\infty(\Omega)}^2. \tag{21}$$

Theorem 1 (global reliability) *Let* $(\bar{\mathsf{y}}, \bar{\mathsf{p}}, \bar{\mathsf{u}}) \in H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ *be the solution to* (7) *and* $(\bar{\mathsf{y}}_{\mathscr{T}}, \bar{\mathsf{p}}_{\mathscr{T}}, \bar{\mathsf{u}}_{\mathscr{T}}) \in \mathbb{V}(\mathscr{T}) \times \mathbb{V}(\mathscr{T}) \times \mathbb{U}_{\text{ad}}(\mathscr{T})$ *its numerical approximation obtained as the solution to* (10). *Then*

$$\|(e_{\bar{\mathsf{y}}}, e_{\bar{\mathsf{p}}}, e_{\bar{\mathsf{u}}})\|_\Omega \lesssim \max\{1, \ell_{\mathscr{T}}\} \mathscr{E}(\bar{\mathsf{y}}_{\mathscr{T}}, \bar{\mathsf{p}}_{\mathscr{T}}, \bar{\mathsf{u}}_{\mathscr{T}}; \mathscr{T}). \tag{22}$$

Proof We proceed in five steps.

Step 1. First we control the error $\|\bar{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(\Omega)}$. Define $\tilde{\mathsf{u}} = \Pi(-\lambda^{-1}\bar{\mathsf{p}}_{\mathscr{T}})$, which can be equivalently characterized by

$$(\bar{\mathsf{p}}_{\mathscr{T}} + \lambda \tilde{\mathsf{u}}, \mathsf{u}-\tilde{\mathsf{u}}) \geq 0 \quad \forall \mathsf{u} \in \mathbb{U}_{\mathrm{ad}}. \tag{23}$$

We first bound $\|\bar{\mathsf{u}}-\tilde{\mathsf{u}}\|_{L^2(\Omega)}$. Set $\mathsf{u}=\tilde{\mathsf{u}}$ in (7), $\mathsf{u}=\bar{\mathsf{u}}$ in (23) and add the results to obtain

$$\lambda \|\bar{\mathsf{u}}-\tilde{\mathsf{u}}\|^2_{L^2(\Omega)} \leq (\bar{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}, \tilde{\mathsf{u}}-\bar{\mathsf{u}}).$$

To bound the right hand side of the previous expression, we let $(\tilde{\mathsf{y}},\tilde{\mathsf{p}}) \in H^1_0(\Omega)\times H^1_0(\Omega)$ be such that

$$(\nabla \tilde{\mathsf{y}}, \nabla \mathsf{v}) = (\mathsf{f}+\tilde{\mathsf{u}}, \mathsf{v}) \quad \forall\, \mathsf{v} \in H^1_0(\Omega), \qquad (\nabla \tilde{\mathsf{p}}, \nabla \mathsf{w}) = (\tilde{\mathsf{y}}-\mathsf{y}_\Omega, \mathsf{w}) \quad \forall\, \mathsf{w} \in H^1_0(\Omega).$$

With the auxilliary adjoint state $\tilde{\mathsf{p}}$ at hand, we thus arrive at

$$\lambda \|\bar{\mathsf{u}}-\tilde{\mathsf{u}}\|^2_{L^2(\Omega)} \leq (\bar{\mathsf{p}}-\tilde{\mathsf{p}}, \tilde{\mathsf{u}}-\bar{\mathsf{u}}) + (\tilde{\mathsf{p}}-\hat{\mathsf{p}}, \tilde{\mathsf{u}}-\bar{\mathsf{u}}) + (\hat{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}, \tilde{\mathsf{u}}-\bar{\mathsf{u}}). \tag{24}$$

