A PDE APPROACH TO FRACTIONAL DIFFUSION IN GENERAL DOMAINS: A PRIORI ERROR ANALYSIS*

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Abstract. The purpose of this work is the study of solution techniques for problems involving fractional powers of symmetric coercive elliptic operators in a bounded domain with Dirichlet boundary conditions. These operators can be realized as the Dirichlet to Neumann map for a degenerate/singular elliptic problem posed on a semi-infinite cylinder, which we analyze in the framework of weighted Sobolev spaces. Motivated by the rapid decay of the solution of this problem, we propose a truncation that is suitable for numerical approximation. We discretize this truncation using first degree tensor product finite elements. We derive a priori error estimates in weighted Sobolev spaces. The estimates exhibit optimal regularity but suboptimal order for quasi-uniform meshes. For anisotropic meshes, instead, they are quasi-optimal in both order and regularity. We present numerical experiments to illustrate the method's performance.

Key words. Fractional diffusion; finite elements; nonlocal operators; degenerate and singular equations; second order elliptic operators; anisotropic elements.

AMS subject classifications. 35S15; 65R20; 65N12; 65N30.

1. Introduction.

Singular integrals and nonlocal operators have been an active area of research in different branches of mathematics such as operator theory and harmonic analysis (see [59]). In addition, they have received significant attention because of their strong connection with real-world problems, since they constitute a fundamental part of the modeling and simulation of complex phenomena that span vastly different length scales.

Nonlocal operators arise in a number of applications such as: boundary control problems [33], finance [23], electromagnetic fluids [49], image processing [38], materials science [8], optimization [33], porous media flow [27], turbulence [5], peridynamics [58], nonlocal continuum field theories [34] and others. Therefore the domain of definition Ω could be rather general.

To make matters precise, in this work we shall be interested in fractional powers of the Dirichlet Laplace operator $(-\Delta)^s$, with $s \in (0,1)$, which for convenience we will simply call the fractional Laplacian. In other words, we shall be concerned with the following problem. Let Ω be an open and bounded subset of $\mathbb{R}^n$ $(n \geq 1)$, with boundary $\partial\Omega$. Given $s \in (0,1)$ and a smooth enough function f, find u such that

$$\begin{cases} (-\Delta)^s u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Our approach, however, is by no means particular to the fractional Laplacian. In

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section 7 we will discuss how, with little modification, our developments can be applied to a general second order, symmetric and uniformly elliptic operator.

The study of boundary value problems involving the fractional Laplacian is important in physical applications where long range or anomalous diffusion is considered. For instance, in the flow in porous media, it is used when modeling the transport of particles that experience very large transitions arising from high heterogeneity and very long spatial autocorrelation (see [10]). In the theory of stochastic processes, the fractional Laplacian is the infinitesimal generator of a stable Lévy process (see [12]).

One of the main difficulties in the study of problem (1.1) is that the fractional Laplacian is a nonlocal operator (see [47, 21, 19]). To localize it, Caffarelli and Silvestre showed in [21] that any power of the fractional Laplacian in $\mathbb{R}^n$ can be realized as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem on the upper half-space $\mathbb{R}^{n+1}_+$. For a bounded domain Ω, the result by Caffarelli and Silvestre has been adapted in [22, 16, 60], thus obtaining an extension problem which is now posed on the semi-infinite cylinder $\mathcal{C} = \Omega \times (0,\infty)$. This extension is the following mixed boundary value problem:

$$\begin{cases} \operatorname{div}(y^\alpha \nabla \mathfrak{u}) = 0, & \text{in } \mathcal{C}, \\ \mathfrak{u} = 0, & \text{on } \partial_L \mathcal{C}, \\ \dfrac{\partial \mathfrak{u}}{\partial \nu^\alpha} = d_s f, & \text{on } \Omega \times \{0\}, \end{cases} \tag{1.2}$$

where $\partial_L \mathcal{C} = \partial\Omega \times [0,\infty)$ denotes the lateral boundary of $\mathcal{C}$, and

$$\frac{\partial \mathfrak{u}}{\partial \nu^\alpha} = -\lim_{y\to 0^+} y^\alpha \mathfrak{u}_y, \tag{1.3}$$

is the the so-called conormal exterior derivative of $\mathfrak{u}$ with ν being the unit outer normal to $\mathcal{C}$ at $\Omega \times \{0\}$. The parameter α is defined as

$$\alpha = 1 - 2s \in (-1, 1). \tag{1.4}$$

Finally, d_s is a positive normalization constant which depends only on s; see [21] for details. We will call y the *extended variable* and the dimension $n+1$ in $\mathbb{R}^{n+1}_+$ the *extended dimension* of problem (1.2).

The limit in (1.3) must be understood in the distributional sense; see [16, 19, 21] or section 2 for more details. As noted in [21, 22, 60], the fractional Laplacian and the Dirichlet to Neumann operator of problem (1.2) are related by

$$d_s(-\Delta)^s u = \frac{\partial \mathfrak{u}}{\partial \nu^\alpha} \quad \text{in } \Omega.$$

Using the aforementioned ideas, we propose the following strategy to find the solution of (1.1): given a sufficiently smooth function f we solve (1.2), thus obtaining a function $\mathfrak{u} : (x', y) \in \mathcal{C} \mapsto \mathfrak{u}(x', y) \in \mathbb{R}$. Setting $u : x' \in \Omega \mapsto u(x') = \mathfrak{u}(x', 0) \in \mathbb{R}$, we obtain the solution of (1.1). The purpose of this work is then to make these ideas rigorous and to analyze a discretization scheme, which consists of approximating the solution of (1.2) via first degree tensor product finite elements. We will show suboptimal error estimates for quasi-uniform discretizations of (1.2) in suitable weighted Sobolev spaces and quasi-optimal error estimates using anisotropic elements.

The main advantage of the proposed algorithm is that we solve the local problem (1.2) instead of dealing with the nonlocal operator $(-\Delta)^s$ of problem (1.1). However, this comes at the expense of incorporating one more dimension to the problem, and raises questions about computational efficiency. The development of efficient computational techniques for the solution of problem (1.2) and issues such as multilevel methods, a posteriori error analysis and adaptivity will be deferred to future reports. In this paper we carry out a complete a priori error analysis of the discretization scheme.

Before proceeding with the analysis of our method, it is instructive to compare it with those advocated in the literature. First of all, for a general Lipschitz domain $\Omega \subset \mathbb{R}^n$ ($n > 1$), we may think of solving problem (1.1) via a spectral decomposition of the operator $-\Delta$. However, to have a sufficiently good approximation, this requires the solution of a large number of eigenvalue problems which, in general, is very time consuming. In [42, 43] the authors studied computationally problem (1.1) in the one-dimensional case and with boundary conditions of Dirichlet, Neumann and Robin type, and introduced the so-called matrix transference technique (MTT). Basically, MTT computes a spatial discretization of the fractional Laplacian by first finding a matrix approximation, A, of the Laplace operator (via finite differences or finite elements) and then computing the s-th power of this matrix. This requires diagonalization of A which, again, amounts to the solution of a large number of eigenvalue problems. For the case $\Omega = (0,1)^2$ and $s \in (1/2, 1)$, [62] applies the MTT technique and avoids diagonalization of A by writing a numerical scheme in terms of the product of a function of the matrix and a vector, $f(A)b$, where b is a suitable vector. This product is then approximated by a preconditioned Lanczos method. Under the same setting, the work [18] makes a computational comparison of three techniques for the computation of $f(A)b$: the contour integral method, extended Krylov subspace methods and the pre-assigned poles and interpolation nodes method.

The outline of this paper is as follows. In § 2 we introduce the functional framework that is suitable for the study of problems (1.1) and (1.2). We recall the definition of the fractional Laplacian on a bounded domain via spectral theory and, in addition, in § 2.6 we study regularity of the solution to (1.2). The numerical analysis of (1.1) begins in § 3. Here we introduce a truncation of problem (1.2) and study some properties of its solution. Having understood the truncation we proceed, in § 4, to study its finite element approximation. We prove interpolation estimates in weighted Sobolev spaces, under mild shape regularity assumptions that allow us to consider anisotropic elements in the extended variable y. Based on the regularity results of § 2.6 we derive, in § 5, a priori error estimates for quasi-uniform meshes which exhibit optimal regularity but suboptimal order. To restore optimal decay, we resort to the so-called principle of error equidistribution and construct graded meshes in the extended variable y. They in turn capture the singular behavior of the solution to (1.2) and allow us to prove a quasi-optimal rate of convergence with respect to both regularity and degrees of freedom. In § 6, to illustrate the method's performance and theory, we provide several numerical experiments. Finally, in § 7 we show that our developments apply to general second order, symmetric and uniformly elliptic operators.

2. Notation and preliminaries. Throughout this work Ω is an open, bounded and connected subset of $\mathbb{R}^n$, $n \geq 1$, with Lipschitz boundary $\partial\Omega$. We define the semi-infinite cylinder

$$\mathcal{C} = \Omega \times (0, \infty), \tag{2.1}$$

and its lateral boundary

$$\partial_L \mathcal{C} = \partial\Omega \times [0, \infty). \tag{2.2}$$

Given $\mathcal{Y} > 0$, we define the truncated cylinder

$$\mathcal{C}_{\mathcal{Y}} = \Omega \times (0, \mathcal{Y}). \tag{2.3}$$

The lateral boundary $\partial_L \mathcal{C}_{\mathcal{Y}}$ is defined accordingly.

Throughout our discussion we will be dealing with objects defined in $\mathbb{R}^{n+1}$ and it will be convenient to distinguish the extended dimension, as it plays a special rôle. A vector $x \in \mathbb{R}^{n+1}$, will be denoted by

$$x = (x^1, \ldots, x^n, x^{n+1}) = (x', x^{n+1}) = (x', y),$$

with $x^i \in \mathbb{R}$ for $i = 1, \ldots, n+1$, $x' \in \mathbb{R}^n$ and $y \in \mathbb{R}$. The upper half-space in $\mathbb{R}^{n+1}$ will be denoted by

$$\mathbb{R}^{n+1}_+ = \{x = (x', y) : x' \in \mathbb{R}^n \; y \in \mathbb{R}, \; y > 0\}.$$

Let $\gamma = (\gamma^1, \gamma^2) \in \mathbb{R}^2$ and $z \in \mathbb{R}^{n+1}$, the binary operation $\odot : \mathbb{R}^2 \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is defined by

$$\gamma \odot z = (\gamma^1 z', \gamma^2 z^{n+1}) \in \mathbb{R}^{n+1}. \tag{2.4}$$

The relation $a \lesssim b$ indicates that $a \leq Cb$, with a constant C that does not depend on neither a nor b but it might depend on s and Ω. The value of C might change at each occurrence. Given two objects X and Y in the same category, we write $X \hookrightarrow Y$ to indicate the existence of a monomorphism between them. Generally, these will be objects in some subcategory of the topological vector spaces (metric, normed, Banach, Hilbert spaces) and, in this case, the monomorphism is nothing more than continuous embedding. If X is a vector space, we denote by X' its dual.

2.1. Fractional Sobolev spaces. Let us recall some function spaces; for details the reader is referred to [48, 50, 28, 61]. For $0 < s < 1$, we introduce the so-called Gagliardo-Slobodeckiĭ seminorm

$$|w|^2_{H^s(\Omega)} = \int_\Omega \int_\Omega \frac{|w(x_1') - w(x_2')|^2}{|x_1' - x_2'|^{n+2s}} \, \mathrm{d}x_1' \, \mathrm{d}x_2'.$$

The Sobolev space $H^s(\Omega)$ of order s is defined by

$$H^s(\Omega) = \left\{ w \in L^2(\Omega) : |w|_{H^s(\Omega)} < \infty \right\}, \tag{2.5}$$

which equipped with the norm

$$\|u\|_{H^s(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + |u|^2_{H^s(\Omega)} \right)^{\frac{1}{2}},$$

is a Hilbert space. An equivalent construction of $H^s(\Omega)$ is obtained by restricting functions in $H^s(\mathbb{R}^n)$ to Ω (cf. [61, Chapter 34]). The space $H^s_0(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^s(\Omega)}$, i.e.,

$$H^s_0(\Omega) = \overline{C_0^\infty(\Omega)}^{H^s(\Omega)}. \tag{2.6}$$

If the boundary of Ω is smooth, an equivalent approach to define fractional Sobolev spaces is given by interpolation in [48, Chapter 1]. Set $H^0(\Omega) = L^2(\Omega)$, then Sobolev spaces with real index $0 \leq s \leq 1$ can be defined as interpolation spaces of index $\theta = 1 - s$ for the pair $[H^1(\Omega), L^2(\Omega)]$, that is

$$H^s(\Omega) = \left[H^1(\Omega), L^2(\Omega)\right]_\theta . \tag{2.7}$$

Analogously, for $s \in [0,1] \setminus \{\frac{1}{2}\}$, the spaces $H_0^s(\Omega)$ are defined as interpolation spaces of index $\theta = 1 - s$ for the pair $[H_0^1(\Omega), L^2(\Omega)]$, in other words

$$H_0^s(\Omega) = \left[H_0^1(\Omega), L^2(\Omega)\right]_\theta , \quad \theta \neq \tfrac{1}{2}. \tag{2.8}$$

The space $[H_0^1(\Omega), L^2(\Omega)]_{\frac{1}{2}}$ is the so-called *Lions-Magenes* space,

$$H_{00}^{\frac{1}{2}}(\Omega) = \left[H_0^1(\Omega), L^2(\Omega)\right]_{\frac{1}{2}} ,$$

which can be characterized as

$$H_{00}^{\frac{1}{2}}(\Omega) = \left\{ w \in H^{\frac{1}{2}}(\Omega) : \int_\Omega \frac{w^2(x')}{\operatorname{dist}(x', \partial\Omega)} \, \mathrm{d}x' < \infty \right\}, \tag{2.9}$$

see [48, Theorem 11.7]. Moreover, we have the strict inclusion $H_{00}^{1/2}(\Omega) \subsetneq H_0^{1/2}(\Omega)$ because $1 \in H_0^{1/2}(\Omega)$ but $1 \notin H_{00}^{1/2}(\Omega)$. If the boundary of Ω is Lipschitz, the characterization (2.9) is equivalent to the definition via interpolation, and definitions (2.7) and (2.8) are also equivalent to definitions (2.5) and (2.6), respectively. To see this, it suffices to notice that when $\Omega = \mathbb{R}^n$ these definitions yield identical spaces and equivalent norms; see [3, Chapter 7]. Consequently, using the well-known extension result of Stein [59] for Lipschitz domains, we obtain the asserted equivalence (see [3, Chapter 7] for details).

When the boundary of Ω is Lipschitz, the space $C_0^\infty(\Omega)$ is dense in $H^s(\Omega)$ if and only if $s \leq \frac{1}{2}$, i.e., $H_0^s(\Omega) = H^s(\Omega)$. If $s > \frac{1}{2}$, we have that $H_0^s(\Omega)$ is strictly contained in $H^s(\Omega)$; see [48, Theorem 11.1]. In particular, we have the inclusions $H_{00}^{1/2}(\Omega) \subsetneq H_0^{1/2}(\Omega) = H^{1/2}(\Omega)$.

2.2. The fractional Laplace operator. It is important to mention that there is no unique way of defining a nonlocal operator related to the fractional Laplacian in a bounded domain. A first possibility is to suitably extend the functions to the whole space $\mathbb{R}^n$ and use Fourier transform

$$\mathcal{F}((-\Delta)^s w)(\xi') = |\xi'|^{2s} \mathcal{F}(w)(\xi').$$

After extension, the following point-wise formula also serves as a definition of the fractional Laplacian

$$(-\Delta)^s w(x') = C_{n,s} \text{p.v.} \int_{\mathbb{R}^n} \frac{w(x') - w(z')}{|x' - z'|^{n+2s}} \, \mathrm{d}z', \tag{2.10}$$

where p.v. stands for the Cauchy principal value and $C_{n,s}$ is a positive normalization constant that depends only on n and s which is introduced to guarantee that the symbol of the resulting operator is $|\xi'|^{2s}$. For details we refer the reader to [19, 47, 28] and, in particular, to [47, Section 1.1] or [28, Proposition 3.3] for a proof of the equivalence of these two definitions.

Even if we restrict ourselves to definitions that do not require extension, there is more than one possibility. For instance, the so-called regional fractional Laplacian ([40, 14]) is defined by restricting the Riesz integral to Ω, leading to an operator related to a Neumann problem. A different operator is obtained by using the spectral decomposition of the Dirichlet Laplace operator $-\Delta$, see [16, 20, 22]. This approach is also different to the integral formula (2.10). Indeed, the spectral definition depends on the domain Ω considered, while the integral one at any point is independent of the domain in which the equation is set. For more details see the discussion in [57].

The definition that we shall adopt is as in [16, 20, 22] and is based on the spectral theory of the Dirichlet Laplacian ([35, 37]) as we summarize below.

We define $-\Delta : L^2(\Omega) \to L^2(\Omega)$ with domain $\mathrm{Dom}(-\Delta) = \{v \in H_0^1(\Omega) : \Delta v \in L^2(\Omega)\}$. This operator is positive, unbounded, closed and its inverse is compact. This implies that the spectrum of the operator $-\Delta$ is discrete, positive and accumulates at infinity. Moreover, there exist $\{\lambda_k, \varphi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+ \times H_0^1(\Omega)$ such that $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and, for $k \in \mathbb{N}$,

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k, & \text{in } \Omega, \\ \varphi_k = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

Consequently, $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthogonal basis of $H_0^1(\Omega)$ and $\|\nabla_{x'}\varphi_k\|_{L^2(\Omega)} = \sqrt{\lambda_k}$.

With this spectral decomposition at hand, fractional powers of the Dirichlet Laplacian $(-\Delta)^s$ can be defined for $u \in C_0^\infty(\Omega)$ by

$$(-\Delta)^s u = \sum_{k=1}^{\infty} u_k \lambda_k^s \varphi_k, \tag{2.12}$$

where the coefficients u_k are defined by $u_k = \int_\Omega u\varphi_k$. Therefore, if $f = \sum_{k=1}^\infty f_k\varphi_k$, and $(-\Delta)^s u = f$, then $u_k = \lambda_k^{-s} f_k$ for all $k \geq 1$.

By density the operator $(-\Delta)^s$ can be extended to the Hilbert space

$$\mathbb{H}^s(\Omega) = \left\{ w = \sum_{k=1}^{\infty} w_k\varphi_k \in L^2(\Omega) : \|w\|^2_{\mathbb{H}^s(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^s |w_k|^2 < \infty \right\}.$$

The theory of Hilbert scales presented in [48, Chapter 1] shows that

$$\left[H_0^1(\Omega), L^2(\Omega)\right]_\theta = \mathrm{Dom}(-\Delta)^{\frac{s}{2}},$$

where $\theta = 1 - s$. This implies the following characterization of the space $\mathbb{H}^s(\Omega)$,

$$\mathbb{H}^s(\Omega) = \begin{cases} H^s(\Omega), & s \in (0, \frac{1}{2}), \\ H_{00}^{1/2}(\Omega), & s = \frac{1}{2}, \\ H_0^s(\Omega), & s \in (\frac{1}{2}, 1). \end{cases} \tag{2.13}$$

We denote by $\mathbb{H}^{-s}(\Omega)$ the dual space of $\mathbb{H}^s(\Omega)$ for $0 < s < 1$.

2.3. Weighted Sobolev spaces. To exploit the Caffarelli-Silvestre extension [21], or its variants [16, 20, 22], we need to deal with a degenerate/singular elliptic equation on $\mathbb{R}_+^{n+1}$. To this end, we consider weighted Sobolev spaces (see, for instance, [36, 41, 46]), with the specific weight $|y|^\alpha$ with $\alpha \in (-1, 1)$.

Let $\mathcal{D} \subset \mathbb{R}^{n+1}$ be an open set and $\alpha \in (-1,1)$. We define $L^2(\mathcal{D},|y|^\alpha)$ as the space of all measurable functions defined on $\mathcal{D}$ such that

$$\|w\|^2_{L^2(\mathcal{D},|y|^\alpha)} = \int_{\mathcal{D}} |y|^\alpha w^2 < \infty.$$

Similarly we define the weighted Sobolev space

$$H^1(\mathcal{D},|y|^\alpha) = \left\{ w \in L^2(\mathcal{D},|y|^\alpha) : |\nabla w| \in L^2(\mathcal{D},|y|^\alpha) \right\},$$

where ∇w is the distributional gradient of w. We equip $H^1(\mathcal{D},|y|^\alpha)$ with the norm

$$\|w\|_{H^1(\mathcal{D},|y|^\alpha)} = \left(\|w\|^2_{L^2(\mathcal{D},|y|^\alpha)} + \|\nabla w\|^2_{L^2(\mathcal{D},|y|^\alpha)} \right)^{\frac{1}{2}}. \qquad (2.14)$$

Notice that taking $\alpha = 0$ in the definition above, we obtain the classical $H^1(\mathcal{D})$.

Properties of this weighted Sobolev space can be found in classical references like [41, 46]. It is remarkable that most of the properties of classical Sobolev spaces have a weighted counterpart not so because of the specific form of the weight but rather due to the fact that the weight $|y|^\alpha$ belongs to the so-called Muckenhoupt class $A_2(\mathbb{R}^{n+1})$; see [36, 39, 52]. We recall the definition of Muckenhoupt classes.

Definition 2.1 (Muckenhoupt class A_p). *Let ω be a positive and measurable function such that $\omega \in L^1_{loc}(\mathbb{R}^N)$ with $N \geq 1$. We say $\omega \in A_p(\mathbb{R}^N)$, $1 < p < \infty$, if there exists a positive constant $C_{p,\omega}$ such that*

$$\sup_B \left(\frac{1}{|B|} \int_B \omega \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)} \right)^{p-1} = C_{p,\omega} < \infty, \qquad (2.15)$$

where the supremum is taken over all balls B in $\mathbb{R}^N$ and $|B|$ denotes the Lebesgue measure of B.