We now observe that $(\nabla(\tilde{\mathsf{y}}-\bar{\mathsf{y}}), \nabla \mathsf{v}) = (\tilde{\mathsf{u}}-\bar{\mathsf{u}}, \mathsf{v})$ for all $\mathsf{v} \in H^1_0(\Omega)$ and $(\nabla \mathsf{w}, \nabla(\bar{\mathsf{p}}-\tilde{\mathsf{p}})) = (\bar{\mathsf{y}}-\tilde{\mathsf{y}}, \mathsf{w})$ $\forall \mathsf{w} \in H^1_0(\Omega)$. Hence,

$$(\bar{\mathsf{p}}-\tilde{\mathsf{p}}, \tilde{\mathsf{u}}-\bar{\mathsf{u}}) = (\nabla(\tilde{\mathsf{y}}-\bar{\mathsf{y}}), \nabla(\bar{\mathsf{p}}-\tilde{\mathsf{p}})) = -\|\bar{\mathsf{y}}-\tilde{\mathsf{y}}\|^2_{L^2(\Omega)} \leq 0.$$

In view of this, an application of the Cauchy–Schwarz and Young's inequalities to (24) yields

$$\|\bar{\mathsf{u}}-\tilde{\mathsf{u}}\|^2_{L^2(\Omega)} \lesssim \|\tilde{\mathsf{p}}-\hat{\mathsf{p}}\|^2_{L^2(\Omega)} + \|\hat{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|^2_{L^2(\Omega)}. \tag{25}$$

We now control $\|\tilde{\mathsf{p}}-\hat{\mathsf{p}}\|^2_{L^2(\Omega)}$. Since $(\nabla \mathsf{w}, \nabla(\tilde{\mathsf{p}}-\hat{\mathsf{p}})) = (\tilde{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}, \mathsf{w})$, for all $\mathsf{w} \in H^1_0(\Omega)$, we have that

$$\begin{aligned}\|\tilde{\mathsf{p}}-\hat{\mathsf{p}}\|^2_{L^2(\Omega)} \lesssim &(\nabla(\tilde{\mathsf{p}}-\hat{\mathsf{p}}), \nabla(\tilde{\mathsf{p}}-\hat{\mathsf{p}})) \\ =&(\tilde{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}, \tilde{\mathsf{p}}-\hat{\mathsf{p}}) \lesssim \left(\|\tilde{\mathsf{y}}-\hat{\mathsf{y}}\|_{L^2(\Omega)} + \|\hat{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|_{L^2(\Omega)}\right) \|\tilde{\mathsf{p}}-\hat{\mathsf{p}}\|_{L^2(\Omega)}.\end{aligned}$$

Consequently, $\|\tilde{\mathsf{p}}-\hat{\mathsf{p}}\|_{L^2(\Omega)} \lesssim \|\tilde{\mathsf{y}}-\hat{\mathsf{y}}\|_{L^2(\Omega)} + \|\hat{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|_{L^2(\Omega)}$. This, in conjunction with (25), yields

$$\|\bar{\mathsf{u}}-\tilde{\mathsf{u}}\|^2_{L^2(\Omega)} \lesssim \|\tilde{\mathsf{y}}-\hat{\mathsf{y}}\|^2_{L^2(\Omega)} + \|\hat{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|^2_{L^2(\Omega)} + \|\hat{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|^2_{L^2(\Omega)}. \tag{26}$$

Upon observing that $(\nabla(\tilde{\mathsf{y}}-\hat{\mathsf{y}}), \nabla \mathsf{v}) = (\tilde{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}, \mathsf{v})$, for all $\mathsf{v} \in H^1_0(\Omega)$, we obtain that

$$\|\tilde{\mathsf{y}}-\hat{\mathsf{y}}\|^2_{L^2(\Omega)} \lesssim (\nabla(\tilde{\mathsf{y}}-\hat{\mathsf{y}}), \nabla(\tilde{\mathsf{y}}-\hat{\mathsf{y}})) = (\tilde{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}, \tilde{\mathsf{y}}-\hat{\mathsf{y}}) \leq \|\tilde{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(\Omega)} \|\tilde{\mathsf{y}}-\hat{\mathsf{y}}\|_{L^2(\Omega)}$$

and hence, $\|\tilde{\mathsf{y}}-\hat{\mathsf{y}}\|_{L^2(\Omega)} \lesssim \|\tilde{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(\Omega)}$. Combining this with (26) implies that

$$\|\bar{\mathsf{u}}-\tilde{\mathsf{u}}\|^2_{L^2(\Omega)} \lesssim \|\tilde{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|^2_{L^2(\Omega)} + \|\hat{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|^2_{L^2(\Omega)} + \|\hat{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|^2_{L^2(\Omega)}.$$