Since $\alpha \in (-1,1)$ it is immediate that $|y|^\alpha \in A_2(\mathbb{R}^{n+1})$, which implies the following important result (see [39, Theorem 1]).

Proposition 2.2 (Properties of weighted Sobolev spaces). *Let $\mathcal{D} \subset \mathbb{R}^{n+1}$ be an open set and $\alpha \in (-1,1)$. Then $H^1(\mathcal{D},|y|^\alpha)$, equipped with the norm (2.14), is a Hilbert space. Moreover, the set $C^\infty(\mathcal{D}) \cap H^1(\mathcal{D},|y|^\alpha)$ is dense in $H^1(\mathcal{D},|y|^\alpha)$.*

Remark 2.3 (Weighted L^2 vs L^1). If $\mathcal{D}$ is a bounded domain and $\alpha \in (-1,1)$ then, $L^2(\mathcal{D},|y|^\alpha) \subset L^1(\mathcal{D})$. Indeed, since $|y|^{-\alpha} \in L^1_{loc}(\mathbb{R}^{n+1})$,

$$\int_{\mathcal{D}} |w| = \int_{\mathcal{D}} |w||y|^{\alpha/2}|y|^{-\alpha/2} \leq \left(\int_{\mathcal{D}} |w|^2 |y|^\alpha \right)^{\frac{1}{2}} \left(\int_{\mathcal{D}} |y|^{-\alpha} \right)^{\frac{1}{2}} \lesssim \|w\|_{L^2(\mathcal{D},|y|^\alpha)}.$$

The following result is given in [46, Theorem 6.3]. For completeness we present here a version of the proof on the truncated cylinder $\mathcal{C}_{\mathcal{Y}}$, which will be important for the numerical approximation of problem (1.2).

Proposition 2.4 (Embeddings in weighted Sobolev spaces). *Let Ω be a bounded domain in $\mathbb{R}^n$ and $\mathcal{Y} > 0$. Then*

$$H^1(\mathcal{C}_{\mathcal{Y}}) \hookrightarrow H^1(\mathcal{C}_{\mathcal{Y}}, y^\alpha), \quad \textit{for } \alpha \in (0,1), \qquad (2.16)$$

and

$$H^1(\mathcal{C}_{\mathcal{Y}}, y^\alpha) \hookrightarrow H^1(\mathcal{C}_{\mathcal{Y}}), \quad \textit{for } \alpha \in (-1,0). \qquad (2.17)$$

Proof. Let us prove (2.16), the proof of (2.17) being similar. Since $\alpha > 0$ we have $y^\alpha \leq \mathcal{Y}^\alpha$, whence $y^\alpha w^2 \leq \mathcal{Y}^\alpha w^2$ and $y^\alpha |\nabla w|^2 \leq \mathcal{Y}^\alpha |\nabla w|^2$ a.e. on $\mathcal{C}_\mathcal{Y}$ for all $w \in H^1(\mathcal{C}_\mathcal{Y})$. This implies $\|w\|_{H^1(\mathcal{C}_\mathcal{Y}, y^\alpha)} \leq \sqrt{2}\mathcal{Y}^{\alpha/2} \|w\|_{H^1(\mathcal{C}_\mathcal{Y})}$, which is (2.16). □

Define

$$\mathring{H}^1_L(\mathcal{C}, y^\alpha) = \left\{ w \in H^1(y^\alpha; \mathcal{C}) : w = 0 \text{ on } \partial_L \mathcal{C} \right\}. \tag{2.18}$$

This space can be equivalently defined as the set of measurable functions $w : \mathcal{C} \to \mathbb{R}$ such that $w \in H^1(\Omega \times (s,t))$ for all $0 < s < t < \infty$, $w = 0$ on $\partial_L \mathcal{C}$ and for which the following seminorm is finite

$$\|w\|^2_{\mathring{H}^1_L(\mathcal{C}, y^\alpha)} = \int_\mathcal{C} y^\alpha |\nabla w|^2; \tag{2.19}$$

see [22]. As a consequence of the usual Poincaré inequality, for any $k \in \mathbb{Z}$ and any function $w \in H^1(\Omega \times (2^k, 2^{k+1}))$ with $w = 0$ on $\partial\Omega \times (2^k, 2^{k+1})$, we have

$$\int_{\Omega \times (2^k, 2^{k+1})} y^\alpha w^2 \leq C_\Omega \int_{\Omega \times (2^k, 2^{k+1})} y^\alpha |\nabla w|^2, \tag{2.20}$$

where C_Ω denotes a positive constant that depends only on Ω. Summing up over $k \in \mathbb{Z}$, we obtain the following *weighted Poincaré inequality*:

$$\int_\mathcal{C} y^\alpha w^2 \lesssim \int_\mathcal{C} y^\alpha |\nabla w|^2. \tag{2.21}$$

Hence, the seminorm (2.19) is a norm on $\mathring{H}^1_L(\mathcal{C}, y^\alpha)$, equivalent to (2.14).

For a function $w \in H^1(\mathcal{C}, y^\alpha)$, we shall denote by $\operatorname{tr}_\Omega w$ its trace onto $\Omega \times \{0\}$. It is well known that $\operatorname{tr}_\Omega H^1(\mathcal{C}) = H^{1/2}(\Omega)$; see [3, 61]. In the subsequent analysis we need a characterization of the trace of functions in $H^1(\mathcal{C}, y^\alpha)$. For a smooth domain this was given in [20, Proposition 1.8] for $s = 1/2$ and in [22, Proposition 2.1] for any $s \in (0,1) \setminus \{\frac{1}{2}\}$. However, since the eigenvalue decomposition (2.12) of the Dirichlet Laplace operator holds true on a Lipschitz domain, we are able to extend this trace characterization to such domains. In summary, we have the following result.

Proposition 2.5 (Characterization of $\operatorname{tr}_\Omega \mathring{H}^1_L(\mathcal{C}, y^\alpha)$). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. The trace operator tr_Ω satisfies $\operatorname{tr}_\Omega \mathring{H}^1_L(\mathcal{C}, y^\alpha) = \mathbb{H}^s(\Omega)$ and*

$$\|\operatorname{tr}_\Omega v\|_{\mathbb{H}^s(\Omega)} \lesssim \|v\|_{\mathring{H}^1_L(\mathcal{C}, y^\alpha)} \qquad \forall\, v \in \mathring{H}^1_L(\mathcal{C}, y^\alpha),$$

where the space $\mathbb{H}^s(\Omega)$ is defined in (2.13) and $\alpha = 1 - 2s$.

2.4. The Caffarelli-Silvestre extension problem. It has been shown in [21] that any power of the fractional Laplacian in $\mathbb{R}^n$ can be determined as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem posed on $\mathbb{R}^{n+1}_+$. For a bounded domain, an analogous result has been obtained in [20] for $s = \frac{1}{2}$, and in [16, 22, 60] for any $s \in (0,1)$.

Let us briefly describe these results. Consider a function u defined on Ω. We define the α-harmonic extension of u to the cylinder $\mathcal{C}$, as the function $\mathfrak{u}$ that solves the boundary value problem

$$\begin{cases} \operatorname{div}(y^\alpha \nabla \mathfrak{u}) = 0, & \text{in } \mathcal{C}, \\ \mathfrak{u} = 0, & \text{on } \partial_L \mathcal{C}, \\ \mathfrak{u} = u, & \text{on } \Omega \times \{0\}. \end{cases} \tag{2.22}$$

From Proposition 2.5 and the Lax Milgram lemma we can conclude that this problem has a unique solution $\mathfrak{u} \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$ whenever $u \in \mathbb{H}^s(\Omega)$. We define the *Dirichlet-to-Neumann* operator $\Gamma_{\alpha,\Omega} : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega)$

$$u \in \mathbb{H}^s(\Omega) \longmapsto \Gamma_{\alpha,\Omega}(u) = \frac{\partial \mathfrak{u}}{\partial \nu^\alpha} \in \mathbb{H}^{-s}(\Omega),$$

where $\mathfrak{u}$ solves (2.22) and $\frac{\partial \mathfrak{u}}{\partial \nu^\alpha}$ is given in (1.3). The space $\mathbb{H}^{-s}(\Omega)$ can be characterized as the space of distributions $h = \sum_k h_k \varphi_k$ such that $\sum_k |h_k|^2 \lambda_k^{-s} < \infty$. The fundamental result of [21], see also [22, Lemma 2.2], is stated below.

Theorem 2.6 (Caffarelli–Silvestre extension). *If $s \in (0,1)$ and $u \in \mathbb{H}^s(\Omega)$, then*

$$d_s(-\Delta)^s u = \Gamma_{\alpha,\Omega}(u),$$

in the sense of distributions. Here $\alpha = 1 - 2s$ and d_s is given by

$$d_s = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}. \tag{2.23}$$

It seems remarkable that he constant d_s does not depend on the dimension. This was proved originally in [21] and its precise value appears in several references, for instance [16, 19].

The relation between the fractional Laplacian and the extension problem is now clear. Given $f \in \mathbb{H}^{-s}(\Omega)$, a function $u \in \mathbb{H}^s(\Omega)$ solves (1.1) if and only if its α-harmonic extension $\mathfrak{u} \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$ solves (1.2).

If $u = \sum_k u_k \varphi_k$, then, as shown in the proofs of [22, Proposition 2.1] and [16, Lemma 2.2], $\mathfrak{u}$ can be expressed as

$$\mathfrak{u}(x) = \sum_{k=1}^{\infty} u_k \varphi_k(x')\psi_k(y), \tag{2.24}$$

where the functions ψ_k solve

$$\begin{cases} \psi_k'' + \dfrac{\alpha}{y}\psi_k' - \lambda_k \psi_k = 0, & \text{in } (0,\infty), \\ \psi_k(0) = 1, & \lim\limits_{y\to\infty} \psi_k(y) = 0. \end{cases} \tag{2.25}$$

If $s = \frac{1}{2}$, then clearly $\psi_k(y) = e^{-\sqrt{\lambda_k}y}$ (see [20, Lemma 2.10]). For $s \in (0,1) \setminus \{\frac{1}{2}\}$ instead (cf. [22, Proposition 2.1])

$$\psi_k(y) = c_s\left(\sqrt{\lambda_k}y\right)^s K_s(\sqrt{\lambda_k}y),$$

where K_s denotes the modified Bessel function of the second kind (see [1, Chapter 9.6]). Using the condition $\psi_k(0) = 1$, and formulas for small arguments of the function K_s (see for instance § 2.5) we obtain

$$c_s = \frac{2^{1-s}}{\Gamma(s)}.$$

The function $\mathfrak{u} \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$ is the unique solution of

$$\int_{\mathcal{C}} y^\alpha \nabla \mathfrak{u} \cdot \nabla \phi = d_s \langle f, \operatorname{tr}_\Omega \phi \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)}, \quad \forall \phi \in \mathring{H}^1_L(\mathcal{C}, y^\alpha), \tag{2.26}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)}$ denotes the duality pairing between $\mathbb{H}^s(\Omega)$ and $\mathbb{H}^{-s}(\Omega)$ which, in light of Proposition 2.5 is well defined for all $f \in \mathbb{H}^{-s}(\Omega)$ and $\phi \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$. This implies the following equalities (see [22, Proposition 2.1] for $s \in (0,1) \setminus \{\frac{1}{2}\}$ and [20, Proposition 2.1] for $s = \frac{1}{2}$):

$$\|\mathfrak{u}\|^2_{\mathring{H}^1_L(\mathcal{C},y^\alpha)} = d_s \|u\|^2_{\mathbb{H}^s(\Omega)} = d_s \|f\|^2_{\mathbb{H}^{-s}(\Omega)}. \tag{2.27}$$

Notice that for $s = \frac{1}{2}$, or equivalently $\alpha = 0$, problem (2.26) reduces to the weak formulation of the Laplace operator with mixed boundary conditions, which is posed on the classical Sobolev space $\mathring{H}^1_L(\mathcal{C})$. Therefore, the value $s = \frac{1}{2}$ becomes a special case for problem (2.26). In addition, $d_{1/2} = 1$, and $\|\mathfrak{u}\|_{\mathring{H}^1_L(\mathcal{C})} = \|u\|_{H^{1/2}_{00}(\Omega)}$.

At this point it is important to give a precise meaning to the Dirichlet boundary condition in (1.1). For $s = \frac{1}{2}$, the boundary condition is interpreted in the sense of the Lions–Magenes space. If $\frac{1}{2} < s \leq 1$, there is a trace operator from $\mathbb{H}^s(\Omega)$ into $L^2(\partial\Omega)$ and the boundary condition can be interpreted in this sense. For $0 < s < 1/2$ this interpretation is no longer possible and thus, for an arbitrary $f \in \mathbb{H}^{-s}(\Omega)$ the boundary condition does not have a clear meaning. For instance, for every $s \in (0, \frac{1}{2})$, $f = (-\Delta)^s 1 \in \mathbb{H}^{-s}(\Omega)$ and the solution to (1.1) for this right hand side is $u = 1$. If $f \in H^\zeta(\Omega)$ with $\zeta > \frac{1}{2} - 2s > -s$, using that $(-\Delta)^s$ is a pseudo-differential operator of order $2s$ a shift-type result is valid, i.e., $u \in H^\varrho(\Omega)$ with $\varrho = \zeta + 2s > 1/2$. In this case, the trace of u on $\partial\Omega$ is well defined and the boundary condition is meaningful. Finally, we comment that it has been proved in [22, Lemma 2.10], that if $f \in L^\infty(\Omega)$ then the solution of (1.1) belongs to $C^{0,\varkappa}(\overline{\Omega})$ with $\varkappa \in (0, \min\{2s, 1\})$.

2.5. Asymptotic estimates. It is important to understand the behavior of the solution $\mathfrak{u}$ of problem (1.2), given by (2.24). Consequently, it becomes necessary to recall some of the main properties of the modified Bessel function of the second kind $K_\nu(z)$, $\nu \in \mathbb{R}$; see [1, Chapter 9.6] for (i)-(iv) and [51, Theorem 5] for (v):

(i) For $\nu > -1$, $K_\nu(z)$ is real and positive.

(ii) For $\nu \in \mathbb{R}$, $K_\nu(z) = K_{-\nu}(z)$.

(iii) For $\nu > 0$,

$$\lim_{z \downarrow 0} \frac{K_\nu(z)}{\frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu}} = 1. \tag{2.28}$$

(iv) For $k \in \mathbb{N}$,

$$\left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^k (z^\nu K_\nu(z)) = (-1)^k z^{\nu-k} K_{\nu-k}(z).$$

In particular, for $k = 1$ and $k = 2$, respectively, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^\nu K_\nu(z)) = -z^\nu K_{\nu-1}(z) = -z^\nu K_{1-\nu}(z), \tag{2.29}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}(z^\nu K_\nu(z)) = z^\nu K_{2-\nu}(z) - z^{\nu-1} K_{1-\nu}(z). \tag{2.30}$$

(v) For $z > 0$, $z^{\min\{\nu,1/2\}} e^z K_\nu(z)$ is a decreasing function.

As an application we obtain the following important properties of the function ψ_k, defined in (2.25). First, for $s \in (0,1)$, properties (ii), (iii) and (iv) imply

$$\lim_{y \downarrow 0^+} \frac{y^\alpha \psi_k'(y)}{d_s \lambda_k^s} = -1, \tag{2.31}$$

Property (v) provides the following asymptotic estimate for $s \in (0,1)$ and $y \geq 1$:

$$|y^\alpha \psi_k(y) \psi_k'(y)| \leq C(s) \lambda_k^s \left(\sqrt{\lambda_k} y\right)^{\left|s - \frac{1}{2}\right|} e^{-2\sqrt{\lambda_k} y}. \tag{2.32}$$

Multiplying the differential equation of problem (2.25) by $y^\alpha \psi_k(y)$ and integrating by parts yields

$$\int_a^b y^\alpha \left(\lambda_k \psi_k(y)^2 + \psi_k'(y)^2\right) \mathrm{d}y = y^\alpha \psi_k(y) \psi_k'(y)\big|_a^b , \tag{2.33}$$

where a and b are real and positive constants.

Let us conclude this section with some remarks on the asymptotic behavior of the function $\mathfrak{u}$ that solves (2.26). Using (2.24) we obtain

$$\mathfrak{u}(x)|_{y=0} = \sum_{k=1}^{\infty} u_k \varphi_k(x') \psi_k(0) = \sum_{k=1}^{\infty} u_k \varphi_k(x') = u(x').$$

For $s \in (0,1)$, using formula (2.31) together with (2.12), we arrive at

$$\frac{\partial \mathfrak{u}}{\partial \nu^\alpha}(x', 0) = -\lim_{y \downarrow 0} y^\alpha \mathfrak{u}_y(x', y) = d_s f(x'), \quad \text{on } \Omega \times \{0\}. \tag{2.34}$$

Notice that, if $s = \frac{1}{2}$, then $\alpha = 0$, $d_{1/2} = 1$ and thus (2.34) reduces to

$$\left.\frac{\partial \mathfrak{u}}{\partial \nu}\right|_{\Omega \times \{0\}} = f(x').$$

For $s \in (0,1) \setminus \{\frac{1}{2}\}$ the asymptotic behavior of the second derivative $\mathfrak{u}_{yy}$ as $y \approx 0^+$ is a consequence of (2.30) applied to the function $\psi_k(y)$. For $s = \frac{1}{2}$ the behavior follows from $\psi_k(y) = e^{-\sqrt{\lambda_k} y}$. In conclusion, for $y \approx 0^+$, we have

$$\mathfrak{u}_{yy} \approx y^{-\alpha - 1} \quad \text{for } s \in (0,1) \setminus \{\tfrac{1}{2}\}, \qquad \mathfrak{u}_{yy} \approx 1 \quad \text{for } s = \tfrac{1}{2}. \tag{2.35}$$

2.6. Regularity of the solution. Since we are interested in the approximation of the solution of problem (2.26), and this is closely related to its regularity, let us now study the behavior of its derivatives. According to (2.34), $\mathfrak{u}_y \approx y^{-\alpha}$ for $y \approx 0^+$. This clearly shows the necessity of introducing the weight, as this behavior, together with the exponential decay given by (v) of § 2.5, imply that $\mathfrak{u}_y \in L^2(\mathcal{C}, y^\alpha) \setminus L^2(\mathcal{C})$ for $s \in (0, 1/4]$.

However, the situation with second derivatives is much more delicate. To see this, let us first argue heuristically and compute how these derivatives scale with y. From the asymptotic formula (2.35), we see that, for $0 < \delta \ll 1$ and $s \in (0,1) \setminus \{\frac{1}{2}\}$,

$$\int_{\Omega \times (0,\delta)} y^\alpha |\mathfrak{u}_{yy}|^2 \, \mathrm{d}x' \, \mathrm{d}y \approx \int_0^\delta y^\alpha y^{-2-2\alpha} \, \mathrm{d}y = \int_0^\delta y^{-2-\alpha} \, \mathrm{d}y, \tag{2.36}$$

which, since $\alpha \in (-1,1) \setminus \{0\}$, does not converge. However,

$$\int_{\Omega \times (0,\delta)} y^{\beta} |\mathfrak{u}_{yy}|^2 \, \mathrm{d}x \, \mathrm{d}y \approx \int_0^{\delta} y^{\beta-2-2\alpha} \, \mathrm{d}y,$$

converges for $\beta > 2\alpha + 1$, hinting at the fact that $\mathfrak{u} \in H^2(\mathcal{C}, y^{\beta}) \setminus H^2(\mathcal{C}, y^{\alpha})$. The following result makes these considerations rigorous.

Theorem 2.7 (Global regularity of the α-harmonic extension). *Let $f \in \mathbb{H}^{1-s}(\Omega)$, where $\mathbb{H}^{1-s}(\Omega)$ is defined in (2.13) for $s \in (0,1)$. Let $\mathfrak{u} \in \mathring{H}^1_L(\mathcal{C}, y^{\alpha})$ solve (2.26) with f as data. Then, for $s \in (0,1) \setminus \{\frac{1}{2}\}$, we have*

$$\|\Delta_{x'} \mathfrak{u}\|^2_{L^2(\mathcal{C}, y^{\alpha})} + \|\partial_y \nabla_{x'} \mathfrak{u}\|^2_{L^2(\mathcal{C}, y^{\alpha})} = d_s \|f\|^2_{\mathbb{H}^{1-s}(\Omega)}, \tag{2.37}$$

and

$$\|\mathfrak{u}_{yy}\|_{L^2(\mathcal{C}, y^{\beta})} \lesssim \|f\|_{L^2(\Omega)},$$

with $\beta > 2\alpha + 1$. For the special case $s = \frac{1}{2}$, we obtain

$$\|\mathfrak{u}\|_{H^2(\mathcal{C})} \lesssim \|f\|_{\mathbb{H}^{1/2}(\Omega)}.$$

Remark 2.8 (Compatibility of f). It is possible to interpret the result of Theorem 2.7 as follows. Consider $s \in (\frac{1}{2}, 1)$, or equivalently $\alpha \in (-1, 0)$. Then the conormal exterior derivative condition for $\mathfrak{u}$ gives us that $\mathfrak{u}_y \approx -d_s y^{-\alpha} f$ as $y \approx 0^+$ on $\Omega \times \{0\}$, which in turn implies that $\mathfrak{u}_y \to 0$ as $y \to 0^+$ on $\Omega \times \{0\}$. This is compatible with $\mathfrak{u} = 0$ on $\partial_L \mathcal{C}$ since this implies $\mathfrak{u}_y = 0$ on $\partial_L \mathcal{C}$. Consequently, we do not need any compatibility condition on the data $f \in H^{1-s}(\Omega)$ to avoid a jump on the derivative $\mathfrak{u}_y$. On the other hand, when $\alpha \in (0,1)$, we have that, for a general f, $\mathfrak{u}_y \nrightarrow 0$ as $y \to 0^+$ on $\Omega \times \{0\}$. To compensate this behavior we need the data f to vanish at the boundary $\partial\Omega$ at a certain rate. This condition is expressed by the requirement $f \in H_0^{1-s}(\Omega)$.