The triangle inequality then allows us to conclude that

$$\begin{aligned}\|\bar{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|^2_{L^2(\Omega)} \lesssim &\|\tilde{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|^2_{L^2(\Omega)} + \|\hat{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|^2_{L^2(\Omega)} + \|\hat{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|^2_{L^2(\Omega)} \\ \lesssim &\|\tilde{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)} + \|\hat{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)} + \|\hat{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)}.\end{aligned} \tag{27}$$

Step 2. In this step we control $\|\bar{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|_{L^\infty(\Omega)}$. In view of the results of [6] there exists $r>d$ such that

$$\|\bar{\mathsf{y}}-\hat{\mathsf{y}}\|_{L^\infty(\Omega)} \lesssim \|\bar{\mathsf{y}}-\hat{\mathsf{y}}\|_{W^{1,r}(\Omega)} \lesssim \|\bar{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(\Omega)}.$$

Consequently, the triangle inequality and (27) give us that

$$\|\bar{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)} \lesssim \|\hat{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)} + \|\hat{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)} + \|\tilde{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)}. \tag{28}$$

Step 3. To bound $\|\bar{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|_{L^\infty(\Omega)}$ we again use [6] to conclude that there exists $r>d$ such that

$$\|\bar{\mathsf{p}}-\hat{\mathsf{p}}\|_{L^\infty(\Omega)} \lesssim \|\bar{\mathsf{p}}-\hat{\mathsf{p}}\|_{W^{1,r}(\Omega)} \lesssim \|\bar{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|_{L^2(\Omega)} \lesssim \|\bar{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|_{L^\infty(\Omega)}.$$

Thus, this estimate and (28) imply that

$$\begin{aligned}\|\bar{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)} &\lesssim \|\bar{\mathsf{p}}-\hat{\mathsf{p}}\|^2_{L^\infty(\Omega)} + \|\hat{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)} \\ &\lesssim \|\hat{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)} + \|\hat{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)} + \|\tilde{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)}.\end{aligned} \tag{29}$$

Step 4. The goal of this step is to control the error $\|\bar{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^\infty(\Omega)}$. We begin with the basic estimate

$$\|\bar{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^\infty(\Omega)} \leq \|\bar{\mathsf{u}}-\tilde{\mathsf{u}}\|_{L^\infty(\Omega)} + \|\tilde{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^\infty(\Omega)}. \tag{30}$$

Using (9) we have that $\|\bar{\mathsf{u}}-\tilde{\mathsf{u}}\|_{L^\infty(\Omega)} = \|\Pi(-\lambda^{-1}\bar{\mathsf{p}})-\Pi(-\lambda^{-1}\bar{\mathsf{p}}_{\mathscr{T}})\|_{L^\infty(\Omega)} \lesssim \|\bar{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|_{L^\infty(\Omega)}$. Therefore, upon combining this with (30) and (29), we can conclude that

$$\|\bar{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)} \lesssim \|\hat{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)} + \|\hat{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)} + \|\tilde{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|^2_{L^\infty(\Omega)}. \tag{31}$$

Step 5. The claimed result follows upon gathering (28), (29) and (31), and using (17) and (20).

5 A posteriori error analysis: efficiency

Let $\mathscr{P}_{\mathscr{T}}$ denote the L^2-projection onto piecewise linear, over $\mathscr{T}$, functions. For $g \in L^2(\Omega)$ and $\mathscr{M} \subset \mathscr{T}$ we define

$$\operatorname{osc}_{\mathscr{T}}(g;\mathscr{M})^2 = \sum_{T\in\mathscr{M}} h_T^{4-d}\|g-\mathscr{P}_{\mathscr{T}}g\|^2_{L^2(T)}. \tag{32}$$

Lemma 1 (local efficiency of $\mathscr{E}_{\mathsf{y}}$) *In the setting of Theorem 1 we have that*

$$\mathscr{E}_{\mathsf{y}}(\bar{\mathsf{y}}_{\mathscr{T}},\bar{\mathsf{u}}_{\mathscr{T}};T) \lesssim \|\bar{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|_{L^\infty(\mathscr{N}_T)} + h_T^2\|\bar{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^\infty(\mathscr{N}_T)} + \operatorname{osc}_{\mathscr{T}}(\mathsf{f};\mathscr{N}_T), \tag{33}$$

where the hidden constant is independent of the size of the elements in the mesh $\mathscr{T}$ and $\#\mathscr{T}$.

Proof Let $\mathsf{v} \in H_0^1(\Omega)$ be such that $\mathsf{v}_{|T} \in C^2(T)$ for all $T \in \mathscr{T}$. Using (7) and integrating by parts yields

$$\int_\Omega \nabla(\bar{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}})\cdot\nabla\mathsf{v} = \sum_{T\in\mathscr{T}}\int_T(\mathsf{f}+\bar{\mathsf{u}})\mathsf{v} + \sum_{S\in\mathscr{S}}\int_S [\![\nabla\bar{\mathsf{y}}_{\mathscr{T}}\cdot\nu]\!]\mathsf{v}.$$

Since on each $T \in \mathscr{T}$ we have that $\mathsf{v} \in C^2(T)$, we again apply integration by parts to conclude that

$$\int_\Omega \nabla(\bar{\mathsf{y}} - \bar{\mathsf{y}}_{\mathscr{T}}) \cdot \nabla \mathsf{v} = -\sum_{T \in \mathscr{T}} \int_T \Delta \mathsf{v} (\bar{\mathsf{y}} - \bar{\mathsf{y}}_{\mathscr{T}}) - \sum_{S \in \mathscr{S}} \int_S [\![\nabla \mathsf{v} \cdot \nu]\!] (\bar{\mathsf{y}} - \bar{\mathsf{y}}_{\mathscr{T}}).$$

In conclusion, since the left hand sides of the previous expressions coincide, we arrive at the identity

$$\sum_{T \in \mathscr{T}} \int_T (\mathsf{f} + \bar{\mathsf{u}}) \mathsf{v} + \sum_{S \in \mathscr{S}} \int_S [\![\nabla \bar{\mathsf{y}}_{\mathscr{T}} \cdot \nu]\!] \mathsf{v} = -\sum_{T \in \mathscr{T}} \int_T \Delta \mathsf{v} (\bar{\mathsf{y}} - \bar{\mathsf{y}}_{\mathscr{T}}) - \sum_{S \in \mathscr{S}} \int_S [\![\nabla \mathsf{v} \cdot \nu]\!] (\bar{\mathsf{y}} - \bar{\mathsf{y}}_{\mathscr{T}}), \qquad (34)$$

for every $\mathsf{v} \in H_0^1(\Omega)$ such that $\mathsf{v}_{|T} \in C^2(T)$ for all $T \in \mathscr{T}$. We now proceed, on the basis of (15), in two steps.

Step 1. Let $T \in \mathscr{T}$. We begin with the basic estimate

$$h_T^{2-d/2} \|\mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(T)} \leq h_T^{2-d/2} \|\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(T)} + h_T^{2-d/2} \|\mathsf{f} - \mathscr{P}_{\mathscr{T}} \mathsf{f}\|_{L^2(T)}. \qquad (35)$$

By letting $\mathsf{v} = \beta_T = (\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}) \varphi_T^2$ in (34), where φ_T is the standard bubble function over T [1,9], we obtain that

$$\int_T (\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}) \beta_T = -\int_T \Delta \beta_T (\bar{\mathsf{y}} - \bar{\mathsf{y}}_{\mathscr{T}}) - \int_T (\bar{\mathsf{u}} - \bar{\mathsf{u}}_{\mathscr{T}}) \beta_T - \int_T (\mathsf{f} - \mathscr{P}_{\mathscr{T}} \mathsf{f}) \beta_T, \qquad (36)$$