Proof of Theorem 2.7. Let us first consider $s = \frac{1}{2}$. In this case (2.26) reduces to the Poisson problem with mixed boundary conditions. In general, the solution of a mixed boundary value problem is not smooth, even for C^{∞} data. The singular behavior occurs near the points of intersection between the Dirichlet and Neumann boundary. For instance, the solution $w = \sqrt{r} \sin(\theta/2)$ of $\Delta w = 0$ in $\mathbb{R}^2_+$, with $w_{x_2} = 0$ for $\{x_1 < 0,\ x_2 = 0\}$ and $w = 0$ for $\{x_1 \geq 0,\ x_2 = 0\}$ does not belong to $H^2(\mathbb{R}^2_+)$. To obtain more regular solutions, a compatibility condition between the data, the operator and the boundary must be imposed (see, for instance, [55]). Since in our case we have the representation (2.24), we can explicitly compute the second derivatives and, using that $\{\varphi_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and $\{\varphi_k / \sqrt{\lambda_k}\}_{k \in \mathbb{N}}$ of $H^1_0(\Omega)$, it is not difficult to show that $f \in H^{1/2}_{00}(\Omega)$ implies $\mathfrak{u} \in H^2(\mathcal{C})$, and $\|\mathfrak{u}\|_{H^2(\mathcal{C})} \lesssim \|f\|_{H^{1/2}_{00}(\Omega)}$.

In the general case $s \in (0,1) \setminus \{\frac{1}{2}\}$, i.e., $\alpha \in (-1,1) \setminus \{0\}$, using (2.33) as well as the asymptotic properties (2.31) and (2.32), we obtain

$$\begin{aligned}
\|\Delta_{x'} \mathfrak{u}\|^2_{L^2(\mathcal{C}, y^{\alpha})} + \|\partial_y \nabla_{x'} \mathfrak{u}\|^2_{L^2(\mathcal{C}, y^{\alpha})} &= \sum_{k=1}^{\infty} u_k^2 \lambda_k \int_0^{\infty} y^{\alpha} \left(\lambda_k \psi_k(y)^2 + \psi_k'(y)^2 \right) \mathrm{d}y \\
&= d_s \sum_{k=1}^{\infty} u_k^2 \lambda_k^{1+s} = d_s \sum_{k=1}^{\infty} f_k^2 \lambda_k^{1-s} = d_s \|f\|^2_{\mathbb{H}^{1-s}(\Omega)},
\end{aligned}$$

which is exactly the regularity estimate given in (2.37). To obtain the regularity estimate on $\mathfrak{u}_{yy}$ we, again, use the exact representation (2.24) and properties of Bessel functions to conclude that any derivative with respect to the extended variable y is smooth away from the Neumann boundary $\Omega \times \{0\}$. By virtue of (2.25) we deduce that the following partial differential equation holds in the strong sense

$$\operatorname{div}(y^\alpha \nabla \mathfrak{u}) = 0 \Longleftrightarrow \mathfrak{u}_{yy} = -\Delta_{x'}\mathfrak{u} - \frac{\alpha}{y}\mathfrak{u}_y. \tag{2.38}$$

Consider sequences $\{a_k = 1/\sqrt{\lambda_k}\}_{k\geq 1}$, $\{b_k\}_{k\geq 1}$ and $\{\delta_k\}_{k\geq 1}$ with $0 < \delta_k \leq a_k \leq b_k$. Using (2.24) we have, for $k \geq 1$,

$$\|\mathfrak{u}_{yy}\|^2_{L^2(\mathcal{C},y^\beta)} = \sum_{k=1}^{\infty} u_k^2 \left(\lim_{\delta_k \downarrow 0} \int_{\delta_k}^{a_k} y^\beta |\psi_k''(y)|^2 \,\mathrm{d}y + \lim_{b_k \uparrow \infty} \int_{a_k}^{b_k} y^\beta |\psi_k''(y)|^2 \,\mathrm{d}y \right) \tag{2.39}$$

Let us now estimate the first integral on the right hand side of (2.39). Formulas (2.30) and (2.28) yield

$$\begin{aligned}\lim_{\delta_k \downarrow 0} \int_{\delta_k}^{a_k} y^\beta |\psi_k''(y)|^2 \,\mathrm{d}y &= c_s^2 \lambda_k^{2-\beta/2-1/2} \lim_{\delta_k \downarrow 0} \int_{\sqrt{\lambda_k}\delta_k}^{1} z^\beta \left| \frac{\mathrm{d}^2}{\mathrm{d}z^2} \left(z^s K_s(z) \right) \right|^2 \mathrm{d}z \\ &\lesssim c_s^2 \lambda_k^{2-\beta/2-1/2} \lim_{\delta_k \downarrow 0} \int_{\sqrt{\lambda_k}\delta_k}^{1} z^{\beta-2-2\alpha} \,\mathrm{d}z \approx \lambda_k^{2-\beta/2-1/2}\end{aligned} \tag{2.40}$$

where the integral converges because $\beta > 2\alpha + 1$. Let us now look at the second integral. Using property (v) of the modified Bessel functions, we have

$$\begin{aligned}\lim_{b_k \uparrow \infty} \int_{a_k}^{b_k} y^\beta |\psi_k''(y)|^2 \,\mathrm{d}y &= c_s^2 \lambda_k^{2-\beta/2-1/2} \lim_{b_k \uparrow \infty} \int_{1}^{\sqrt{\lambda_k}b_k} z^\beta \left| \frac{\mathrm{d}^2}{\mathrm{d}z^2} \left(z^s K_s(z) \right) \right|^2 \mathrm{d}z \\ &\lesssim c_s^2 \lambda_k^{2-\beta/2-1/2}.\end{aligned} \tag{2.41}$$

Replacing (2.40) and (2.41) into (2.39), and using that $u_k = \lambda_k^{-s} f_k$, we deduce

$$\|\mathfrak{u}_{yy}\|^2_{L^2(\mathcal{C},y^\beta)} \lesssim \sum_{k=1}^{\infty} \lambda_k^{2-\beta/2-1/2-2s} f_k^2 \leq \|f\|^2_{L^2(\Omega)},$$

because $2 - 2s - \frac{\beta}{2} - \frac{1}{2} = \frac{1}{2}(1 + 2\alpha - \beta) < 0$. This concludes the proof. ◻

For the design of graded meshes later in § 5.2 we also need the following local regularity result in the extended variable.

THEOREM 2.9 (Local regularity of the α-harmonic extension). *Let $\mathcal{C}(a,b) := \Omega \times (a,b)$ for $0 \leq a < b \leq 1$. The solution $\mathfrak{u} \in \mathring{H}^1_L(\mathcal{C},y^\alpha)$ of (2.26) satisfies for all a, b*

$$\|\Delta_{x'}\mathfrak{u}\|^2_{L^2(\mathcal{C}(a,b),y^\alpha)} + \|\partial_y \nabla_{x'}\mathfrak{u}\|^2_{L^2(\mathcal{C}(a,b),y^\alpha)} \lesssim (b-a)\, \|f\|^2_{\mathbb{H}^{1-s}(\Omega)}, \tag{2.42}$$

and, with $\delta := \beta - 2\alpha - 1 > 0$,

$$\|\mathfrak{u}_{yy}\|^2_{L^2(\mathcal{C}(a,b),y^\beta)} \lesssim \left(b^\delta - a^\delta \right) \|f\|^2_{L^2(\Omega)}. \tag{2.43}$$

Proof. To derive (2.42) we proceed as in Theorem 2.7. Since $0 \leq a < b \leq 1$, property (iii) of § 2.5, together with (2.31) imply that

$$|y^\alpha \psi_k(y) \psi_k'(y)| \lesssim \lambda_k^s.$$

This, together with (2.33) and the property $u_k = \lambda_k^{-s} f_k$, allows us to conclude

$$\|\Delta_{x'} \mathfrak{u}\|^2_{L^2(\mathcal{C}(a,b),y^\alpha)} + \|\partial_y \nabla_{x'} \mathfrak{u}\|^2_{L^2(\mathcal{C}(a,b),y^\alpha)} = \sum_{k=1}^{\infty} u_k^2 \lambda_k \int_a^b y^\alpha \left(\lambda_k \psi_k(y)^2 + \psi_k'(y)^2 \right) \mathrm{d}y$$
$$\lesssim (b-a) \sum_{k=1}^{\infty} u_k^2 \lambda_k^{1+s} = (b-a) \|f\|^2_{\mathbb{H}^{1-s}(\Omega)}.$$

To prove (2.43) we observe that the same argument used in (2.40) gives

$$\int_a^b y^\beta \left| \psi_k''(y) \right|^2 \mathrm{d}y \lesssim \lambda_k^{2-\beta/2-1/2} \left(b^\delta - a^\delta \right),$$

whence

$$\|\mathfrak{u}_{yy}\|^2_{L^2(\mathcal{C}(a,b),y^\alpha)} \lesssim \left(b^\delta - a^\delta \right) \sum_{k=1}^{\infty} f_k^2 \lambda_k^{2-\beta/2-1/2-2s} \lesssim \left(b^\delta - a^\delta \right) \|f\|^2_{L^2(\Omega)},$$

because $2 - 2s - \frac{\beta}{2} - \frac{1}{2} < 0$. ∎

Remark 2.10 (Domain and data regularity). The results of Theorem 2.7 and Theorem 2.9 are meaningful only if $f \in \mathbb{H}^{1-s}(\Omega)$ and the domain Ω is such that

$$\|w\|_{H^2(\Omega)} \lesssim \|\Delta_{x'} w\|_{L^2(\Omega)}, \quad \forall w \in H^2(\Omega) \cap H^1_0(\Omega),$$

holds. In the analysis that follows we will, without explicit mention, make this assumption. Let us, however, remark that our method works even when these conditions are not satisfied. We refer to § 6.3 for an illustration of that case.

3. Truncation. The solution $\mathfrak{u}$ of problem (2.26) is defined on the infinite domain $\mathcal{C}$ and, consequently, it cannot be directly approximated with finite element-like techniques. In this section we will show that $\mathfrak{u}$ decays sufficiently fast – in fact exponentially – in the extended direction. This suggests truncating the cylinder $\mathcal{C}$ to $\mathcal{C}_\mathcal{Y}$, for a suitably defined $\mathcal{Y}$. The exponential decay is the content of the next result.

Proposition 3.1 (Exponential decay). *For every $\mathcal{Y} \geq 1$, the solution $\mathfrak{u}$ of* (2.26) *satisfies*

$$\|\nabla \mathfrak{u}\|_{L^2(\Omega \times (\mathcal{Y},\infty),y^\alpha)} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/2} \|f\|_{\mathbb{H}^{-s}(\Omega)}. \tag{3.1}$$

Proof. Recall that if $u \in \mathbb{H}^s(\Omega)$ has the decomposition $u = \sum_k u_k \varphi_k(x')$, the solution $\mathfrak{u} \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$ to (2.26) has the representation $\mathfrak{u} = \sum_k u_k \varphi(x') \psi_k(y)$, where the functions ψ_k solve (2.25).

Consider $s = \frac{1}{2}$. In this case $\psi_k(y) = e^{-\sqrt{\lambda_k} y}$. Using the fact that $\{\varphi_k\}_{k=1}^\infty$ are eigenfunctions of Dirichlet Laplacian on Ω, orthonormal in $L^2(\Omega)$ and orthogonal in $H^1_0(\Omega)$ with $\|\nabla_{x'} \varphi_k\|_{L^2(\Omega)} = \sqrt{\lambda_k}$, we get

$$\int_\mathcal{Y}^\infty \int_\Omega |\nabla \mathfrak{u}|^2 = \int_\mathcal{Y}^\infty \int_\Omega \left(|\nabla_{x'} \mathfrak{u}|^2 + |\partial_y \mathfrak{u}|^2 \right) = \sum_{k=1}^\infty \lambda_k^{\frac{1}{2}} |u_k|^2 e^{-2\sqrt{\lambda_k}\mathcal{Y}} \leq e^{-2\sqrt{\lambda_1}\mathcal{Y}} \|u\|^2_{\mathbb{H}^{1/2}(\Omega)}.$$

Since $\|u\|_{\mathbb{H}^{1/2}(\Omega)} = \|f\|_{\mathbb{H}^{-1/2}(\Omega)}$, this implies (3.1).

Consider now $s \in (0,1) \setminus \{\frac{1}{2}\}$ and $\psi_k(y) = c_s \left(\sqrt{\lambda_k} y\right)^s K_s(\sqrt{\lambda_k} y)$. To be able to argue as before, we need the estimates on K_s and its derivative for sufficiently large arguments discussed in § 2.5. In fact, using (2.32) and (2.33), we obtain

$$\begin{aligned}\int_{\mathcal{Y}}^{\infty} \int_{\Omega} y^{\alpha} |\nabla \mathfrak{u}|^2 &= \int_{\mathcal{Y}}^{\infty} y^{\alpha} \int_{\Omega} \left(|\nabla_{x'} \mathfrak{u}|^2 + |\partial_y \mathfrak{u}|^2 \right) \\ &= \sum_{k=1}^{\infty} |u_k|^2 \int_{\mathcal{Y}}^{\infty} y^{\alpha} \left(\lambda_k \psi_k(y)^2 + \psi_k'(y)^2 \right) \, \mathrm{d}y \\ &= \sum_{k=1}^{\infty} |u_k|^2 y^{\alpha} \psi_k(y) \psi_k'(y) \Big|_{\mathcal{Y}}^{\infty} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}} \|u\|_{\mathbb{H}^s(\Omega)}^2 .\end{aligned}$$

Again, since $\|u\|_{\mathbb{H}^s(\Omega)} = \|f\|_{\mathbb{H}^{-s}(\Omega)}$ we get (3.1). $\square$

Expression (3.1) motivates the approximation of $\mathfrak{u}$ by a function v that solves

$$\begin{cases} \operatorname{div}(y^{\alpha} \nabla v) = 0, & \text{in } \mathcal{C}_{\mathcal{Y}}, \\ v = 0, & \text{on } \partial_L \mathcal{C}_{\mathcal{Y}} \cup \Omega \times \{\mathcal{Y}\}, \\ \dfrac{\partial v}{\partial \nu^{\alpha}} = d_s f, & \text{on } \Omega \times \{0\}, \end{cases} \tag{3.2}$$

with $\mathcal{Y}$ sufficiently large. Problem (3.2) is understood in the weak sense, i.e., we define the space

$$\mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^{\alpha}) = \left\{ v \in H^1(\mathcal{C}, y^{\alpha}) : v = 0 \text{ on } \partial_L \mathcal{C}_{\mathcal{Y}} \cup \Omega \times \{\mathcal{Y}\} \right\},$$

and seek for $v \in \mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^{\alpha})$ such that

$$\int_{\mathcal{C}_{\mathcal{Y}}} y^{\alpha} \nabla v \cdot \nabla \phi = d_s \langle f, \operatorname{tr}_{\Omega} \phi \rangle, \quad \forall \phi \in \mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^{\alpha}). \tag{3.3}$$

Existence and uniqueness of v follows from the Lax-Milgram lemma.

Remark 3.2 (Zero extension). For every $\mathcal{Y} > 0$ we have the embedding

$$\mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^{\alpha}) \hookrightarrow \mathring{H}_L^1(\mathcal{C}, y^{\alpha}). \tag{3.4}$$

To see this, it suffices to consider the extension by zero for $y > \mathcal{Y}$.

The next result shows the approximation properties of v, solution of (3.3) in $\mathcal{C}_{\mathcal{Y}}$.

Lemma 3.3 (Exponential convergence in $\mathcal{Y}$). *For any positive $\mathcal{Y} \geq 1$, we have*

$$\|\nabla(\mathfrak{u} - v)\|_{L^2(\mathcal{C}_{\mathcal{Y}}, y^{\alpha})} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/4} \|f\|_{\mathbb{H}^{-s}(\Omega)}. \tag{3.5}$$

Proof. Given $\phi \in \mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^{\alpha})$ denote by ϕ_e its extension by zero to $\mathcal{C}$. By Remark 3.2, $\phi_e \in \mathring{H}_L^1(\mathcal{C}, y^{\alpha})$. Take ϕ_e and ϕ as test functions in (2.26) and (3.3), respectively. Subtract the resulting expressions to obtain

$$\int_{\mathcal{C}_{\mathcal{Y}}} y^{\alpha} (\nabla \mathfrak{u} - \nabla v) \cdot \nabla \phi = 0 \quad \forall \phi \in \mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^{\alpha}),$$

which implies that v is the best approximation of $\mathfrak{u}$ in $\mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^{\alpha})$, i.e.,

$$\|\nabla(\mathfrak{u} - v)\|_{L^2(\mathcal{C}_{\mathcal{Y}}, y^{\alpha})} = \inf_{\phi \in \mathring{H}_L^1(\mathcal{C}_{\mathcal{Y}}, y^{\alpha})} \|\nabla(\mathfrak{u} - \phi)\|_{L^2(\mathcal{C}_{\mathcal{Y}}, y^{\alpha})}. \tag{3.6}$$

Let us construct explicitly a function $\phi_0 \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha)$ to use in (3.6). Define

$$\rho(y) = \begin{cases} 1, & 0 \le y \le \mathcal{Y}/2, \\ \dfrac{2}{\mathcal{Y}}(\mathcal{Y} - y), & \mathcal{Y}/2 < y < \mathcal{Y}, \\ 0, & y \ge \mathcal{Y}. \end{cases} \tag{3.7}$$

Notice that $\rho \in W^1_\infty(0,\infty)$, $|\rho(y)| \le 1$ and $|\rho'(y)| \le 2/\mathcal{Y}$ for all $y > 0$. Set $\phi_0(x',y) = \mathfrak{u}(x',y)\rho(y)$ for $x' \in \Omega$ and $y > 0$. A straightforward computation shows

$$|\nabla\left((1-\rho)\mathfrak{u}\right)|^2 \le 2\left(|\rho'|^2|\mathfrak{u}|^2 + (1-\rho)^2|\nabla \mathfrak{u}|^2\right) \le 2\left(\frac{4}{\mathcal{Y}^2}\mathfrak{u}^2 + |\nabla\mathfrak{u}|^2\right),$$

so that

$$\|\nabla(\mathfrak{u} - \phi_0)\|^2_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \le 2\left(\frac{4}{\mathcal{Y}^2}\int_{\mathcal{Y}/2}^{\mathcal{Y}}\int_\Omega y^\alpha|\mathfrak{u}|^2 + \int_{\mathcal{Y}/2}^{\mathcal{Y}}\int_\Omega y^\alpha|\nabla\mathfrak{u}|^2\right). \tag{3.8}$$

To estimate the first term on the right hand side of (3.8) we use the Poincaré inequality (2.20) over a dyadic partition that covers the interval $[\mathcal{Y}/2, \mathcal{Y}]$ (see the derivation of (2.21) in § 2.3), to obtain

$$\int_{\mathcal{Y}/2}^{\mathcal{Y}} y^\alpha \int_\Omega |\mathfrak{u}|^2 \lesssim \int_{\mathcal{Y}/2}^{\mathcal{Y}} y^\alpha \int_\Omega |\nabla\mathfrak{u}|^2. \tag{3.9}$$

To bound the second integral in (3.8) we use (2.33) as in the proof of Proposition 3.1:

$$\int_{\mathcal{Y}/2}^{\mathcal{Y}} y^\alpha \int_\Omega |\nabla\mathfrak{u}|^2 = \sum_{k=1}^\infty |u_k|^2 y^\alpha \psi_k(y)\psi_k'(y)\Big|_{\mathcal{Y}/2}^{\mathcal{Y}} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/2}\|f\|^2_{\mathbb{H}^{-s}(\Omega)}.$$

Inserting these estimates into (3.6) implies (3.5). □

The following result is a direct consequence of Lemma 3.3.

Remark 3.4 (Stability). Let $\mathcal{Y} \ge 1$, then

$$\|\nabla v\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \lesssim \|f\|_{\mathbb{H}^{-s}(\Omega)}. \tag{3.10}$$

Indeed, by the triangle inequality

$$\|\nabla v\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \le \|\nabla(v - \mathfrak{u})\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} + \|\nabla\mathfrak{u}\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \lesssim \left(e^{-\sqrt{\lambda_1}\mathcal{Y}/4} + 1\right)\|f\|_{\mathbb{H}^{-s}(\Omega)}.$$

The previous two results allow us to show a full approximation estimate.

Theorem 3.5 (Global exponential estimate). *Let $\mathcal{Y} \ge 1$, then*

$$\|\nabla(\mathfrak{u} - v)\|_{L^2(\mathcal{C}, y^\alpha)} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/4}\|f\|_{\mathbb{H}^{-s}(\Omega)}. \tag{3.11}$$

In particular, for every $\epsilon > 0$, let

$$\mathcal{Y}_0 = \frac{2}{\sqrt{\lambda_1}}\left(\log C + 2\log\frac{1}{\epsilon}\right),$$

where C depends only on s and Ω. Then, for $\mathcal{Y} \ge \max\{\mathcal{Y}_0, 1\}$, we have

$$\|\nabla(\mathfrak{u} - v)\|_{L^2(\mathcal{C}, y^\alpha)} \le \epsilon\|f\|_{\mathbb{H}^{-s}(\Omega)}. \tag{3.12}$$

Proof. Extending v by zero outside of $\mathcal{C}_\mathcal{Y}$ we obtain

$$\|\nabla(\mathfrak{u}-v)\|^2_{L^2(\mathcal{C},y^\alpha)} = \|\nabla(\mathfrak{u}-v)\|^2_{L^2(\mathcal{C}_\mathcal{Y},y^\alpha)} + \|\nabla\mathfrak{u}\|^2_{L^2(\Omega\times(\mathcal{Y},\infty),y^\alpha)}.$$

Hence Lemma 3.3 and Proposition 3.1 imply

$$\|\nabla(\mathfrak{u}-v)\|^2_{L^2(\mathcal{C},y^\alpha)} \leq Ce^{-\sqrt{\lambda_1}\mathcal{Y}/2}\|f\|^2_{\mathbb{H}^{-s}(\Omega)} \leq \epsilon^2\|f\|^2_{\mathbb{H}^{-s}(\Omega)}, \tag{3.13}$$

for all $\mathcal{Y} \geq \max\{\mathcal{Y}_0, 1\}$. □

4. Finite element discretization and interpolation estimates. In this section we prove error estimates for a piecewise $\mathbb{Q}_1$ interpolation operator on anisotropic elements in the extended variable y. We consider elements of the form $T = K \times I$, where $K \subset \mathbb{R}^n$ is an element isoparametrically equivalent to the unit cube $[0,1]^n$, via a $\mathbb{Q}_1$ mapping and, $I \subset \mathbb{R}$ is an interval. The anisotropic character of the mesh $\mathscr{T}_\mathcal{Y} = \{T\}$ will be given by the family of intervals I.