since $\int_S [\![\nabla \beta_T \cdot \nu]\!] (\bar{\mathsf{y}} - \bar{\mathsf{y}}_{\mathscr{T}}) = 0$ for all $S \in \mathscr{S}$. We now bound each term on the right–hand side of (36) separately. Since $\Delta(\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}) = 0$ on T, we have that $\Delta \beta_T = 4\nabla(\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}) \cdot \nabla \varphi_T \varphi_T + 2(\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}})(\varphi_T \Delta \varphi_T + \nabla \varphi_T \cdot \nabla \varphi_T)$. This equality, the properties of the bubble function φ_T and an inverse inequality allow us to conclude that

$$\begin{aligned}
&\left| \int_T \Delta \beta_T (\bar{\mathsf{y}} - \bar{\mathsf{y}}_{\mathscr{T}}) \right| \\
&\lesssim \left(h_T^{d/2-1} \|\nabla(\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}})\|_{L^2(T)} + h_T^{d/2-2} \|\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(T)} \right) \|\bar{\mathsf{y}} - \bar{\mathsf{y}}_{\mathscr{T}}\|_{L^\infty(T)} \\
&\lesssim h_T^{d/2-2} \|\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(T)} \|\bar{\mathsf{y}} - \bar{\mathsf{y}}_{\mathscr{T}}\|_{L^\infty(T)}.
\end{aligned}$$

In addition, we have that

$$\begin{aligned}
\left| \int_T (\bar{\mathsf{u}} - \bar{\mathsf{u}}_{\mathscr{T}}) \beta_T \right| &\lesssim \|\bar{\mathsf{u}} - \bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(T)} \|\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(T)} \\
&\lesssim h_T^{d/2} \|\bar{\mathsf{u}} - \bar{\mathsf{u}}_{\mathscr{T}}\|_{L^\infty(T)} \|\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(T)}
\end{aligned}$$

and

$$\left| \int_T (\mathsf{f} - \mathscr{P}_{\mathscr{T}} \mathsf{f}) \beta_T \right| \lesssim \|\mathsf{f} - \mathscr{P}_{\mathscr{T}} \mathsf{f}\|_{L^2(T)} \|\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(T)}.$$

In view of the fact that $\|\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(T)}^2 \lesssim \int_T (\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}) \beta_T$, the previous findings allow us to state that

$$h_T^{2-d/2} \|\mathscr{P}_{\mathscr{T}} \mathsf{f} + \bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(T)} \lesssim \|\bar{\mathsf{y}} - \bar{\mathsf{y}}_{\mathscr{T}}\|_{L^\infty(T)} + h_T^2 \|\bar{\mathsf{u}} - \bar{\mathsf{u}}_{\mathscr{T}}\|_{L^\infty(T)} + h_T^{2-d/2} \|\mathsf{f} - \mathscr{P}_{\mathscr{T}} \mathsf{f}\|_{L^2(T)}.$$

Consequently, using (35) we conclude that

$$h_T^{2-d/2}\|\mathscr{P}_{\mathscr{T}}\mathsf{f}+\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(T)} \lesssim \|\bar{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|_{L^\infty(T)} + h_T^2\|\bar{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^\infty(T)} + \operatorname{osc}_{\mathscr{T}}(\mathsf{f};T). \tag{37}$$

Step 2. Let $T\in\mathscr{T}$ and $S\in\mathscr{S}_T$. We proceed to control $h_T\|[\![\nabla\bar{\mathsf{y}}_{\mathscr{T}}\cdot\nu]\!]\|_{L^\infty(S)}$ in (15). To do this, we use the property

$$|S|\|[\![\nabla\bar{\mathsf{y}}_{\mathscr{T}}\cdot\nu]\!]\|_{L^\infty(S)} \lesssim \left|\int_S [\![\nabla\bar{\mathsf{y}}_{\mathscr{T}}\cdot\nu]\!]\varphi_S\right|, \tag{38}$$

of φ_S, the standard bubble function over S [1,9]. We now let $\mathsf{v}=\varphi_S$ in (34) and arrive at