The error estimates are derived in the weighted Sobolev spaces $L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)$ and $H^1(\mathcal{C}_\mathcal{Y}, y^\alpha)$, and they are valid under the condition that neighboring elements have comparable size in the extended $(n+1)$–dimension (see [30]). This is a mild assumption that includes general meshes which do not satisfy the so-called shape-regularity assumption, i.e., mesh refinements for which the quotient between outer and inner diameter of the elements does not remain bounded (see [17, Chapter 4]).

Anisotropic or narrow elements are elements with disparate sizes in each direction. They arise naturally when approximating solutions of problems with a strong directional-dependent behavior since, using anisotropy, the local mesh size can be adapted to capture such features. Examples of this include boundary layers, shocks and edge singularities (see [30, 31]). In our problem, anisotropic elements are essential in order to capture the singular/degenerate behavior of the solution $\mathfrak{u}$ to problem (2.26) at $y \approx 0^+$ given in (2.34). These elements will provide optimal error estimates, which cannot be obtained using shape-regular elements.

Error estimates for weighted Sobolev spaces have been obtained in several works; see, for instance, [4, 9, 30]. The type of weight considered in [4, 9] is related to the distance to a point or an edge, and the type of quasi-interpolators are modifications of the well known Clément [26] and Scott-Zhang [56] operators. These works are developed in 3D and 2D respectively, and the analysis developed in [4] allows for anisotropy. Our approach follows the work of Durán and Lombardi [30], and is based on a piecewise $\mathbb{Q}_1$ averaged interpolator on anisotropic elements. It allows us to obtain anisotropic interpolation estimates in the extended variable y and in weighted Sobolev spaces, using only that $|y|^\alpha \in A_2(\mathbb{R}^{n+1})$, the Muckenhoupt class A_2 of Definition 2.1. We develop a general interpolation theory for weights of class A_p with $1 < p < \infty$ in [54].

4.1. Finite element discretization. Let us now describe the discretization of problem (3.2). To avoid technical difficulties we assume that the boundary of Ω is polygonal. The difficulties inherent to curved boundaries could be handled, for instance, with the methods of [11] (see also [44, 45]). Let $\mathscr{T}_\Omega = \{K\}$ be a mesh of Ω made of isoparametric quadrilaterals K in the sense of Ciarlet [24] and Ciarlet and Raviart [25]. In other words, given $\hat{K} = [0,1]^n$ and a family of mappings $\{\mathcal{F}_K \in \mathbb{Q}_1(\hat{K})^n\}$ we have

$$K = \mathcal{F}_K(\hat{K}) \tag{4.1}$$

and

$$\bar{\Omega} = \bigcup_{K \in \mathscr{T}_\Omega} K, \qquad |\Omega| = \sum_{K \in \mathscr{T}_\Omega} |K|.$$

The collection of triangulations is denoted by $\mathbb{T}_\Omega$.

The mesh $\mathscr{T}_\Omega$ is assumed to be conforming or compatible, i.e., the intersection of any two isoparametric elements K and K' in $\mathscr{T}_\Omega$ is either empty or a common lower dimensional isoparametric element.

In addition, we assume that $\mathscr{T}_\Omega$ is shape regular (cf. [24, Chapter 4.3]). This means that $\mathcal{F}_K$ can be decomposed as $\mathcal{F}_K = \mathcal{A}_K + \mathcal{B}_K$, where $\mathcal{A}_K$ is affine and $\mathcal{B}_K$ is a perturbation map and, if we define $\tilde{K} = \mathcal{A}_K(\hat{K})$, $h_K = \operatorname{diam}(\tilde{K})$, ρ_K as the diameter of the largest sphere inscribed in $\tilde{K}$ and the shape coefficient of K as the ratio $\sigma_K = h_K/\rho_K$, then the following two conditions are satisfied:

(a) There exists a constant $\sigma_\Omega > 1$ such that for all $\mathscr{T}_\Omega \in \mathbb{T}_\Omega$,

$$\max\{\sigma_K : K \in \mathscr{T}_\Omega\} \le \sigma_\Omega.$$

(b) For all $K \in \mathscr{T}_\Omega$ the mapping $\mathcal{B}_K$ is Fréchet differentiable and

$$\|D\mathcal{B}_K\|_{L^\infty(\hat{K})} = \mathcal{O}(h_K^2),$$

for all $K \in \mathscr{T}_\Omega$ and all $\mathscr{T}_\Omega \in \mathbb{T}_\Omega$.

As a consequence of these conditions, if h_K is small enough, the mapping $\mathcal{F}_K$ is one-to-one, its Jacobian $J_{\mathcal{F}_K}$ does not vanish, and

$$J_{\mathcal{F}_K} \lesssim h_K^n, \quad \|D\mathcal{F}_K\|_{L^\infty(\hat{K})} \lesssim h_K. \tag{4.2}$$

The set $\mathbb{T}_\Omega$ is called quasi-uniform if for all $\mathscr{T}_\Omega \in \mathbb{T}_\Omega$,

$$\max\{\rho_K : K \in \mathscr{T}_\Omega\} \lesssim \min\{h_K : K \in \mathscr{T}_\Omega\}.$$

In this case, we define $h_{\mathscr{T}_\Omega} = \max_{K \in \mathscr{T}} h_K$.

We define $\mathscr{T}_\mathcal{Y}$ as a triangulation of $\mathcal{C}_\mathcal{Y}$ into cells of the form $T = K \times I$, where $K \in \mathscr{T}_\Omega$, and I denotes an interval in the extended dimension. Notice that each discretization of the truncated cylinder $\mathcal{C}_\mathcal{Y}$ depends on the truncation parameter $\mathcal{Y}$. The set of all such triangulations is denoted by $\mathbb{T}$. In order to obtain a global regularity assumption for $\mathbb{T}$ we assume the aforementioned conditions on $\mathbb{T}_\Omega$, besides the following weak regularity condition:

(c) There is a constant σ such that, for all $\mathscr{T}_\mathcal{Y} \in \mathbb{T}$, if $T_1 = K_1 \times I_1, T_2 = K_2 \times I_2 \in \mathscr{T}_\mathcal{Y}$ have nonempty intersection, then

$$\frac{h_{I_1}}{h_{I_2}} \le \sigma,$$

where $h_I = |I|$.

Notice that the assumptions imposed on $\mathbb{T}$ are weaker than the standard shape-regularity assumptions, since they allow for anisotropy in the extended variable (cf. [30]). It is also important to notice that, given the Cartesian product structure of the cells $T \in \mathscr{T}_\mathcal{Y}$, they are isoparametrically equivalent to $\hat{T} = [0,1]^{n+1}$. We will denote the corresponding mappings by $\mathcal{F}_T$. Then,

$$\mathcal{F}_T : \hat{x} = (\hat{x}', \hat{y}) \in \hat{T} \longmapsto x = (x', y) = (\mathcal{F}_K(\hat{x}'), \mathcal{F}_I(\hat{y})) \in T = K \times I,$$

where $\mathcal{F}_K$ is the bilinear mapping defined in (4.1) for K and, if $I=(c,d)$, $\mathcal{F}_I(y)=(y-c)/(d-c)$. From (4.2), we immediately conclude that

$$J_{\mathcal{F}_T} \lesssim h_K^n h_I, \quad \|D\mathcal{F}_T\|_{L^\infty(\hat{T})} \lesssim h_T, \tag{4.3}$$

for all elements $T \in \mathscr{T}_\mathcal{Y}$ where $h_T = \max\{h_K, h_I\}$.

Given $\mathscr{T}_\mathcal{Y} \in \mathbb{T}$, we define the finite element space $\mathbb{V}(\mathscr{T}_\mathcal{Y})$ by

$$\mathbb{V}(\mathscr{T}_\mathcal{Y}) = \left\{ W \in \mathcal{C}^0(\overline{\mathcal{C}_\mathcal{Y}}) : W|_T \in \mathbb{Q}_1(T)\ \forall T \in \mathscr{T}_\mathcal{Y},\ W|_{\Gamma_D} = 0 \right\}.$$

where $\Gamma_D = \partial_L \mathcal{C}_\mathcal{Y} \cup \Omega \times \{\mathcal{Y}\}$ is called the Dirichlet boundary. The Galerkin approximation of (3.3) is given by the unique function $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ such that

$$\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla V_{\mathscr{T}_\mathcal{Y}} \cdot \nabla W = d_s \langle f, \operatorname{tr}_\Omega W \rangle, \quad \forall W \in \mathbb{V}(\mathscr{T}_\mathcal{Y}). \tag{4.4}$$

Existence and uniqueness of $V_{\mathscr{T}_\mathcal{Y}}$ follows from $\mathbb{V}(\mathscr{T}_\mathcal{Y}) \subset \mathring{H}_L^1(\mathcal{C}_\mathcal{Y}, y^\alpha)$ and the Lax-Milgram lemma.

We define the space $\mathbb{U}(\mathscr{T}_\Omega) = \operatorname{tr}_\Omega \mathbb{V}(\mathscr{T}_\mathcal{Y})$, which is nothing more than a $\mathbb{Q}_1$ finite element space over the mesh $\mathscr{T}_\Omega$. The finite element approximation of $u \in \mathbb{H}^s(\Omega)$, solution of (1.1), is then given by

$$U_{\mathscr{T}_\Omega} = \operatorname{tr}_\Omega V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{U}(\mathscr{T}_\Omega), \tag{4.5}$$

and we have the following result.

THEOREM 4.1 (Energy error estimate). *Let v solve* (3.3) *with $\mathcal{Y} \geq \max\{\mathcal{Y}_0, 1\}$. If $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ solves* (4.4) *and $U_{\mathscr{T}_\Omega} \in \mathbb{U}(\mathscr{T}_\Omega)$ is defined in* (4.5)*, then we have*

$$\|u - U_{\mathscr{T}_\Omega}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\mathfrak{u} - V_{\mathscr{T}_\mathcal{Y}}\|_{\mathring{H}_L^1(\mathcal{C}, y^\alpha)}, \tag{4.6}$$

and

$$\|\mathfrak{u} - V_{\mathscr{T}_\mathcal{Y}}\|_{\mathring{H}_L^1(\mathcal{C}, y^\alpha)} \lesssim \epsilon \|f\|_{\mathbb{H}^{-s}(\Omega)} + \|v - V_{\mathscr{T}_\mathcal{Y}}\|_{\mathring{H}_L^1(\mathcal{C}_\mathcal{Y}, y^\alpha)}. \tag{4.7}$$

Proof. Estimate (4.6) is just an application of the trace estimate of Proposition 2.5. Inequality (4.7) is obtained by the triangle inequality and (3.12). □

By Galerkin orthogonality

$$\|v - V_{\mathscr{T}_\mathcal{Y}}\|_{\mathring{H}_L^1(\mathcal{C}_\mathcal{Y}, y^\alpha)} = \inf_{W \in \mathbb{V}(\mathscr{T}_\mathcal{Y})} \|v - W\|_{\mathring{H}_L^1(\mathcal{C}_\mathcal{Y}, y^\alpha)}.$$

Theorem 4.1 and Galerkin orthogonality imply that the approximation estimate (4.7) depends on the regularity of $\mathfrak{u}$. To see this we introduce

$$\rho(y) = \begin{cases} 1, & 0 \leq y < \mathcal{Y}/2, \\ p, & \mathcal{Y}/2 \leq y \leq \mathcal{Y}, \end{cases} \tag{4.8}$$

where p is the unique cubic polynomial on $[\mathcal{Y}/2, \mathcal{Y}]$ defined by the conditions $p(\mathcal{Y}/2) = 1$, $p(\mathcal{Y}) = 0$, $p'(\mathcal{Y}/2) = 0$ and $p'(\mathcal{Y}) = 0$. Notice that $\rho \in W^2_\infty(0, \mathcal{Y})$, $|\rho(y)| \leq 1$, $|\rho'(y)| \lesssim 1$ and $|\rho''(y)| \lesssim 1$. Set $\mathfrak{u}_0(x', y) = \rho(y)\mathfrak{u}(x', y)$ for $x' \in \Omega$ and $y \in [0, \mathcal{Y}]$, and notice that $\mathfrak{u}_0 \in \mathring{H}_L^1(\mathcal{C}_\mathcal{Y}, y^\alpha)$. With this construction at hand, repeating the arguments used in the proof of Lemma 3.3, we have that

$$\begin{aligned} \|\Delta_{x'} \mathfrak{u}_0\|_{L^2(\mathcal{C}_\mathcal{Y},y^\alpha)} &\lesssim \|\Delta_{x'} \mathfrak{u}\|_{L^2(\mathcal{C}_\mathcal{Y},y^\alpha)}, \\ \|\partial_y \nabla_{x'} \mathfrak{u}_0\|_{L^2(\mathcal{C}_\mathcal{Y},y^\alpha)} &\lesssim \|\partial_y \nabla_{x'} \mathfrak{u}\|_{L^2(\mathcal{C}_\mathcal{Y},y^\alpha)} + \|f\|_{\mathbb{H}^{-s}(\Omega)}, \\ \|\partial_{yy} \mathfrak{u}_0\|_{L^2(\mathcal{C}_\mathcal{Y},y^\beta)} &\lesssim \|\partial_{yy} \mathfrak{u}\|_{L^2(\mathcal{C}_\mathcal{Y},y^\beta)} + \|f\|_{\mathbb{H}^{-s}(\Omega)}. \end{aligned} \tag{4.9}$$

In addition, if we assume that there is an operator

$$\Pi_{\mathscr{T}_\mathcal{Y}} : \mathring{H}_L^1(\mathcal{C}_\mathcal{Y}, y^\alpha) \to \mathbb{V}(\mathscr{T}_\mathcal{Y}),$$

that is stable, i.e., $\|\Pi_{\mathscr{T}_\mathcal{Y}} w\|_{\mathring{H}_L^1(\mathcal{C}_\mathcal{Y},y^\alpha)} \lesssim \|w\|_{\mathring{H}_L^1(\mathcal{C}_\mathcal{Y},y^\alpha)}$, for all $w \in \mathring{H}_L^1(\mathcal{C}_\mathcal{Y}, y^\alpha)$, then the following estimate holds

$$\|\mathfrak{u} - V_{\mathscr{T}_\mathcal{Y}}\|_{\mathring{H}_L^1(\mathcal{C},y^\alpha)} \lesssim \epsilon \|f\|_{\mathbb{H}^{-s}(\Omega)} + \|\mathfrak{u}_0 - \Pi_{\mathscr{T}_\mathcal{Y}} \mathfrak{u}_0\|_{\mathring{H}_L^1(\mathcal{C}_\mathcal{Y},y^\alpha)}. \tag{4.10}$$

To see this, we use (4.7), together with Galerkin orthogonality and the stability of the operator $\Pi_{\mathscr{T}_\mathcal{Y}}$, to obtain

$$\begin{aligned} \|\mathfrak{u} - V_{\mathscr{T}_\mathcal{Y}}\|_{\mathring{H}_L^1(\mathcal{C},y^\alpha)} &\lesssim \epsilon \|f\|_{\mathbb{H}^{-s}(\Omega)} + \|v - \Pi_{\mathscr{T}_\mathcal{Y}} v\|_{\mathring{H}_L^1(\mathcal{C}_\mathcal{Y},y^\alpha)} \\ &\lesssim \epsilon \|f\|_{\mathbb{H}^{-s}(\Omega)} + \|v - \mathfrak{u}_0\|_{\mathring{H}_L^1(\mathcal{C}_\mathcal{Y},y^\alpha)} + \|\mathfrak{u}_0 - \Pi_{\mathscr{T}_\mathcal{Y}} \mathfrak{u}_0\|_{\mathring{H}_L^1(\mathcal{C}_\mathcal{Y},y^\alpha)}. \end{aligned}$$

The second term on the right hand side of the previous inequality is estimated as in Lemma 3.3. We leave the details to the reader.

Estimates for $\mathfrak{u}_0 - \Pi_{\mathscr{T}_\mathcal{Y}} \mathfrak{u}_0$ on weighted Sobolev spaces are derived in § 4.2. Clearly, these depend on the regularity of $\mathfrak{u}_0$ which, in light of (4.9), depends on the regularity of $\mathfrak{u}$. For this reason, and to lighten the notation, we shall in the sequel write $\mathfrak{u}$ and obtain interpolation error estimates for it, even though $\mathfrak{u}$ does not vanish at $y = \mathcal{Y}$.

4.2. Interpolation estimates in weighted Sobolev spaces. Let us begin by introducing some notation and terminology. Given $\mathscr{T}_\mathcal{Y}$, we call $\mathcal{N}$ the set of its nodes and $\mathcal{N}_{\text{in}}$ the set of its interior and Neumann nodes. For each vertex $\mathtt{v} \in \mathcal{N}$, we write $\mathtt{v} = (\mathtt{v}', \mathtt{v}'')$, where $\mathtt{v}'$ corresponds to a node of $\mathscr{T}_\Omega$, and $\mathtt{v}''$ corresponds to a node of the discretization of the $n+1$–dimension. We define $h_{\mathtt{v}'} = \min\{h_K : \mathtt{v}' \text{ is a vertex of } K\}$, and $h_{\mathtt{v}''} = \min\{h_I : \mathtt{v}'' \text{ is a vertex of } I\}$.

Given $\mathtt{v} \in \mathcal{N}$, the *star* or patch around $\mathtt{v}$ is defined as

$$\omega_\mathtt{v} = \bigcup_{T \ni \mathtt{v}} T,$$

and for $T \in \mathscr{T}_\mathcal{Y}$ we define its *patch* as

$$\omega_T = \bigcup_{\mathtt{v} \in T} \omega_\mathtt{v}.$$

Let $\psi \in C^\infty(\mathbb{R}^{n+1})$ be such that $\int \psi = 1$ and $D := \operatorname{supp} \psi \subset B_r \times (0, r_\mathcal{Y})$, where B_r denotes the ball in $\mathbb{R}^n$ of radius r and centered at zero, and $r \leq 1/\sigma_\Omega$ and $r_\mathcal{Y} \leq 1/\sigma$. For $\mathtt{v} \in \mathcal{N}_{\text{in}}$, we rescale ψ as

$$\psi_\mathtt{v}(x) = \frac{1}{h_{\mathtt{v}'}^n h_{\mathtt{v}''}} \psi\left(\frac{x' - \mathtt{v}'}{h_{\mathtt{v}'}}, \frac{y - \mathtt{v}''}{h_{\mathtt{v}''}} \right),$$

and note that $\operatorname{supp}\psi_{\mathtt{v}} \subset \omega_{\mathtt{v}}$ and $\int_{\omega_{\mathtt{v}}} \psi_{\mathtt{v}} = 1$ for any interior and Neumann node $\mathtt{v}$.

REMARK 4.2 (Boundary conditions of Neumann type). For an interior node $\mathtt{v}$, it would be natural to consider $B_r \times (-r_{\mathcal{Y}}, r_{\mathcal{Y}})$ as the support of the smooth function ψ. However, for a Neumann node $\mathtt{v}$, this choice would not provide the important properties $\operatorname{supp}\psi_{\mathtt{v}} \subset \omega_{\mathtt{v}}$ and $\int_{\omega_{\mathtt{v}}} \psi_{\mathtt{v}} = 1$. In order to treat both types of nodes indistinctly in the subsequent analysis, we have considered $\operatorname{supp}\psi \subset B_r \times (0, r_{\mathcal{Y}})$.

Given a function $w \in L^2(\mathcal{C}_{\mathcal{Y}}, y^{\alpha})$ and a node $\mathtt{v}$ in $\mathcal{N}_{\mathrm{in}}$ we define, following Durán and Lombardi [30], the regularized Taylor polynomial of first degree of w about $\mathtt{v}$ as

$$w_{\mathtt{v}}(z) = \int P(x,z)\psi_{\mathtt{v}}(x)\,\mathrm{d}x = \int_{\omega_{\mathtt{v}}} P(x,z)\psi_{\mathtt{v}}(x)\,\mathrm{d}x, \tag{4.11}$$

where P denotes the Taylor polynomial of degree 1 in the variable z of the function w about the point x, i.e.,

$$P(x,z) = w(x) + \nabla w(x)\cdot(z-x). \tag{4.12}$$

As a consequence of Remark 2.3 and the fact that the averaged Taylor polynomial is defined for functions in $L^1(\mathcal{C}_{\mathcal{Y}})$ (cf. [17, Proposition 4.1.12]), we conclude that P is well defined for any function in $L^2(\mathcal{C}_{\mathcal{Y}}, y^{\alpha})$.