$$\begin{aligned}
&\left|\int_S [\![\nabla\bar{\mathsf{y}}_{\mathscr{T}}\cdot\nu]\!]\varphi_S\right| \\
&\leq \sum_{T'\in\mathscr{N}_S}\int_{T'}|\mathsf{f}+\bar{\mathsf{u}}|\varphi_S + \sum_{T'\in\mathscr{N}_S}\int_{T'}|\bar{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}||\Delta\varphi_S| + \sum_{T'\in\mathscr{N}_S}\sum_{S'\in\mathscr{S}_{T'}}\int_{S'}|\bar{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}||[\![\nabla\varphi_S\cdot\nu]\!]| \\
&\lesssim \sum_{T'\in\mathscr{N}_S}|T'|^{1/2}\left(\|\mathsf{f}+\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^2(T')}+|T'|^{1/2}\|\bar{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^\infty(T')}\right) \\
&\quad+\sum_{T'\in\mathscr{N}_S}\left(h_S^{-2}|T'|+h_S^{-1}\sum_{S'\in\mathscr{S}_{T'}}|S'|\right)\|\bar{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|_{L^\infty(T')}.
\end{aligned}$$

In view of the fact that $h_T|S|^{-1}\approx h_T^{2-d}$, the previous estimate combined with (38) and (37) yields the bound

$$h_T\|[\![\nabla\bar{\mathsf{y}}_{\mathscr{T}}\cdot\nu]\!]\|_{L^\infty(S)} \lesssim h_T^2\|\bar{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^\infty(\mathscr{N}_S)} + \|\bar{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|_{L^\infty(\mathscr{N}_S)} + \operatorname{osc}_{\mathscr{T}}(\mathsf{f};\mathscr{N}_S).$$

We finally combine the results of Step 1 and 2 and arrive at the desired estimate (33). This concludes the proof.

Similar arguments to the ones elaborated in the proof of Lemma 1 allow us to conclude the following result.

Lemma 2 (local efficiency of $\mathscr{E}_{\mathsf{p}}$) *In the setting of Theorem 1 we have that*

$$\mathscr{E}_{\mathsf{p}}(\bar{\mathsf{p}}_{\mathscr{T}},\bar{\mathsf{y}}_{\mathscr{T}};T) \lesssim \|\bar{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|_{L^\infty(\mathscr{N}_T)} + h_T^2\|\bar{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|_{L^\infty(\mathscr{N}_T)} + \operatorname{osc}_{\mathscr{T}}(\mathsf{y}_\Omega;\mathscr{N}_T), \tag{39}$$

where the hidden constant is independent of the size of the elements in the mesh $\mathscr{T}$ and $\#\mathscr{T}$.

Lemma 3 (local efficiency of $\mathscr{E}_{\mathsf{u}}$) *In the setting of Theorem 1 we have that*

$$\mathscr{E}_{\mathsf{u}}(\bar{\mathsf{u}}_{\mathscr{T}},\bar{\mathsf{p}}_{\mathscr{T}};T) \lesssim \|\bar{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^\infty(T)} + \|\bar{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|_{L^\infty(T)}, \tag{40}$$

where the hidden constant is independent of the size of the elements in the mesh $\mathscr{T}$ and $\#\mathscr{T}$.

Proof The estimate follows immediately from definition (8) and the Lipschitz property (9).

The results of Lemmas 1, 2 and 3 immediately yield the following result upon observing that Ω is bounded.

Theorem 2 (local and global efficiency of $\mathscr{E}$) *In the setting of Theorem 1 we have that*

$$\mathscr{E}(\bar{\mathsf{y}}_{\mathscr{T}},\bar{\mathsf{p}}_{\mathscr{T}},\bar{\mathsf{u}}_{\mathscr{T}};T) \lesssim \|\bar{\mathsf{y}}-\bar{\mathsf{y}}_{\mathscr{T}}\|_{L^\infty(\mathscr{N}_T)} + \|\bar{\mathsf{p}}-\bar{\mathsf{p}}_{\mathscr{T}}\|_{L^\infty(\mathscr{N}_T)} + \|\bar{\mathsf{u}}-\bar{\mathsf{u}}_{\mathscr{T}}\|_{L^\infty(\mathscr{N}_T)} + \operatorname{osc}_{\mathscr{T}}(\mathsf{f};\mathscr{N}_T) + \operatorname{osc}_{\mathscr{T}}(\mathsf{y}_\Omega;\mathscr{N}_T), \tag{41}$$

and

$$\mathscr{E}(\bar{\mathsf{y}}_{\mathscr{T}},\bar{\mathsf{p}}_{\mathscr{T}},\bar{\mathsf{u}}_{\mathscr{T}};\mathscr{T}) \lesssim \|(e_{\bar{\mathsf{y}}},e_{\bar{\mathsf{p}}},e_{\bar{\mathsf{u}}})\|_\Omega + \max_{T\in\mathscr{T}} \operatorname{osc}_{\mathscr{T}}(\mathsf{f};\mathscr{N}_T) + \max_{T\in\mathscr{T}} \operatorname{osc}_{\mathscr{T}}(\mathsf{y}_\Omega;\mathscr{N}_T), \tag{42}$$

where the hidden constants are independent of the size of the elements in the mesh $\mathscr{T}$ and $\#\mathscr{T}$.

6 Numerical example

We illustrate the performance of the a posteriori error estimator with a numerical example. We set $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$, $\mathsf{a} = 0$, $\mathsf{b} = 1$ and the data f and y_Ω to be such that, in polar coordinates (r,θ) with $\theta \in [0, 3\pi/2]$,

$$\bar{\mathsf{y}} = (1-r^2\cos^2(\theta))(1-r^2\sin^2(\theta))r^{2/3}\sin(2\theta/3)$$

and

$$\bar{\mathsf{p}} = \sin(2\pi r\cos(\theta))\sin(2\pi r\sin(\theta))r^{2/3}\sin(2\theta/3).$$

A sequence of adaptively refined meshes was generated from an initial mesh (consisting of 12 congruent triangles) by using a maximum strategy to mark elements for refinement. The number of degrees of freedom Ndof is three times the number of vertices in the mesh. Figure 1 shows the results and we can observe that, once the mesh has been sufficiently refined, the error and estimator converge at the optimal rate.

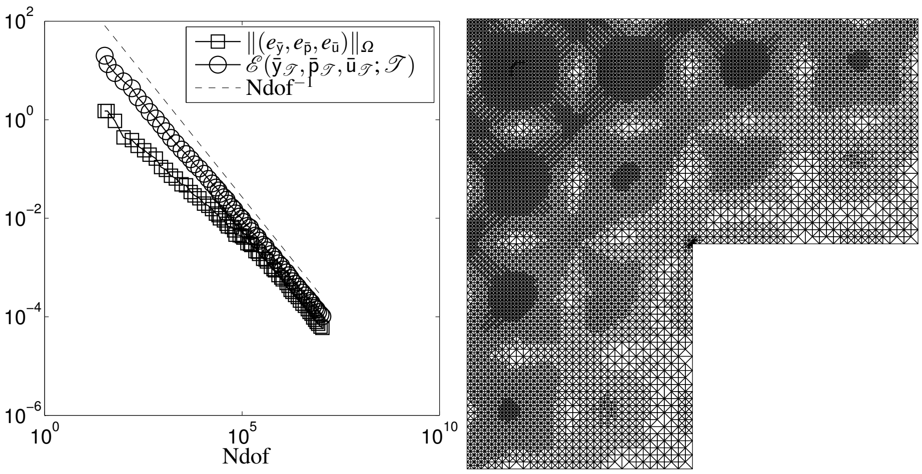

Fig. 1 The error $\|(e_{\bar{\mathsf{y}}},e_{\bar{\mathsf{p}}},e_{\bar{\mathsf{u}}})\|_\Omega$ and the estimator $\mathscr{E}(\bar{\mathsf{y}}_{\mathscr{T}},\bar{\mathsf{p}}_{\mathscr{T}},\bar{\mathsf{u}}_{\mathscr{T}};\mathscr{T})$ (left) and the 24th adaptively refined mesh (right).

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