We define the averaged $\mathbb{Q}_1$ interpolant $\Pi_{\mathscr{T}_{\mathcal{Y}}} w$, as the unique piecewise $\mathbb{Q}_1$ function such that $\Pi_{\mathscr{T}_{\mathcal{Y}}} w(\mathtt{v}) = 0$ if $\mathtt{v}$ lies on the Dirichlet boundary Γ_D and $\Pi_{\mathscr{T}_{\mathcal{Y}}} w(\mathtt{v}) = w_{\mathtt{v}}(\mathtt{v})$ if $\mathtt{v} \in \mathcal{N}_{\mathrm{in}}$. If $\lambda_{\mathtt{v}}$ denotes the Lagrange basis function associated with node $\mathtt{v}$, then

$$\Pi_{\mathscr{T}_{\mathcal{Y}}} w = \sum_{\mathtt{v}\in\mathcal{N}_{\mathrm{in}}} w_{\mathtt{v}}(\mathtt{v})\lambda_{\mathtt{v}}.$$

There are two principal reasons to consider averaged interpolation. First, we are interested in the approximation of singular functions and thus Lagrange interpolation cannot be used since point-wise values become meaningless. In fact, this motivated the introduction of averaged interpolation (see [26, 56]). In addition, averaged interpolation has better approximation properties when narrow elements are used (see [2]).

Finally, for $\mathtt{v} \in \mathcal{N}_{\mathrm{in}}$, we define the weighted regularized average of w as

$$Q_{\mathtt{v}} w = \int w(x)\psi_{\mathtt{v}}(x)\,\mathrm{d}x = \int_{\omega_{\mathtt{v}}} w(x)\psi_{\mathtt{v}}(x)\,\mathrm{d}x. \tag{4.13}$$

4.2.1. Weighted Poincaré inequality. In order to obtain interpolation error estimates in $L^2(\mathcal{C}_{\mathcal{Y}}, y^{\alpha})$ and $H^1(\mathcal{C}_{\mathcal{Y}}, y^{\alpha})$, it is instrumental to have a weighted Poincaré-type inequality. Weighted Poincaré inequalities are particularly pertinent in the study of the nonlinear potential theory of degenerate elliptic equations, see [36, 41]. If the domain is a ball and the weight belongs to A_p, with $1 \le p < \infty$, this result can be found in [36, Theorem 1.3 and Theorem 1.5]. However, to the best of our knowledge, such a result is not available in the literature for more general domains. For our specific weight we present here a constructive proof, i.e., not based on a compactness argument. This allows us to study the dependence of the constant on the domain.

LEMMA 4.3 (Weighted Poincaré inequality I). *Let $\omega \subset \mathbb{R}^{n+1}$ be bounded, star-shaped with respect to a ball B, and $\operatorname{diam}\omega \approx 1$. Let $\chi \in C^0(\bar{\omega})$ with $\int_{\omega} \chi = 1$, and $\xi_{\alpha}(y) := |a|y| + b|^{\alpha}$ for $a, b \in \mathbb{R}$. If $w \in H^1(\omega, \xi_{\alpha}(y))$ is such that $\int_{\omega} \chi w = 0$, then*

$$\|w\|_{L^2(\omega,\xi_{\alpha})} \lesssim \|\nabla w\|_{L^2(\omega,\xi_{\alpha})}, \tag{4.14}$$

where the hidden constant depends only on χ, α *and the radius* r *of* B, *but is independent of both* a *and* b.

Proof. The fact that $\alpha \in (-1,1)$ implies $\xi_\alpha \in A_2(\mathbb{R}^{n+1})$ with a Muckenhoupt constant C_{2,ξ_α} in (2.15) uniform in both a and b. Define

$$\widetilde{w} = \xi_\alpha w - \left(\int_\omega \xi_\alpha w \right) \chi.$$

Clearly $\widetilde{w} \in L^1(\omega)$ and it has vanishing mean value by construction.

Since $\int_\omega \chi w = 0$ we obtain

$$\|w\|^2_{L^2(\omega,\xi_\alpha)} = \int_\omega w\widetilde{w} + \left(\int_\omega \xi_\alpha w \right) \int_\omega \chi w = \int_\omega w\widetilde{w}. \tag{4.15}$$

Consequently, given that ω is star shaped with respect to B, and $\xi_\alpha \in A_2(\mathbb{R}^{n+1})$, there exists $F \in H^1_0(\omega,\xi_\alpha)^{n+1}$ such that $-\mathrm{div} F = \widetilde{w}$, and

$$\|F\|_{H^1_0(\omega,\xi_\alpha^{-1})^{n+1}} \lesssim \|\widetilde{w}\|_{L^2(\omega,\xi_\alpha^{-1})}, \tag{4.16}$$

where the hidden constant in (4.16) depends on r and the constant C_{2,ξ_α} from Definition 2.1 [32, Theorem 3.1].

Replacing $\widetilde{w}$ by $-\mathrm{div} F$ in (4.15), integrating by parts and using (4.16), we get

$$\|w\|^2_{L^2(\omega,\xi_\alpha)} = -\int_\omega w \,\mathrm{div} F = \int_\omega \nabla w \cdot F \lesssim \|\nabla w\|_{L^2(\omega,\xi_\alpha)} \|\widetilde{w}\|_{L^2(\omega,\xi_\alpha^{-1})}. \tag{4.17}$$

To estimate $\|\widetilde{w}\|_{L^2(\omega,\xi_\alpha^{-1})}$ we use the Cauchy-Schwarz inequality and the constant C_{2,ξ_α} from Definition 2.1 as follows:

$$\|\widetilde{w}\|^2_{L^2(\omega,\xi_\alpha^{-1})} \le 2 \left(1 + \int_\omega \xi_\alpha \int_\omega \chi^2 \xi_\alpha^{-1} \right) \|w\|^2_{L^2(\omega,\xi_\alpha)} \lesssim \|w\|^2_{L^2(\omega,\xi_\alpha)}.$$

Inserting the inequality above into (4.17), we obtain (4.14). □

We need a slightly more general form of the Poincaré inequality for the applications below. We now relax the geometric assumption on the domain ω and let the vanishing mean property hold just in a subdomain.

COROLLARY 4.4 (Weighted Poincaré inequality II). *Let* $\omega = \cup_{i=1}^N \omega_i \subset \mathbb{R}^{n+1}$ *be a connected domain and each* ω_i *be a star-shaped domain with respect to a ball* B_i. *Let* $\chi_i \in C^0(\bar{\omega}_i)$ *and* ξ_α *be as in Lemma 4.3. If* $w \in H^1(\omega,\xi_\alpha)$ *and* $w_i := \int_{\omega_i} w\chi_i$, *then*

$$\|w - w_i\|_{L^2(\omega,\xi_\alpha)} \lesssim \|\nabla w\|_{L^2(\omega,\xi_\alpha)} \qquad \forall 1 \le i \le N, \tag{4.18}$$

where the hidden constant depends on $\{\chi_i\}_{i=1}^N$, α, *the radius* r_i *of* B_i, *and the amount of overlap between the subdomains* $\{\omega_i\}_{i=1}^N$, *but is independent of both* a *and* b.

Proof. This is a consequence of Lemma 4.3 and [29, Theorem 7.1]. We sketch the proof here for completeness. It suffices to deal with two subdomains, ω_1, ω_2, and the overlapping region $B = \omega_1 \cap \omega_2$. We observe that

$$\|w - w_1\|_{L^2(\omega_2,\xi_\alpha)} \le \|w - w_2\|_{L^2(\omega_2,\xi_\alpha)} + \|w_1 - w_2\|_{L^2(\omega_2,\xi_\alpha)},$$

together with $\|w_1 - w_2\|_{L^2(\omega_2,\xi_\alpha)} = \left(\frac{\int_{\omega_2} \xi_\alpha}{\int_B \xi_\alpha} \right)^{1/2} \|w_1 - w_2\|_{L^2(B,\xi_\alpha)}$ and

$$\|w_1 - w_2\|_{L^2(B,\xi_\alpha)} \lesssim \|w - w_1\|_{L^2(\omega_1,\xi_\alpha)} + \|w - w_2\|_{L^2(\omega_2,\xi_\alpha)},$$

imply $\|w - w_1\|_{L^2(\omega_2,\xi_\alpha)} \lesssim \|\nabla w\|_{L^2(\omega_1 \cup \omega_2,\xi_\alpha)}$. This, combined with (4.14), gives (4.18) for $i = 1$ with a stability constant depending on the ratio $\frac{\int_{\omega_2} \xi_\alpha}{\int_B \xi_\alpha}$. □

4.2.2. Weighted L^2 interpolation estimates. Owing to the weighted Poincaré inequality of Corollary 4.4, we can adapt the proof of [30, Lemma 2.3] to obtain interpolation estimates in the weighted L^2-norm. These estimates allow a disparate mesh-size on the extended direction, relative to the coordinate directions x_i, $i = 1,\dots,n$, which may in turn be graded. This is the principal difference with [30, Lemma 2.3] where, however, the domain must be a cube.

LEMMA 4.5 (Weighted L^2-based interpolation estimates). *Let* $\mathtt{v} \in \mathcal{N}_{\text{in}}$. *Then, for all* $w \in H^1(\omega_{\mathtt{v}}, y^\alpha)$, *we have*

$$\|w - Q_{\mathtt{v}} w\|_{L^2(\omega_{\mathtt{v}},y^\alpha)} \lesssim h_{\mathtt{v}'} \|\nabla_{x'} w\|_{L^2(\omega_{\mathtt{v}},y^\alpha)} + h_{\mathtt{v}''} \|\partial_y w\|_{L^2(\omega_{\mathtt{v}},y^\alpha)}, \tag{4.19}$$

and, for all $w \in H^2(\omega_{\mathtt{v}}, y^\alpha)$ *and* $j = 1,\dots,n+1$, *we have*

$$\|\partial_{x_j}(w - w_{\mathtt{v}})\|_{L^2(\omega_{\mathtt{v}},y^\alpha)} \lesssim h_{\mathtt{v}'} \sum_{i=1}^{n} \|\partial^2_{x_j x_i} w\|_{L^2(\omega_{\mathtt{v}},y^\alpha)} + h_{\mathtt{v}''} \|\partial^2_{x_j y} w\|_{L^2(\omega_{\mathtt{v}},y^\alpha)}, \tag{4.20}$$

where, in both inequalities, the hidden constant depends only on α, σ_Ω, σ *and* ψ.

Proof. Define by $\mathcal{F}_{\mathtt{v}} : (x', y) \to (\bar{x}', \bar{y})$ the scaling map

$$\bar{x}' = \frac{x' - \mathtt{v}'}{h_{\mathtt{v}'}}, \qquad \bar{y} = \frac{y - \mathtt{v}''}{h_{\mathtt{v}''}},$$

along with $\overline{\omega}_{\mathtt{v}} = \mathcal{F}_{\mathtt{v}}(\omega_{\mathtt{v}})$ and $\bar{w}(\bar{x}) = w(x)$. Define also $\bar{Q}\bar{w} = \int \bar{w}\psi$, where ψ has been introduced in Section 4.2. Since $\operatorname{supp}\psi \subset \overline{\omega}_{\mathtt{v}}$ integration takes place only over $\overline{\omega}_{\mathtt{v}}$, and $\int_{\overline{\omega}_{\mathtt{v}}} \psi = 1$. Then, $\bar{Q}\bar{w}$ satisfies $\bar{Q}\bar{w} = \int_{\overline{\omega}_{\mathtt{v}}} \bar{w}\psi = \int_{\omega_{\mathtt{v}}} w\psi_{\mathtt{v}} = Q_{\mathtt{v}} w$, and

$$\int_{\overline{\omega}_{\mathtt{v}}} (\bar{Q}\bar{w} - \bar{w})\psi \,\mathrm{d}\bar{x} = \bar{Q}\bar{w} - \int_{\overline{\omega}_{\mathtt{v}}} \bar{w}\psi \,\mathrm{d}\bar{x} = 0. \tag{4.21}$$

Simple scaling, using the definition of the mapping $\mathcal{F}_{\mathtt{v}}$, yields

$$\int_{\omega_{\mathtt{v}}} y^\alpha |w - Q_{\mathtt{v}} w|^2 \,\mathrm{d}x = h^n_{\mathtt{v}'} h_{\mathtt{v}''} \int_{\overline{\omega}_{\mathtt{v}}} \xi_\alpha |\bar{w} - \bar{Q}\bar{w}|^2 \,\mathrm{d}\bar{x}, \tag{4.22}$$

where $\xi_\alpha(y) := |\mathtt{v}'' + \bar{y} h_{\mathtt{v}''}|^\alpha$. By shape regularity, the mesh sizes $h_{\mathtt{v}'}, h_{\mathtt{v}''}$ satisfy $1/2\sigma \leq h_{\bar{\mathtt{v}}''} \leq 2\sigma$ and $1/2\sigma_\Omega \leq h_{\bar{\mathtt{v}}'} \leq 2\sigma_\Omega$, respectively, and $\operatorname{diam}\overline{\omega}_{\mathtt{v}} \approx 1$. In view of (4.21), we can apply Lemma 4.3 with the weight ξ_α and $\chi = \psi$, to $\omega = \overline{\omega}_{\mathtt{v}}$ to obtain

$$\|\bar{w} - \bar{Q}\bar{w}\|_{L^2(\bar{\omega}_{\mathtt{v}},\xi_\alpha)} \lesssim \|\bar{\nabla}\bar{w}\|_{L^2(\bar{\omega}_{\mathtt{v}},\xi_\alpha)},$$

where the hidden constant depends only on α, σ_Ω, σ and ψ, but not on $\mathtt{v}''$ and $h_{\mathtt{v}''}$. Applying this to (4.22), together with a change of variables with $\mathcal{F}_{\mathtt{v}}^{-1}$, we get (4.19).

The proof of (4.20) is similar. Notice that

$$\begin{aligned} w_{\mathtt{v}}(z) &= \int_{\omega_{\mathtt{v}}} \left(w(x) + \nabla w(x)\cdot(z - x)\right)\psi_{\mathtt{v}}(x)\,\mathrm{d}x \\ &= \int_{\overline{\omega}_{\mathtt{v}}} \left(\bar{w}(\bar{x}) + \bar{\nabla}\bar{w}(\bar{x})\cdot(\bar{z} - \bar{x})\right)\psi(\bar{x})\,\mathrm{d}\bar{x} =: \bar{w}_0(\bar{z}). \end{aligned}$$

Since $\partial_{\bar{z}_i}\bar{w}_0(\bar{z}) = \int_{\overline{\omega}_{\mathtt{v}}} \partial_{\bar{x}_i}\bar{w}(\bar{x})\psi(\bar{x})\,\mathrm{d}\bar{x}$ is constant, we have the vanishing mean value property

$$\int_{\overline{\omega}_{\mathtt{v}}} \partial_{\bar{z}_i}\left(\bar{w}(\bar{z}) - \bar{w}_0(\bar{z})\right)\psi(\bar{z})\,\mathrm{d}\bar{z} = 0.$$

Applying Lemma 4.3 to $\partial_{\bar{x}_i}(\bar{w}(\bar{x}) - \bar{w}_0(\bar{x}))$, and scaling with $\mathscr{F}_{\mathtt{v}}$ we obtain (4.20). ◻

By shape regularity, for all $\mathtt{v} \in \mathscr{N}_{\text{in}}$ and $T \subset \omega_{\mathtt{v}}$, the quantities $h_{\mathtt{v}'}$ and $h_{\mathtt{v}''}$ are equivalent to h_K and h_I, up to a constant that depends only on σ_Ω and σ, respectively. This fact leads to the following result about interpolation estimates in the weighted L^2-norm on interior elements; we refer to § 4.2.4 for boundary elements.

THEOREM 4.6 (Stability and local interpolation in the weighted L^2-norm). *For all $T \in \mathscr{T}_{\mathcal{Y}}$ such that $\partial T \cap \Gamma_D = \emptyset$, and $w \in L^2(\omega_T, y^\alpha)$ we have*

$$\|\Pi_{\mathscr{T}_{\mathcal{Y}}} w\|_{L^2(T,y^\alpha)} \lesssim \|w\|_{L^2(\omega_T,y^\alpha)}. \tag{4.23}$$

If, in addition, $w \in H^1(\omega_T, y^\alpha)$

$$\|w - \Pi_{\mathscr{T}_{\mathcal{Y}}} w\|_{L^2(T,y^\alpha)} \lesssim h_{\mathtt{v}'} \|\nabla_{x'} w\|_{L^2(\omega_T,y^\alpha)} + h_{\mathtt{v}''} \|\partial_y w\|_{L^2(\omega_T,y^\alpha)}. \tag{4.24}$$

The hidden constants in both inequalities depend only on σ_Ω, σ, ψ and α.

Proof. Let $T \in \mathscr{T}_{\mathcal{Y}}$ be an element such that $\partial T \cap \Gamma_D = \emptyset$. Assume, for the moment, that $\Pi_{\mathscr{T}_{\mathcal{Y}}}$ is uniformly bounded as a mapping from $L^2(\omega_T, y^\alpha)$ to $L^2(T, y^\alpha)$, i.e., (4.23).

Choose a node $\mathtt{v}$ of T. Since $Q_{\mathtt{v}} w$ is constant, we deduce $\Pi_{\mathscr{T}_{\mathcal{Y}}} Q_{\mathtt{v}} w = Q_{\mathtt{v}} w$, whence

$$\|w - \Pi_{\mathscr{T}_{\mathcal{Y}}} w\|_{L^2(T,y^\alpha)} = \|(I - \Pi_{\mathscr{T}_{\mathcal{Y}}})(w - Q_{\mathtt{v}} w)\|_{L^2(T,y^\alpha)} \lesssim \|w - Q_{\mathtt{v}} w\|_{L^2(\omega_T,y^\alpha)},$$

so that (4.24) follows from Corollary 4.4.

It remains to show the local boundedness (4.23) of $\Pi_{\mathscr{T}_{\mathcal{Y}}}$. By definition,

$$\Pi_{\mathscr{T}_{\mathcal{Y}}} w = \sum_{i=1}^{n_T} w_{\mathtt{v}_i}(\mathtt{v}_i) \lambda_{\mathtt{v}_i},$$

where $\{\mathtt{v}_i\}_{i=1}^{n_T}$ denotes the set of interior vertices of T. By the triangle inequality

$$\|\Pi_{\mathscr{T}_{\mathcal{Y}}} w\|_{L^2(T,y^\alpha)} \leq \sum_{i=1}^{n_T} \|w_{\mathtt{v}_i}\|_{L^\infty(T)} \|\lambda_{\mathtt{v}_i}\|_{L^2(T,y^\alpha)}, \tag{4.25}$$

so that we need to estimate $\|w_{\mathtt{v}_i}\|_{L^\infty(T)}$. This follows from (4.11) along with,

$$\left| \int_{\omega_{\mathtt{v}_i}} w \psi_{\mathtt{v}_i} \right| \leq \|w\|_{L^2(\omega_{\mathtt{v}_i},y^\alpha)} \|\psi_{\mathtt{v}_i}\|_{L^2(\omega_{\mathtt{v}_i},y^{-\alpha})}, \tag{4.26}$$

and, for $\ell = 1, \dots, n+1$,

$$\left| \int_{\omega_{\mathtt{v}_i}} \partial_{x_\ell} w(x)(z_\ell - x_\ell) \psi_{\mathtt{v}_i}(x) \,\mathrm{d}x \right| \lesssim \|w\|_{L^2(\omega_{\mathtt{v}_i},y^\alpha)} \|\psi_{\mathtt{v}_i}\|_{L^2(\omega_{\mathtt{v}_i},y^{-\alpha})}. \tag{4.27}$$

We get (4.27) upon integration by parts, and noticing that $\psi_{\mathtt{v}_i} = 0$ on $\partial\omega_{\mathtt{v}_i}$, and $|z_l - x_l||\partial_{x_l}\psi_{\mathtt{v}_i}| \lesssim 1$ for $1 \leq l \leq n+1$. Replacing (4.26) and (4.27) in (4.25), we get

$$\|\Pi_{\mathscr{T}_{\mathcal{Y}}} w\|_{L^2(T,y^\alpha)} \lesssim \|w\|_{L^2(\omega_T,y^\alpha)} \sum_{i=1}^{n_T} \|\lambda_{\mathtt{v}_i}\|_{L^2(T,y^\alpha)} \|\psi_{\mathtt{v}_i}\|_{L^2(\omega_{\mathtt{v}_i},y^{-\alpha})} \lesssim \|w\|_{L^2(\omega_T,y^\alpha)},$$

where the last inequality is a consequence of $\lambda_{\mathtt{v}_i}$ and ψ being bounded in $L^\infty(\omega_T)$,

$$\|\lambda_{\mathtt{v}_i}\|_{L^2(T,y^\alpha)} \|\psi_{\mathtt{v}_i}\|_{L^2(\omega_{\mathtt{v}_i},y^{-\alpha})} \lesssim |\omega_{\mathtt{v}_i}|^{-1} \left(\int_{\omega_{\mathtt{v}_i}} |y|^\alpha \int_{\omega_{\mathtt{v}_i}} |y|^{-\alpha} \right)^{1/2},$$

together with $|y|^\alpha \in A_2(\mathbb{R}^{n+1})$; see (2.15). ◻

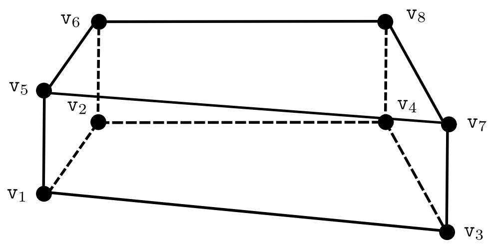

FIG. 4.1. *A generic element $T = K \times I$ in three dimensions: a quadrilateral prism.*

4.2.3. Weighted H^1 interpolation estimates on interior elements. Here we prove interpolation estimates on the first derivatives for interior elements. The, rather technical, proof is an adaption of [30, Theorem 2.5] to our particular geometric setting. In contrast to [30, Theorem 2.5], we do not have the symmetries of a cube. However, exploiting the Cartesian product structure of the elements $T = K \times I$, we are capable of handling the anisotropy in the extended variable y for general shape-regular graded meshes $\mathscr{T}_{\mathcal{Y}}$. This is the content of the following result.

THEOREM 4.7 (Stability and local interpolation: interior elements). *Let $T \in \mathscr{T}_{\mathcal{Y}}$ be such that $\partial T \cap \Gamma_D = \emptyset$. For all $w \in H^1(\omega_T, y^\alpha)$ we have the stability bounds*

$$\|\nabla_{x'} \Pi_{\mathscr{T}_{\mathcal{Y}}} w\|_{L^2(T,y^\alpha)} \lesssim \|\nabla_{x'} w\|_{L^2(\omega_T,y^\alpha)}, \tag{4.28}$$

$$\|\partial_y \Pi_{\mathscr{T}_{\mathcal{Y}}} w\|_{L^2(T,y^\alpha)} \lesssim \|\partial_y w\|_{L^2(\omega_T,y^\alpha)}, \tag{4.29}$$

and, for all $w \in H^2(\omega_T, y^\alpha)$ and $j = 1, \dots, n+1$ we have the error estimates

$$\|\partial_{x_j}(w - \Pi_{\mathscr{T}_{\mathcal{Y}}} w)\|_{L^2(T,y^\alpha)} \lesssim h_{\mathtt{v}'} \|\nabla_{x'} \partial_{x_j} w\|_{L^2(\omega_T,y^\alpha)} + h_{\mathtt{v}''} \|\partial_y \partial_{x_j} w\|_{L^2(\omega_T,y^\alpha)}. \tag{4.30}$$

Proof. To exploit the particular structure of T, we label its vertices in an appropriate way; see Figure 4.1 for the three-dimensional case. In general, if $T = K \times [a,b]$, we first assign a numbering $\{\mathtt{v}_k\}_{k=1,\dots,2^n}$ to the nodes that belong to $K \times \{a\}$. If $(\tilde{\mathtt{v}}', b)$ is a vertex in $K \times \{b\}$, then there is a $\mathtt{v}_k \in K \times \{a\}$ such that $\tilde{\mathtt{v}}' = \mathtt{v}_k'$, and we set $\mathtt{v}_{k+2^n} = \tilde{\mathtt{v}}$. We proceed in three steps.

$\boxed{1}$ *Derivative ∂_y in the extended dimension.* We wish to obtain a bound for the norm $\|\partial_y(w - \Pi_{\mathscr{T}_{\mathcal{Y}}} w)\|_{L^2(T,y^\alpha)}$. Since, $w - \Pi_{\mathscr{T}_{\mathcal{Y}}} w = (w - w_{\mathtt{v}_1}) + (w_{\mathtt{v}_1} - \Pi_{\mathscr{T}_{\mathcal{Y}}} w)$ and an estimate for the difference $w - w_{\mathtt{v}_1}$ is given in Lemma 4.5, it suffices to consider $q := w_{\mathtt{v}_1} - \Pi_{\mathscr{T}_{\mathcal{Y}}} w \in \mathbb{Q}_1$. Thanks to the special labeling of the nodes and the tensor product structure of the elements, i.e., $\partial_y \lambda_{\mathtt{v}_{i+2^n}} = -\partial_y \lambda_{\mathtt{v}_i}$, we get

$$\partial_y q = \sum_{i=1}^{2^{n+1}} q(\mathtt{v}_i) \partial_y \lambda_{\mathtt{v}_i} = \sum_{i=1}^{2^n} (q(\mathtt{v}_i) - q(\mathtt{v}_{i+2^n})) \partial_y \lambda_{\mathtt{v}_i},$$

so that

$$\|\partial_y q\|_{L^2(T,y^\alpha)} \leq \sum_{i=1}^{2^n} |q(\mathtt{v}_i) - q(\mathtt{v}_{i+2^n})| \|\partial_y \lambda_{\mathtt{v}_i}\|_{L^2(T,y^\alpha)}. \tag{4.31}$$

We now set $i = 1$ and proceed to estimate the difference $|q(\mathtt{v}_1) - q(\mathtt{v}_{1+2^n})|$. By the definitions of $\Pi_{\mathscr{T}_{\mathcal{Y}}}$ and q, we have $\Pi_{\mathscr{T}_{\mathcal{Y}}} w(\mathtt{v}_1) = w_{\mathtt{v}_1}(\mathtt{v}_1)$, whence

$$\delta q(\mathtt{v}_1) := q(\mathtt{v}_1) - q(\mathtt{v}_{1+2^n}) = w_{\mathtt{v}_{1+2^n}}(\mathtt{v}_{1+2^n}) - w_{\mathtt{v}_1}(\mathtt{v}_{1+2^n}),$$

and by the definition (4.11) of the averaged Taylor polynomial we have

$$\delta q(\mathtt{v}_1) = \int_{\omega_{\mathtt{v}_{1+2^n}}} P(x, \mathtt{v}_{1+2^n}) \psi_{\mathtt{v}_{1+2^n}}(x) \,\mathrm{d}x - \int_{\omega_{\mathtt{v}_1}} P(x, \mathtt{v}_{1+2^n}) \psi_{\mathtt{v}_1}(x) \,\mathrm{d}x. \tag{4.32}$$

Recalling the operator $\odot$, introduced in (2.4), we notice that, for $h_{\mathtt{v}} = (h_{\mathtt{v}'}, h_{\mathtt{v}''})$ and $z \in \mathbb{R}^{n+1}$, the vector $h_{\mathtt{v}} \odot z$ is uniformly equivalent to $(h_K z', h_I z'')$ for all $T = K \times I$ in the star $\omega_{\mathtt{v}}$. Changing variables in (4.32) yields

$$\delta q(\mathtt{v}_1) = \int \left(P(\mathtt{v}_{1+2^n} - h_{\mathtt{v}_{1+2^n}} \odot z, \mathtt{v}_{1+2^n}) - P(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z, \mathtt{v}_{1+2^n}) \right) \psi(z) \,\mathrm{d}z. \tag{4.33}$$

To estimate this expression define

$$\theta = (0, \theta'') = \left(0, \mathtt{v}''_{1+2^n} - \mathtt{v}''_1 + (h_{\mathtt{v}''_1} - h_{\mathtt{v}''_{1+2^n}}) z'' \right), \tag{4.34}$$

and $F_z(t) = P(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta, \mathtt{v}_{1+2^n})$. Using that $\mathtt{v}'_1 = \mathtt{v}'_{1+2^n}$ and $h_{\mathtt{v}'_1} = h_{\mathtt{v}'_{1+2^n}}$, we easily obtain

$$P(\mathtt{v}_{1+2^n} - h_{\mathtt{v}_{1+2^n}} \odot z, \mathtt{v}_{1+2^n}) - P(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z, \mathtt{v}_{1+2^n}) = F_z(1) - F_z(0).$$

Consequently,

$$\delta q(\mathtt{v}_1) = \int \int_0^1 F'_z(t) \psi(z) \,\mathrm{d}t \,\mathrm{d}z = \int_0^1 \int F'_z(t) \psi(z) \,\mathrm{d}z \,\mathrm{d}t, \tag{4.35}$$

and since ψ is bounded in L^∞ and $\operatorname{supp} \psi = D \subset B_1 \times (-1, 1)$, we need to estimate the integral

$$I(t) = \int_D |F'_z(t)| \,\mathrm{d}z, \quad 0 \le t \le 1.$$

Invoking the definitions of F_z and $P(x, y)$, we deduce

$$F'_z(t) = \nabla_x P(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta, \mathtt{v}_{1+2^n}) \cdot \theta,$$

and

$$\nabla_x P(x, \mathtt{v}) = D^2 w(x) \cdot (\mathtt{v} - x).$$

Using these two expressions, we arrive at

$$\begin{aligned} I(t) \le \int_D \Big(&\left|\partial^2_{yy} w(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta)\right| \left|\mathtt{v}''_{1+2^n} - \mathtt{v}''_1 + h_{\mathtt{v}''_1} z'' - t\theta''\right| \\ &+ |\partial_y \nabla_{x'} w(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta)| \, |\mathtt{v}'_{1+2^n} - \mathtt{v}'_1 + h_{\mathtt{v}'_1} z'| \Big) |\theta''| \,\mathrm{d}z, \end{aligned}$$

Now, since $|z'|, |z''| \le 1$ and $0 \le t \le 1$, we see that

$$|\mathtt{v}'_{1+2^n} - \mathtt{v}'_1 + h_{\mathtt{v}'_1} z'| \lesssim h_{\mathtt{v}'_1}, \qquad |\mathtt{v}''_{1+2^n} - \mathtt{v}''_1 + h_{\mathtt{v}''_1} z'' - t\theta''| \lesssim h_{\mathtt{v}''_1}.$$

Consequently,

$$I(t) \lesssim \int_D \left(\left|\partial^2_{yy} w(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta)\right| h^2_{\mathtt{v}''_1} + |\partial_y \nabla_{x'} w(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta)| \, h_{\mathtt{v}'_1} h_{\mathtt{v}''_1} \right) \mathrm{d}z.$$

Changing variables, via $\tau = \mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta$, we obtain

$$I(t) \lesssim \int_{\omega_T} \left(\frac{h_{\mathtt{v}_1''}}{h_{\mathtt{v}_1'}^n} \left|\partial_{yy}^2 w(\tau)\right| + \frac{1}{h_{\mathtt{v}_1'}^{n-1}} \left|\partial_y \nabla_{x'} w(\tau)\right| \right) \mathrm{d}\tau, \tag{4.36}$$

because the support D of ψ is contained in $B_{1/\sigma_\Omega} \times (-1/\sigma_\mathcal{Y}, 1/\sigma_\mathcal{Y})$, and so is mapped into $\omega_{\mathtt{v}_1} \subset \omega_T$. Notice also that $h_{\mathtt{v}_1''} \lesssim (1-t)h_{\mathtt{v}_1''} + t h_{\mathtt{v}_{1+2^n}''}$. This implies

$$I(t) \lesssim \left(\frac{h_{\mathtt{v}_1''}}{h_{\mathtt{v}_1'}^n} \|\partial_{yy}^2 w\|_{L^2(\omega_T, y^\alpha)} + \frac{1}{h_{\mathtt{v}_1'}^{n-1}} \|\nabla_{x'} \partial_y w\|_{L^2(\omega_T, y^\alpha)} \right) \|1\|_{L^2(\omega_T, y^{-\alpha})}, \tag{4.37}$$

which, together with (4.35), yields

$$\begin{aligned} |\delta q(\mathtt{v}_1)| \|\partial_y \lambda_{\mathtt{v}_1}\|_{L^2(T, y^\alpha)} \lesssim & \left(\frac{h_{\mathtt{v}_1''}}{h_{\mathtt{v}_1'}^n} \|\partial_{yy}^2 w\|_{L^2(\omega_T, y^\alpha)} + \frac{1}{h_{\mathtt{v}_1'}^{n-1}} \|\nabla_{x'} \partial_y w\|_{L^2(\omega_T, y^\alpha)} \right) \\ & \cdot \|1\|_{L^2(\omega_T, y^{-\alpha})} \|\partial_y \lambda_{\mathtt{v}_1}\|_{L^2(T, y^\alpha)}. \end{aligned} \tag{4.38}$$

Since $|y|^\alpha \in A_2(\mathbb{R}^{n+1})$, we have

$$\|1\|_{L^2(\omega_T, y^{-\alpha})} \|\partial_y \lambda_{\mathtt{v}_1}\|_{L^2(T, y^\alpha)} \lesssim h_{\mathtt{v}_1'}^n \frac{1}{h_{\mathtt{v}_1''}} \left(\int_I y^{-\alpha} \right)^{\frac{1}{2}} \left(\int_I y^{\alpha} \right)^{\frac{1}{2}} \lesssim h_{\mathtt{v}_1'}^n.$$

Replacing this into (4.38), we obtain

$$|\delta q(\mathtt{v}_1)| \|\partial_y \lambda_{\mathtt{v}_1}\|_{L^2(T, y^\alpha)} \lesssim h_{\mathtt{v}_1}' \|\nabla_{x'} \partial_y w\|_{L^2(\omega_T, y^\alpha)} + h_{\mathtt{v}_1''} \|\partial_{yy}^2 w\|_{L^2(\omega_T, y^\alpha)}, \tag{4.39}$$

which, in this case, implies (4.30).

We now proceed to estimate the differences $|q(\mathtt{v}_i) - q(\mathtt{v}_{i+2^n})|$ in (4.31) for $i = 2, \dots, 2^n$. We employ the arguments presented in [30, Theorem 2.5] in conjunction with the techniques developed to get the estimate (4.39). We start by writing

$$\begin{aligned} q(\mathtt{v}_i) - q(\mathtt{v}_{i+2^n}) &= w_{\mathtt{v}_1}(\mathtt{v}_i) - w_{\mathtt{v}_i}(\mathtt{v}_i) - (w_{\mathtt{v}_1}(\mathtt{v}_{i+2^n}) - w_{\mathtt{v}_{i+2^n}}(\mathtt{v}_{i+2^n})) \\ &= w_{\mathtt{v}_1}(\mathtt{v}_i) - w_{\mathtt{v}_1}(\mathtt{v}_{i+2^n}) - (w_{\mathtt{v}_i}(\mathtt{v}_i) - w_{\mathtt{v}_i}(\mathtt{v}_{i+2^n})) \\ &\quad + (w_{\mathtt{v}_{i+2^n}}(\mathtt{v}_{i+2^n}) - w_{\mathtt{v}_i}(\mathtt{v}_{i+2^n})) = I - II + III. \end{aligned}$$

Term III is identical to (4.32). The novelty here is the presence of terms I and II which, in view of (4.11) and the fact that $\mathtt{v}_i' = \mathtt{v}_{i+2^n}'$ for $i = 2, \dots, 2^n$, can be rewritten as

$$\begin{aligned} I - II &= \int_{\omega_{\mathtt{v}_1}} (\mathtt{v}_i'' - \mathtt{v}_{i+2^n}'') \partial_y w(x) \psi_{\mathtt{v}_1}(x) \, \mathrm{d}x - \int_{\omega_{\mathtt{v}_i}} (\mathtt{v}_i'' - \mathtt{v}_{i+2^n}'') \partial_y w(x) \psi_{\mathtt{v}_i}(x) \, \mathrm{d}x \\ &= (\mathtt{v}_i'' - \mathtt{v}_{i+2^n}'') \int \left(\partial_y w(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z) - \partial_y w(\mathtt{v}_i - h_{\mathtt{v}_i} \odot z) \right) \psi(z) \, \mathrm{d}z. \end{aligned}$$

To estimate this expression, we define $\vartheta = (\mathtt{v}_1' - \mathtt{v}_i' - (h_{\mathtt{v}_1'} - h_{\mathtt{v}_i'}) z', 0)$, and the function $G_z(t) = \partial_y w(\mathtt{v}_i - h_{\mathtt{v}_i} \odot z + t\vartheta)$. Then, by using $\mathtt{v}_1'' = \mathtt{v}_i''$ and $h_{\mathtt{v}_1''} = h_{\mathtt{v}_i''}$ for $i = 2, \dots, 2^n$, we arrive at

$$I - II = (\mathtt{v}_i'' - \mathtt{v}_{i+2^n}'') \int_0^1 \int G_z'(t) \psi(z) \, \mathrm{d}z \, \mathrm{d}t.$$

Proceeding as in the case $i=1$, we obtain

$$|I-II|\|\partial_y \lambda_{\mathtt{v}_i}\|_{L^2(T,y^\alpha)} \lesssim h_{\mathtt{v}'_i} \|\partial_y \nabla_{x'} w\|_{L^2(\omega_T,y^\alpha)}.$$

Collecting the estimates above for $i=2,\dots,2^n$, we finally get

$$|q(\mathtt{v}_i)-q(\mathtt{v}_{i+2^n})|\|\partial_y \lambda_{\mathtt{v}_i}\|_{L^2(T,y^\alpha)} \lesssim h_{\mathtt{v}'} \|\partial_y \nabla_{x'} w\|_{L^2(\omega_T,y^\alpha)} + h_{\mathtt{v}''}\|\partial_{yy} w\|_{L^2(\omega_T,y^\alpha)}.$$

This together with (4.39) implies the desired estimate (4.30) for $j=n+1$.

$\boxed{2}$ *Derivatives* $\nabla_{x'}$ *in the domain* Ω. To prove an estimate for $\nabla_{x'}(w-\Pi_{\mathscr{T}_\mathcal{Y}} w)$ we notice that, given a vertex $\mathtt{v}$, the associated basis function $\lambda_\mathtt{v}$ can be written as $\lambda_\mathtt{v}(x) = \Lambda_{\mathtt{v}'}(x')\mu_{\mathtt{v}''}(y)$, where $\Lambda_{\mathtt{v}'}$ is the canonical $\mathbb{Q}_1$ basis function on the variable x' associated to the node $\mathtt{v}'$ in the triangulation $\mathscr{T}_\Omega$, and $\mu_{\mathtt{v}''}$ corresponds to the piecewise $\mathbb{P}_1$ basis function associated to the node $\mathtt{v}''$. Recall that, by construction, the basis $\{\Lambda_i\}_{i=1}^{2^n}$ possesses the so-called partition of unity property, i.e.,

$$\sum_{i=1}^{2^n} \Lambda_{\mathtt{v}'_i}(x') = 1 \quad \forall x' \in K, \qquad \Longrightarrow \qquad \sum_{i=1}^{2^n} \nabla_{x'}\Lambda_{\mathtt{v}'_i}(x') = 0 \quad \forall x' \in K. \tag{4.40}$$

This implies that, for every $q \in \mathbb{Q}_1(T)$,

$$\begin{aligned}
\nabla_{x'} q &= \sum_{i=1}^{2^{n+1}} q(\mathtt{v}_i)\nabla_{x'}\lambda_{\mathtt{v}_i} = \sum_{i=1}^{2^n} \Big(q(\mathtt{v}_i)\mu_{\mathtt{v}''_i}(y) + q(\mathtt{v}_{i+2^n})\mu_{\mathtt{v}''_{i+2^n}}(y) \Big) \nabla_{x'}\Lambda_{\mathtt{v}'_i}(x') \\
&= \sum_{i=1}^{2^n} \Big[(q(\mathtt{v}_i)-q(\mathtt{v}_1))\mu_{\mathtt{v}''_i}(y) + (q(\mathtt{v}_{i+2^n}) - q(\mathtt{v}_{1+2^n}))\mu_{\mathtt{v}''_{i+2^n}}(y) \Big] \nabla_{x'}\Lambda_{\mathtt{v}'_i}(x'),
\end{aligned}$$

whence, for $j=1,\dots,n$,

$$\begin{aligned}
\|\partial_{x_j} q\|_{L^2(T,y^\alpha)} \lesssim & \sum_{i=1}^{2^n} |q(\mathtt{v}_i)-q(\mathtt{v}_1)| \|\mu_{\mathtt{v}''_i}\partial_{x_j}\Lambda_{\mathtt{v}'_i}\|_{L^2(T,y^\alpha)} \\
&+ \sum_{i=1}^{2^n} |q(\mathtt{v}_{1+2^n}) - q(\mathtt{v}_{i+2^n})| \|\mu_{\mathtt{v}''_{i+2^n}}\partial_{x_j}\Lambda_{\mathtt{v}'_i}\|_{L^2(T,y^\alpha)}.
\end{aligned}$$

This expression shows that the same techniques developed for the previous step lead to (4.30). In fact, we let $q = w_{\mathtt{v}_1} - \Pi_{\mathscr{T}_\mathcal{Y}} w \in \mathbb{Q}_1$ and estimate $\delta q(\mathtt{v}_i) := q(\mathtt{v}_i) - q(\mathtt{v}_1)$ and $\delta q(\mathtt{v}_{i+2^n}) := q(\mathtt{v}_{i+2^n}) - q(\mathtt{v}_{1+2^n})$ for $i=2,\dots,2^n$ as follows; we deal with $\delta q(\mathtt{v}_i)$ only because the same argument applies to $\delta q(\mathtt{v}_{i+2^n})$. Using (4.11) and changing variables, we derive

$$\delta q(\mathtt{v}_i) = w_{\mathtt{v}_1}(\mathtt{v}_i) - w_{\mathtt{v}_i}(\mathtt{v}_i) = \int \left(P(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z, \mathtt{v}_i) - P(\mathtt{v}_i - h_{\mathtt{v}_i} \odot z, \mathtt{v}_i) \right) \psi(z) \, \mathrm{d}z.$$

Defining the vector $\varrho := (\varrho_1, 0) = (\mathtt{v}'_1 - \mathtt{v}'_i + (h'_{\mathtt{v}_1} - h'_{\mathtt{v}_i})z', 0)$ and $H_z(t) := P(\mathtt{v}_i - h_{\mathtt{v}_i} \odot z + t\varrho, \mathtt{v}_i)$ yields

$$\delta q(\mathtt{v}_i) = \int_0^1 \int H'_z(t)\psi(z) \, \mathrm{d}z \, \mathrm{d}t.$$

Since ψ is bounded in L^∞ and $\operatorname{supp}\psi \subset D$, we next invoke the definitions of H_z and the polynomial P, to deduce

$$\int |H_z'(t)\psi(z)|\,\mathrm{d}z \lesssim \int_D |\nabla_{x'}\partial_{x_j} w(\mathtt{v}_i - h_{\mathtt{v}_i} \odot z + t\varrho)||h_{\mathtt{v}_i'} z' + t\varrho_1||\varrho_1|\,\mathrm{d}z \\ + \int_D |\partial_y \partial_{x_j} w(\mathtt{v}_i - h_{\mathtt{v}_i} \odot z + t\varrho)||h_{\mathtt{v}_i''} z''||\varrho_1|\,\mathrm{d}z.$$

Arguing as with the estimate (4.39), and using the scaling result

$$\|1\|_{L^2(\omega_T, y^\alpha)} \|\mu_{\mathtt{v}_i''} \partial_{x_j} \Lambda_{\mathtt{v}_i'}\|_{L^2(T,y^\alpha)} \lesssim h_{\mathtt{v}_i'}^{n-1} h_{\mathtt{v}_i''},$$

we infer that

$$|\delta q(\mathtt{v}_i)| \|\mu_{\mathtt{v}_i''} \partial_{x_j} \Lambda_{\mathtt{v}_i'}\|_{L^2(T,y^\alpha)} \lesssim h_{\mathtt{v}_i'} \|\nabla_{x'} \partial_{x_j} w\|_{L^2(\omega_T,y^\alpha)} + h_{\mathtt{v}_i''} \|\partial_y \partial_{x_j} w\|_{L^2(\omega_T,y^\alpha)}.$$

Finally, collecting the above estimates we obtain (4.30) for ∂_{x_j} with $j = 1, \dots, n$.

$\boxed{3}$ *Stability.* It remains to prove (4.28) and (4.29). By the triangle inequality,

$$\|\partial_y \Pi_{\mathscr{T}_\mathcal{Y}} w\|_{L^2(T,y^\alpha)} \le \|\partial_y (w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(T,y^\alpha)} + \|\partial_y w\|_{L^2(T,y^\alpha)},$$

so that it suffices to estimate the first term. Add and subtract $w_{\mathtt{v}_1}$,

$$\|\partial_y (w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(T,y^\alpha)} \le \|\partial_y (w - w_{\mathtt{v}_1})\|_{L^2(T,y^\alpha)} + \|\partial_y (w_{\mathtt{v}_1} - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(T,y^\alpha)}. \quad (4.41)$$

Let us estimate the first term. The definition of $\psi_{\mathtt{v}_1}$, together with $|y|^\alpha \in A_2(\mathbb{R}^{n+1})$ implies $\|\psi_{\mathtt{v}_1}\|_{L^2(\omega_{\mathtt{v}_1}, y^{-\alpha})} \|1\|_{L^2(\omega_{\mathtt{v}_1}, y^\alpha)} \lesssim 1$, whence invoking the definition (4.11) of the regularized Taylor polynomial $w_{\mathtt{v}_1}$ yields

$$\|\partial_y w_{\mathtt{v}_1}\|_{L^2(T,y^\alpha)} \le \|\partial_y w\|_{L^2(\omega_{\mathtt{v}_1}, y^\alpha)},$$

and

$$\|\partial_y (w - w_{\mathtt{v}_1})\|_{L^2(T,y^\alpha)} \le \|\partial_y w\|_{L^2(T,y^\alpha)} + \|\partial_y w_{\mathtt{v}_1}\|_{L^2(T,y^\alpha)} \lesssim \|\partial_y w\|_{L^2(\omega_{\mathtt{v}_1}, y^\alpha)}. \quad (4.42)$$

To estimate the second term of the right hand side of (4.41), we repeat the steps used to obtain (4.30), starting from (4.32). We recall $\delta q(\mathtt{v}_i) = q(\mathtt{v}_i) - q(\mathtt{v}_{i+2^n})$, and we proceed to estimate $\delta q(\mathtt{v}_1)$. Integrating by parts and using that $\psi_{\mathtt{v}_i} = 0$ on $\partial \omega_{\mathtt{v}_i}$, we get, for $\ell = 1, \dots, n+1$,

$$\int_{\omega_{\mathtt{v}_i}} \partial_{x_\ell} w(x)(z_\ell - x_\ell)\psi_{\mathtt{v}_i}(x)\,\mathrm{d}x = \int_{\omega_{\mathtt{v}_i}} w(x)\psi_{\mathtt{v}_i}(x)\,\mathrm{d}x \\ - \int_{\omega_{\mathtt{v}_i}} w(x)(z_\ell - x_\ell)\partial_{x_\ell}\psi_{\mathtt{v}_i}(x)\,\mathrm{d}x,$$

whence

$$\begin{aligned} \delta q(\mathtt{v}_1) &= (n+2)\left(\int w(x)\psi_{\mathtt{v}_{1+2^n}}\,\mathrm{d}x - \int w(x)\psi_{\mathtt{v}_1}\,\mathrm{d}x\right) \\ &- \int w(x)(\mathtt{v}_{1+2^n} - x)\cdot\nabla\psi_{\mathtt{v}_{1+2^n}}(x)\,\mathrm{d}x + \int w(x)(\mathtt{v}_1 - x)\cdot\nabla\psi_{\mathtt{v}_1}(x)\,\mathrm{d}x \\ &= I_1 + I_2. \end{aligned} \quad (4.43)$$

To estimate I_1 we consider the same change of variables used to obtain (4.33). Define $G_z(t) = (n+2)\cdot w(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta)$, with θ as in (4.34), and observe that

$$I_1 = \int_0^1 \int G_z'(t)\psi(z)\,\mathrm{d}z\,\mathrm{d}t = (n+2)\int_0^1 \int \partial_y w(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta)\theta''\psi(z)\,\mathrm{d}z\,\mathrm{d}t.$$

Introducing the change of variables $\tau = \mathtt{v}_1 - h_{\mathtt{v}_1} \odot z + t\theta$, we obtain

$$|I_1| \lesssim \int_{\omega_T} \frac{1}{h^n_{\mathtt{v}_1'}} |\partial_y w(\tau)|\,\mathrm{d}\tau \le \frac{1}{h^n_{\mathtt{v}_1'}} \|\partial_y w\|_{L^2(\omega_T, y^\alpha)} \|1\|_{L^2(\omega_T, y^{-\alpha})}. \tag{4.44}$$

We now estimate I_2. Changing variables,

$$\begin{aligned} I_2 &= \int \big(w(\mathtt{v}_{1+2^n} - h_{\mathtt{v}_{1+2^n}} \odot z) - w(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z)\big)\, z' \cdot \nabla_{x'}\psi(z)\,\mathrm{d}z \\ &+ \int \big(w(\mathtt{v}_{1+2^n} - h_{\mathtt{v}_{1+2^n}} \odot z)z'' - w(\mathtt{v}_1 - h_{\mathtt{v}_1} \odot z)(\vartheta + z'')\big)\,\partial_y \psi(z)\,\mathrm{d}z \\ &= I_{2,1} + I_{2,2}, \end{aligned}$$

where $\vartheta = (\mathtt{v}''_{1+2^n} - \mathtt{v}''_1)/h_{\mathtt{v}''_1}$. Arguing as in the derivation of (4.44) we obtain

$$|I_{2,1}|, |I_{2,2}| \lesssim \int_{\omega_T} \frac{1}{h^n_{\mathtt{v}_1'}} |\partial_y w(\tau)|\,\mathrm{d}\tau \le \frac{1}{h^n_{\mathtt{v}_1'}} \|\partial_y w\|_{L^2(\omega_T, y^\alpha)} \|1\|_{L^2(\omega_T, y^{-\alpha})}. \tag{4.45}$$

Inserting (4.44) and (4.45) in (4.43) we deduce

$$|\delta q(\mathtt{v}_1)| \lesssim \frac{1}{h^n_{\mathtt{v}_1'}} \|\partial_y w\|_{L^2(\omega_T, y^\alpha)} \|1\|_{L^2(\omega_T, y^{-\alpha})},$$

whence

$$|\delta q(\mathtt{v}_1)| \|\partial_y \lambda_{\mathtt{v}_1}\|_{L^2(T, y^\alpha)} \lesssim \|\partial_y w\|_{L^2(\omega_T, y^\alpha)}, \tag{4.46}$$

because $h^{-n}_{\mathtt{v}_1'} \|\partial_y \lambda_{\mathtt{v}_1}\|_{L^2(\omega_T, y^\alpha)} \|1\|_{L^2(\omega_T, y^{-\alpha})} \le C$. Replacing (4.46) in (4.31), we get

$$\|\partial_y (w_{\mathtt{v}_1} - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(T, y^\alpha)} \lesssim \|\partial_y w\|_{L^2(\omega_T, y^\alpha)},$$

which, together with (4.41) and (4.42), imply the desired result (4.29) for $i = 1$. For $i = 2, \dots, 2^n$, the estimates for $\delta q(\mathtt{v}_i)$ follow the same steps as in $\boxed{1}$. To prove the stability bound (4.28) we proceed as in $\boxed{2}$ to estimate the interpolation errors for the x'-derivatives, but we skip the details. □

4.2.4. Weighted interpolation estimates on boundary elements. Let us now extend the interpolation estimates of § 4.2.2 and § 4.2.3 to elements that intersect the Dirichlet boundary, where the functions to be approximated vanish. To do so, we start by adapting the results of [30, Theorem 3.1] to our particular case.

We consider, as in [30, Section 3], different cases according to the relative position of the element T in $\mathscr{T}_\mathcal{Y}$. We define the three sets

$$\begin{aligned} \mathcal{C}_1 &= \{T \in \mathscr{T}_\mathcal{Y} : \partial T \cap \Gamma_D = \emptyset\}, \\ \mathcal{C}_2 &= \{T \in \mathscr{T}_\mathcal{Y} : \partial T \cap \partial_L \mathcal{C}_\mathcal{Y} \neq \emptyset\}, \\ \mathcal{C}_3 &= \{T \in \mathscr{T}_\mathcal{Y} : \partial T \cap (\Omega \times \{\mathcal{Y}\}) \neq \emptyset\}. \end{aligned}$$

The elements in $\mathcal{C}_1$ are interior, so the corresponding interpolation estimate is given in Theorem 4.7. Interpolation estimates on elements in $\mathcal{C}_3$ are a direct consequence of [30, Theorem 3.1] and Theorem 4.8 below. This is due to the fact that, since $\mathcal{Y} \geq 1$, the weight y^α over $\mathcal{C}_3$ is no longer singular nor degenerate. It remains only to provide interpolation estimates for elements in $\mathcal{C}_2$.

THEOREM 4.8 (Weighted H^1 interpolation estimates over elements in $\mathcal{C}_2$). *Let $T \in \mathcal{C}_2$ and $w \in H^1(\omega_T, y^\alpha)$ vanish on $\partial T \cap \partial_L \mathcal{C}_\mathcal{Y}$. Then, we have the stability bounds*

$$\|\nabla_{x'} \Pi_{\mathscr{T}_\mathcal{Y}} w\|_{L^2(T,y^\alpha)} \lesssim \|\nabla_{x'} w\|_{L^2(\omega_T,y^\alpha)}, \tag{4.47}$$

$$\|\partial_y \Pi_{\mathscr{T}_\mathcal{Y}} w\|_{L^2(T,y^\alpha)} \lesssim \|\partial_y w\|_{L^2(\omega_T,y^\alpha)}, \tag{4.48}$$

If, in addition, $w \in H^2(\omega_T, y^\alpha)$, then, for $j = 1, \ldots, n+1$,

$$\|\partial_{x_j}(w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(T,y^\alpha)} \lesssim h_{\mathtt{v}'} \|\partial_{x_j} \nabla_{x'} w\|_{L^2(\omega_T,y^\alpha)} + h_{\mathtt{v}''} \|\partial_{x_j y} w\|_{L^2(\omega_T,y^\alpha)}. \tag{4.49}$$

Proof. For simplicity we present the proof in two dimensions. Let $T = (0,a) \times (0,b) \in \mathcal{C}_2$ and let us label its vertices according to Figure 4.1: $\mathtt{v}_1 = (0,0)$, $\mathtt{v}_2 = (a,0)$, $\mathtt{v}_3 = (0,b)$, $\mathtt{v}_4 = (a,b)$. Notice that this is the worst situation because over such an element the weight becomes degenerate or singular; estimates over other elements of $\mathcal{C}_2$ are simpler. We proceed now to exploit the symmetry of T. By the definition of $\Pi_{\mathscr{T}_\mathcal{Y}}$ we have

$$\Pi_{\mathscr{T}_\mathcal{Y}} w|_T = w_{\mathtt{v}_2}(\mathtt{v}_2)\lambda_{\mathtt{v}_2} + w_{\mathtt{v}_4}(\mathtt{v}_4)\lambda_{\mathtt{v}_4}. \tag{4.50}$$

The proofs of (4.47) and (4.48) are similar to Step 3 of Theorem 4.7. To show (4.49), we write the local difference between a function and its interpolant as $(w - \Pi_{\mathscr{T}_\mathcal{Y}} w)|_T = (w - w_{\mathtt{v}_2})|_T + (w_{\mathtt{v}_2} - \Pi_{\mathscr{T}_\mathcal{Y}})|_T$. Proceeding as in the proof of Lemma 4.5, we can bound $\partial_{x_j}(w - w_{\mathtt{v}_2})|_T$ for $j = 1, 2$, in the $L^2(T, y^\alpha)$-norm, by the right hand side of (4.49) because this is independent of the trace of w. It remains then to derive a bound for $(w_{\mathtt{v}_2} - \Pi_{\mathscr{T}_\mathcal{Y}} w)|_T$, for which we consider two separate cases.

$\boxed{1}$ *Derivative in the extended direction.* We use $w_{\mathtt{v}_2} \in \mathbb{Q}_1$, (4.50) and $\Pi_{\mathscr{T}_\mathcal{Y}} w(\mathtt{v}_1) = \Pi_{\mathscr{T}_\mathcal{Y}} w(\mathtt{v}_3) = 0$, to write

$$\partial_y (w_{\mathtt{v}_2} - \Pi_{\mathscr{T}_\mathcal{Y}} w)|_T = (w_{\mathtt{v}_2}(\mathtt{v}_3) - w_{\mathtt{v}_2}(\mathtt{v}_1))\, \partial_y \lambda_{\mathtt{v}_3} + (w_{\mathtt{v}_2}(\mathtt{v}_4) - w_{\mathtt{v}_4}(\mathtt{v}_4))\, \partial_y \lambda_{\mathtt{v}_4}.$$

Since $w \equiv 0$ on $\{0\} \times (0,b)$, then $\partial_y w \equiv 0$ on $\{0\} \times (0,b)$. By the definition of the Taylor polynomial P, given in (4.12), and the fact that $\mathtt{v}_1' = \mathtt{v}_3'$, we obtain

$$\begin{aligned} w_{\mathtt{v}_2}(\mathtt{v}_3) - w_{\mathtt{v}_2}(\mathtt{v}_1) &= (\mathtt{v}_3'' - \mathtt{v}_1'') \int_{\omega_{\mathtt{v}_2}} \partial_y w(x) \psi_{\mathtt{v}_2}(x)\, \mathrm{d}x \\ &= (\mathtt{v}_3'' - \mathtt{v}_1'') \int_{\omega_{\mathtt{v}_2}} \int_0^{x'} \partial_{x'y} w(\sigma, y) \psi_{\mathtt{v}_2}(x', y)\, \mathrm{d}\sigma\, \mathrm{d}x'\, \mathrm{d}y. \end{aligned}$$

Therefore

$$\begin{aligned} |w_{\mathtt{v}_2}(\mathtt{v}_3) - w_{\mathtt{v}_2}(\mathtt{v}_1)| &\lesssim h_{\mathtt{v}_1''} h_{\mathtt{v}_1'} \|\partial_{x'y} w\|_{L^2(\omega_T,y^\alpha)} \|\psi_{\mathtt{v}_2}\|_{L^2(\omega_T,y^{-\alpha})} \\ &\lesssim h_{\mathtt{v}_1''} h_{\mathtt{v}_1'} \frac{h_{\mathtt{v}_1'}^{\frac{1}{2}}}{h_{\mathtt{v}_2'} h_{\mathtt{v}_2''}} \left(\int_0^b y^{-\alpha}\, \mathrm{d}y \right)^{\frac{1}{2}} \|\partial_{x'y} w\|_{L^2(\omega_T,y^\alpha)}. \end{aligned}$$

Since, in view of the weak shape regularity assumption on the mesh $\mathscr{T}_{\mathcal{Y}}$, $h_{\mathtt{v}_1'} \approx h_{\mathtt{v}_2'}$, $h_{\mathtt{v}_1''} = h_{\mathtt{v}_2''}$, and $y^\alpha \in A_2(\mathbb{R}^{n+1}_+)$, we conclude that

$$
\begin{aligned}
|w_{\mathtt{v}_2}(\mathtt{v}_3) - w_{\mathtt{v}_2}(\mathtt{v}_1)| \|\partial_y \lambda_{\mathtt{v}_3}\|_{L^2(T,y^\alpha)} &\lesssim \frac{h_{\mathtt{v}_1'}}{h_{\mathtt{v}_1''}} \left(\int_0^b y^{-\alpha} \,\mathrm{d}y \int_0^b y^\alpha \,\mathrm{d}y \right)^{\frac{1}{2}} \times \\
&\quad \times \|\partial_{x'y} w\|_{L^2(\omega_T,y^\alpha)} \\
&\lesssim h_{\mathtt{v}_1'} \|\partial_{x'y} w\|_{L^2(\omega_T,y^\alpha)}.
\end{aligned}
\tag{4.51}
$$

Finally, to bound $w_{\mathtt{v}_2}(\mathtt{v}_4) - w_{\mathtt{v}_4}(\mathtt{v}_4)$, we proceed as in Step 1 of the proof of Theorem 4.7, which is valid regardless of the trace of w, and deduce

$$
|w_{\mathtt{v}_2}(\mathtt{v}_4) - w_{\mathtt{v}_4}(\mathtt{v}_4)| \|\partial_y \lambda_{\mathtt{v}_3}\|_{L^2(T,y^\alpha)} \lesssim h_{\mathtt{v}_1'} \|\partial_{x'y} w\|_{L^2(\omega_T,y^\alpha)} + h_{\mathtt{v}_1''} \|\partial_{yy} w\|_{L^2(\omega_T,y^\alpha)}.
$$

This, in conjunction with the previous estimate, yields (4.49) for the derivative in the extended direction.

[2] *Derivative in the x' direction.* To estimate $\partial_{x'}(w_{\mathtt{v}_2} - \Pi_{\mathscr{T}_{\mathcal{Y}}} w)|_T$ we proceed as in Theorem 4.7 and [30, Theorem 3.1], but we cannot exploit the symmetry of the tensor product structure now. For brevity, we shall only point out the main technical differences. Using, again, that $(w_{\mathtt{v}_2} - \Pi_{\mathscr{T}_{\mathcal{Y}}} w) \in \mathbb{Q}_1$,

$$
\begin{aligned}
\partial_{x'}(w_{\mathtt{v}_2} - \Pi_{\mathscr{T}_{\mathcal{Y}}} w)|_T &= w_{\mathtt{v}_2}(\mathtt{v}_1) \partial_{x'} \lambda_{\mathtt{v}_1} + w_{\mathtt{v}_2}(\mathtt{v}_3) \partial_{x'} \lambda_{\mathtt{v}_3} + (w_{\mathtt{v}_2}(\mathtt{v}_4) - w_{\mathtt{v}_4}(\mathtt{v}_4)) \partial_{x'} \lambda_{\mathtt{v}_4} \\
&= w_{\mathtt{v}_2}(\mathtt{v}_1) \partial_{x'} \lambda_{\mathtt{v}_1} + (w_{\mathtt{v}_2}(\mathtt{v}_4) - w_{\mathtt{v}_2}(\mathtt{v}_3)) \partial_{x'} \lambda_{\mathtt{v}_4} \\
&\quad - (w_{\mathtt{v}_4}(\mathtt{v}_4) - w_{\mathtt{v}_4}(\mathtt{v}_3)) \partial_{x'} \lambda_{\mathtt{v}_4} - w_{\mathtt{v}_4}(\mathtt{v}_3) \partial_{x'} \lambda_{\mathtt{v}_4} \\
&= J(w_{\mathtt{v}_2}, w_{\mathtt{v}_4}) \partial_{x'} \lambda_{\mathtt{v}_4} + w_{\mathtt{v}_2}(\mathtt{v}_1) \partial_{x'} \lambda_{\mathtt{v}_1} - w_{\mathtt{v}_4}(\mathtt{v}_3) \partial_{x'} \lambda_{\mathtt{v}_4},
\end{aligned}
$$

where

$$
J(w_{\mathtt{v}_2}, w_{\mathtt{v}_4}) = (w_{\mathtt{v}_2}(\mathtt{v}_4) - w_{\mathtt{v}_2}(\mathtt{v}_3)) - (w_{\mathtt{v}_4}(\mathtt{v}_4) - w_{\mathtt{v}_4}(\mathtt{v}_3)).
$$

Define $\theta = (0, \theta'') = (0, \mathtt{v}_4'' - \mathtt{v}_2'' - (h_{\mathtt{v}_4''} - h_{\mathtt{v}_2''}) z'')$, and rewrite $J(w_{\mathtt{v}_2}, w_{\mathtt{v}_4})$ as follows:

$$
\begin{aligned}
J(w_{\mathtt{v}_2}, w_{\mathtt{v}_4}) &= (\mathtt{v}_4' - \mathtt{v}_3') \int_D \left(\partial_{x'} w(\mathtt{v}_2 - h_{\mathtt{v}_2} \odot z) - \partial_{x'} w(\mathtt{v}_4 - h_{\mathtt{v}_4} \odot z) \right) \psi(z) \,\mathrm{d}z \\
&= -(\mathtt{v}_4' - \mathtt{v}_3') \int_D \int_0^1 \partial_{x'y} w(\mathtt{v}_2 - h_{\mathtt{v}_2} \odot z + \theta t) \theta'' \psi(z) \,\mathrm{d}t \,\mathrm{d}z,
\end{aligned}
$$

where $D = \operatorname{supp} \psi$. Denote

$$
I(t) = \int |\partial_{x'y} w(\mathtt{v}_2 - h_{\mathtt{v}_2} \odot z + \theta t) \theta''| \,\mathrm{d}z.
$$

Using the change of variables $z \mapsto \tau = \mathtt{v}_2 - h_{\mathtt{v}_2} \odot z + \theta t$, results in

$$
\begin{aligned}
|I(t)| &\lesssim \frac{1}{h_{\mathtt{v}_2'}} \int_{\omega_T} |\partial_{x'y} w(\tau)| \psi(\tau) \,\mathrm{d}\tau \lesssim \frac{1}{h_{\mathtt{v}_2'}} \|\partial_{x'y} w\|_{L^2(\omega_T,y^\alpha)} \|1\|_{L^2(\omega_T,y^{-\alpha})} \\
&\lesssim h_{\mathtt{v}_2'}^{-\frac{1}{2}} \|\partial_{x'y} w\|_{L^2(\omega_T,y^\alpha)} \left(\int_0^b y^{-\alpha} \,\mathrm{d}y \right)^{\frac{1}{2}},
\end{aligned}
$$

whence $|J(w_{\mathtt{v}_2}, w_{\mathtt{v}_4})| \lesssim h_{\mathtt{v}_2'}^{\frac{1}{2}} \|\partial_{x'y} w\|_{L^2(\omega_T, y^\alpha)} \left(\int_0^b y^{-\alpha} \, \mathrm{d}y \right)^{\frac{1}{2}}$. This implies

$$\|J(w_{\mathtt{v}_2}, w_{\mathtt{v}_4}) \partial_{x'} \lambda_{\mathtt{v}_4}\|_{L^2(T, y^\alpha)} \lesssim \left(\int_0^b y^{-\alpha} \, \mathrm{d}y \right)^{\frac{1}{2}} \left(\int_0^b y^{\alpha} \, \mathrm{d}y \right)^{\frac{1}{2}} \|\partial_{x'y} w\|_{L^2(\omega_T, y^\alpha)}$$
$$\lesssim h_{\mathtt{v}_2''} \|\partial_{x'y} w\|_{L^2(\omega_T, y^\alpha)},$$

which follows from the fact that $y^\alpha \in A_2(\mathbb{R}^+)$, and then (4.49) holds true.

The estimate of $w_{\mathtt{v}_2}(\mathtt{v}_1)\partial_{x'}\lambda_{\mathtt{v}_2}$ exploits the fact that the trace of w vanishes on $\partial_L \mathcal{C}_\mathcal{Y}$; the same happens with $w_{\mathtt{v}_4}(\mathtt{v}_3)\partial_{x'}\lambda_{\mathtt{v}_4}$. In fact, we can write

$$w_{\mathtt{v}_2}(\mathtt{v}_1) = \int_{\omega_{\mathtt{v}_2}} \int_0^{x'} \left(\partial_{x'} w(\tau, y) - \partial_{x'} w(x', y) \right) \psi_{\mathtt{v}_2}(x', y) \, \mathrm{d}\tau \, \mathrm{d}x' \, \mathrm{d}y$$
$$+ \int_{\omega_{\mathtt{v}_2}} \left(\partial_y w(0, y) - \partial_y w(x', y) \right) y \psi_{\mathtt{v}_2}(x', y) \, \mathrm{d}x' \, \mathrm{d}y.$$

To derive (4.49) we finally proceed as in the proofs of Theorem 4.7 and [30, Theorem 3.1]. We omit the details. ▯

We now conclude with a result involving weighted L^2 interpolation estimates on boundary elements. As in the weighted H^1 case, the elements in $\mathcal{C}_1$ are interior, and then, the interpolation estimates are given by Theorem (4.6). It remains, to analyze the interpolation estimates on the sets $\mathcal{C}_2$ and $\mathcal{C}_3$.

THEOREM 4.9 (Weighted L^2 interpolation estimates over elements in $\mathcal{C}_2$ and $\mathcal{C}_3$). *If $T \in \mathcal{C}_2 \cup \mathcal{C}_3$ and $w \in H^1(\omega_T, y^\alpha)$ vanish on $\partial T \cap \partial_L \mathcal{C}_\mathcal{Y}$ and $\partial T \cap (\{\Omega\} \times \mathcal{Y})$, then*

$$\|w - \Pi_{\mathscr{T}_\mathcal{Y}} w\|_{L^2(T, y^\alpha)} \lesssim h_{\mathtt{v}'} \|\nabla_{x'} w\|_{L^2(\omega_T, y^\alpha)} + h_{\mathtt{v}''} \|\partial_y w\|_{L^2(\omega_T, y^\alpha)}. \tag{4.52}$$

Proof. We consider $T \in \mathcal{C}_2$, and the same geometric setting as in the proof of Theorem 4.8; we skip the case $T \in \mathcal{C}_3$ as in Theorem 4.8. We write the difference $w - \Pi_{\mathscr{T}_\mathcal{Y}} w|_T = (w - w_{\mathtt{v}_2})|_T + (w_{\mathtt{v}_2} - \Pi_{\mathscr{T}_\mathcal{Y}} w)|_T$. Applying Lemma 4.5, we can bound the term $(w - w_{\mathtt{v}_2})|_T$ in the $L^2(T, y^\alpha)$-norm by the right hand side of (4.52). Then, it suffices to estimate $(w_{\mathtt{v}_2} - \Pi_{\mathscr{T}_\mathcal{Y}} w)|_T \in \mathbb{Q}_1(T)$. Writing

$$(w_{\mathtt{v}_2} - \Pi_{\mathscr{T}_\mathcal{Y}} w)|_T = w_{\mathtt{v}_2}(\mathtt{v}_1)\lambda_{\mathtt{v}_1} + w_{\mathtt{v}_2}(\mathtt{v}_3)\lambda_{\mathtt{v}_3} + (w_{\mathtt{v}_2}(\mathtt{v}_4) - w_{\mathtt{v}_4}(\mathtt{v}_4))\lambda_{\mathtt{v}_4},$$

and using the fact that the trace of w vanishes on $\partial_L \mathcal{C}_\mathcal{Y}$, we see that

$$w_{\mathtt{v}_2}(\mathtt{v}_1) = \int_{\omega_{\mathtt{v}_2}} \int_0^{x'} \partial_{x'} w(\sigma, y) \psi_{\mathtt{v}_2} \, \mathrm{d}\sigma \, \mathrm{d}x' \, \mathrm{d}y + \int_{\omega_{\mathtt{v}_2}} (\mathtt{v}_1 - x) \cdot \nabla w(x) \psi_{\mathtt{v}_2}(x) \, \mathrm{d}x; \tag{4.53}$$

the same argument holds for $w_{\mathtt{v}_2}(\mathtt{v}_3)$. On the other hand, we handle $w_{\mathtt{v}_2}(\mathtt{v}_4) - w_{\mathtt{v}_4}(\mathtt{v}_4)$ with the same techniques as in the proof of Theorem 4.7. ▯

5. Error estimates. The estimates of § 4.2.3 and § 4.2.4 are obtained under the local assumption that $w \in H^2(\omega_T, y^\alpha)$. However, the solution $\mathfrak{u}$ of (2.26) satisfies $\mathfrak{u}_{yy} \in L^2(\mathcal{C}, y^\beta)$ only when $\beta > 2\alpha + 1$, according to Theorem 2.7. For this reason, in this section we derive error estimates for both quasi-uniform and graded meshes. The estimates of § 5.1 for quasi-uniform meshes are quasi-optimal in terms of regularity but suboptimal in terms of order. The estimates of § 5.2 for graded meshes are, instead, quasi-optimal in both regularity and order. Mesh anisotropy is able to capture the singular behavior of the solution and restore optimal decay rates.

5.1. Quasi-uniform meshes. We start with a simple one dimensional case ($n = 1$) and assume that we need to approximate over the interval $[0, \mathcal{Y}]$ the function $w(y) = y^{1-\alpha}$. Notice that $w_y(y) \approx y^{-\alpha}$ as $y \approx 0^+$ has the same behavior as the derivative in the extended direction of the α-harmonic extension $\mathfrak{u}$.

Given $M \in \mathbb{N}$ we consider the uniform partition of the interval $[0, \mathcal{Y}]$

$$y_k = \frac{k}{M}\mathcal{Y}, \quad k = 0, \dots, M. \tag{5.1}$$

and corresponding elements $I_k = [y_k, y_{k+1}]$ of size $h_k = h = \mathcal{Y}/M$ for $k = 0, \dots, M-1$.

We can adapt the definition of $\Pi_{\mathscr{T}_\mathcal{Y}}$ of § 4.2 to this setting, and bound the local interpolation errors $E_k = \|\partial_y(w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2(I_k, y^\alpha)}$. For $k = 2, \dots, M-1$, since $y \geq h$ and $\alpha < 2\alpha + 1 < \beta$, (4.30) implies

$$E_k^2 \lesssim h^2 \int_{\omega_{I_k}} y^\alpha |w_{yy}|^2 \,\mathrm{d}y \lesssim h^{2+\alpha-\beta} \int_{\omega_{I_k}} y^\beta |w_{yy}|^2 \,\mathrm{d}y, \tag{5.2}$$

because $\left(\frac{y}{h}\right)^\alpha \leq \left(\frac{y}{h}\right)^\beta$. The estimate for $E_0^2 + E_1^2$ follows from from the stability of the operator $\Pi_{\mathscr{T}_\mathcal{Y}}$ (4.29) and (4.48):

$$E_0^2 + E_1^2 \lesssim \int_0^{3h} y^\alpha |w_y|^2 \lesssim h^{1-\alpha}, \tag{5.3}$$

because $w(y) \approx y^{-\alpha}$ as $y \approx 0^+$. Using (5.2) and (5.3) in conjunction with $2 + \alpha - \beta < 1 - \alpha$, we obtain a global interpolation estimate

$$\|\partial_y(w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|_{L^2((0,\mathcal{Y}), y^\alpha)} \lesssim h^{(2+\alpha-\beta)/2}. \tag{5.4}$$

These ideas can be extended to prove an error estimate for $\mathfrak{u}$ on uniform meshes.

Theorem 5.1 (Error estimate for quasi-uniform meshes). *Let* $\mathfrak{u}$ *solve* (2.26), *and* $V_{\mathscr{T}_\mathcal{Y}}$ *be the solution of* (4.4), *constructed over a quasi-uniform mesh of size* h. *If* $f \in \mathbb{H}^{1-s}(\Omega)$ *and* $\mathcal{Y} \approx |\log h|$, *then for all* $\epsilon > 0$

$$\|\nabla(\mathfrak{u} - V_{\mathscr{T}_\mathcal{Y}})\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \lesssim h^{s-\varepsilon} \|f\|_{\mathbb{H}^{1-s}(\Omega)}, \tag{5.5}$$

where the hidden constant blows up if ε *tends to* 0.

Proof. Use first Theorem 3.5 and Theorem 4.1, combined with (4.10), to reduce the approximation error to the interpolation error of $\mathfrak{u}$. Repeat next the steps leading to (5.2)–(5.3), but combining the interpolation estimates of Theorems 4.7 and 4.8 with the regularity results of Theorem 2.7, which are valid because $f \in \mathbb{H}^{1-s}(\Omega)$. □

Remark 5.2 (Sharpness of (5.5) for $s \neq \frac{1}{2}$). According to (2.34) and (2.37), $\partial_y \mathfrak{u} \approx y^{-\alpha}$, and this formally implies $\partial_y \mathfrak{u} \in H^{s-\varepsilon}(\mathcal{C}, y^\alpha)$ for all $\varepsilon > 0$ provided $f \in \mathbb{H}^{1-s}(\Omega)$. In this sense (5.5) appears to be sharp with respect to regularity even though it does not exhibit the optimal rate. We verify this argument via a simple numerical illustration for dimension $n = 1$. We let $\Omega = (0, 1)$, $s = 0.2$, right hand side $f = \pi^{2s} \sin(\pi x)$, and note that $u(x) = \sin(\pi x)$, and the solution $\mathfrak{u}$ to (1.2) is

$$\mathfrak{u}(x, y) = \frac{2^{1-s}\pi^s}{\Gamma(s)} \sin(\pi x) K_s(\pi y).$$

Figure 5.1 shows the rate of convergence for the $H^1(\mathcal{C}_\mathcal{Y}, y^\alpha)$-seminorm. Estimate (5.5) predicts a rate of $h^{-0.2-\varepsilon}$. We point out that for the α-harmonic extension

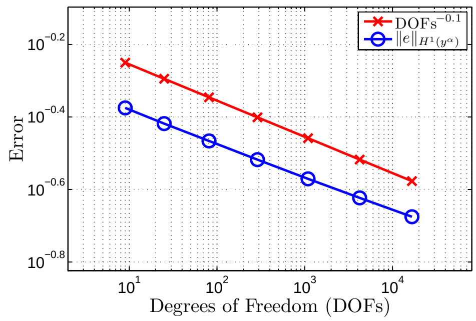

FIG. 5.1. *Computational rate of convergence* $\#(\mathscr{T}_\mathcal{Y})^{-s/(n+1)}$ *for quasi-uniform meshes* $\mathscr{T}_\mathcal{Y}$*, with* $s = 0.2$ *and* $n = 1$*.*

we are solving a two dimensional problem and, since the mesh $\mathscr{T}_\mathcal{Y}$ is quasi-uniform, $\#\mathscr{T}_\mathcal{Y} \approx h^{-2}$. In other words the rate of convergence, when measured in terms of degrees of freedom, is $(\#\mathscr{T}_\mathcal{Y})^{-0.1-\varepsilon}$, which is what Figure 5.1 displays.

REMARK 5.3 (Case $s = \frac{1}{2}$). Estimate (5.5) does not hold for $s = \frac{1}{2}$. In this case there is no weight and the scaling issues in (5.2) are no longer present, so that $E_k \lesssim h\|v\|_{H^2(I_k)}$. We thus obtain the optimal error estimate

$$\|\nabla(\mathfrak{u} - V_{\mathscr{T}_\mathcal{Y}})\|_{L^2(\mathcal{C}_\mathcal{Y})} \lesssim h\|f\|_{H^{1/2}_{00}(\Omega)}.$$

5.2. Graded meshes. The estimate (5.5) can be written equivalently

$$\|\nabla(\mathfrak{u} - V_{\mathscr{T}_\mathcal{Y}})\|_{L^2(\mathcal{C}_\mathcal{Y}, y^\alpha)} \lesssim (\#\mathscr{T}_\mathcal{Y})^{-\frac{s-\varepsilon}{n+1}}\|f\|_{\mathbb{H}^{1-s}(\Omega)},$$

for quasi-uniform meshes in dimension $n + 1$. We now show how to compensate the singular behavior in the extended variable y by anisotropic meshes and restore the optimal convergence rate $-1/(n + 1)$.

As in § 5.1 we start the discussion in dimension $n = 1$ with the function $w(y) = y^{1-\alpha}$ over $[0, \mathcal{Y}]$. We consider the graded partition $\mathscr{T}_\mathcal{Y}$ of the interval $[0, \mathcal{Y}]$

$$y_k = \left(\frac{k}{M}\right)^\gamma \mathcal{Y}, \quad k = 0, \ldots, M, \tag{5.6}$$

where $\gamma = \gamma(\alpha) > 3/(1 - \alpha) > 1$. If we denote by h_k the length of the interval

$$I_k = [y_k, y_{k+1}] = \left[\left(\frac{k}{M}\right)^\gamma \mathcal{Y}, \left(\frac{k+1}{M}\right)^\gamma \mathcal{Y}\right],$$

then

$$h_k = y_{k+1} - y_k \lesssim \frac{\mathcal{Y}}{M^\gamma} k^{\gamma-1}, \quad k = 1, \ldots, M - 1.$$

We again consider the operator $\Pi_{\mathscr{T}_\mathcal{Y}}$ of § 4.2 on the one dimensional mesh $\mathscr{T}_\mathcal{Y}$ and wish to bound the local interpolation errors E_k of § 5.1. We apply estimate (4.30) to interior elements to obtain that, for $k = 2, \dots, M-1$,

$$
\begin{aligned}
E_k^2 &\lesssim h_k^2 \int_{\omega_{I_k}} y^\alpha |w_{yy}|^2 \, \mathrm{d}y \lesssim \mathcal{Y}^2 \frac{k^{2(\gamma-1)}}{M^{2\gamma}} \int_{\omega_{I_k}} y^\alpha |w_{yy}|^2 \, \mathrm{d}y \\
&\lesssim \mathcal{Y}^{2+\alpha-\beta} \frac{k^{2(\gamma-1)}}{M^{2\gamma}} \left(\frac{k}{M}\right)^{\gamma(\alpha-\beta)} \int_{\omega_{I_k}} y^\beta |w_{yy}|^2 \, \mathrm{d}y \lesssim \mathcal{Y}^{1-\alpha} \frac{k^{\gamma(1-\alpha)-3}}{M^{\gamma(1-\alpha)}}.
\end{aligned}
\tag{5.7}
$$

because $y^\alpha \lesssim \left(\frac{k}{M}\right)^{\gamma(\alpha-\beta)} \mathcal{Y}^{\alpha-\beta} y^\beta$ and $w(y) = y^{1-\alpha}$ over $[0, \mathcal{Y}]$. Adding (5.7) over $k = 2, \dots, M-1$, and using that $\gamma(1-\alpha) > 3$, we arrive at

$$
\|\partial_y (w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|^2_{L^2((y_2,\mathcal{Y}),y^\alpha)} \lesssim \mathcal{Y}^{1-\alpha} M^{-2}. \tag{5.8}
$$

For the errors E_0^2, E_1^2 we resort to the stability bounds (4.29) and (4.48) to write

$$
\|\partial_y (w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|^2_{L^2((0,y_3),y^\alpha)} \lesssim \int_0^{\left(\frac{3}{M}\right)^\gamma \mathcal{Y}} y^{-\alpha} \, \mathrm{d}y \lesssim \frac{\mathcal{Y}^{1-\alpha}}{M^{\gamma(1-\alpha)}}, \tag{5.9}
$$

where we have used (5.6). Finally, adding (5.8) and (5.9) gives

$$
\|\partial_y (w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\|^2_{L^2((0,\mathcal{Y}),y^\alpha)} \lesssim \mathcal{Y}^{1-\alpha} M^{-2},
$$

and shows that the interpolation error exhibits optimal decay rate.

We now apply this idea to the numerical solution of problem (3.3). We assume $\mathscr{T}_\Omega$ to be quasi-uniform in $\mathbb{T}_\Omega$ with $\#\mathscr{T}_\Omega \approx M^n$ and construct $\mathscr{T}_\mathcal{Y} \in \mathbb{T}$ as the tensor product of $\mathscr{T}_\Omega$ and the partition given in (5.6), with $\gamma > 3/(1-\alpha)$. Consequently, $\#\mathscr{T}_\mathcal{Y} = M \cdot \#\mathscr{T}_\Omega \approx M^{n+1}$. Finally, we notice that since $\mathscr{T}_\Omega$ is shape regular and quasi-uniform, $h_{\mathscr{T}_\Omega} \approx (\#\mathscr{T}_\Omega)^{-1/n} \approx M^{-1}$.

Theorem 5.4 (Error estimate for graded meshes). *Let $V_\mathscr{T} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ solve* (4.4) *and $U_{\mathscr{T}_\Omega} \in \mathbb{U}(\mathscr{T}_\Omega)$ be defined as in* (4.5). *If $f \in \mathbb{H}^{1-s}(\Omega)$, then*

$$
\|\mathfrak{u} - V_{\mathscr{T}_\mathcal{Y}}\|_{\mathring{H}^1_L(\mathcal{C},y^\alpha)} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/4} \|f\|_{\mathbb{H}^{-s}(\Omega)} + \mathcal{Y}^{(1-\alpha)/2} (\#\mathscr{T}_\mathcal{Y})^{-1/(n+1)} \|f\|_{\mathbb{H}^{1-s}(\Omega)}, \tag{5.10}
$$

Proof. In light of (4.10), with $\epsilon \approx e^{-\sqrt{\lambda_1}\mathcal{Y}/4}$, it suffices to bound the interpolation error $\mathfrak{u} - \Pi_{\mathscr{T}_\mathcal{Y}} \mathfrak{u}$ on the mesh $\mathscr{T}_\mathcal{Y}$. To do so we, first of all, notice that if I_1 and I_2 are neighboring cells on the partition of $[0, \mathcal{Y}]$, then there is a constant $\sigma = \sigma(\gamma)$ such that $h_{I_1} \le \sigma h_{I_2}$, whence the weak regularity condition (c) holds. We can thus apply the polynomial interpolation theory of § 4.2. We decompose the mesh $\mathscr{T}_\mathcal{Y}$ into the sets

$$
\mathcal{T}_0 := \left\{ T \in \mathscr{T}_\mathcal{Y} : \ \omega_T \cap (\bar\Omega \times \{0\}) = \emptyset \right\}, \quad \mathcal{T}_1 := \left\{ T \in \mathscr{T}_\mathcal{Y} : \ \omega_T \cap (\bar\Omega \times \{0\}) \neq \emptyset \right\}.
$$

We observe that for all $T = K \times I_k \in \mathcal{T}_0$ we have $k \ge 2$ and $y^\alpha \lesssim \left(\frac{k}{M}\right)^{\gamma(\alpha-\beta)} \mathcal{Y}^{\alpha-\beta} y^\beta$. Applying Theorem 4.7 and Theorem 4.8 to elements in $\mathcal{T}_0$ we obtain

$$
\begin{aligned}
\sum_{T \in \mathcal{T}_0} \|\nabla(\mathfrak{u} - \Pi_{\mathscr{T}_\mathcal{Y}} \mathfrak{u})\|^2_{L^2(T,y^\alpha)} \lesssim \sum_{T = K \times I \in \mathcal{T}_0} &\Big(h_K^2 \|\nabla_{x'} \nabla \mathfrak{u}\|^2_{L^2(\omega_T,y^\alpha)} \\
&+ h_I^2 \|\partial_y \nabla_{x'} \mathfrak{u}\|^2_{L^2(\omega_T,y^\alpha)} + h_I^2 \|\partial_{yy} \mathfrak{u}\|^2_{L^2(\omega_T,y^\beta)} \Big) = S_1 + S_2 + S_3.
\end{aligned}
$$

We examine first the most problematic third term S_3, which we rewrite as follows:

$$S_3 \lesssim \sum_{k=2}^{M} \mathcal{Y}^{2+\alpha-\beta} \frac{k^{2(\gamma-1)}}{M^{2\gamma}} \left(\frac{k}{M}\right)^{\gamma(\alpha-\beta)} \int_{a_k}^{b_k} y^{\beta} \int_{\Omega} |\partial_{yy}\mathfrak{u}|^2 \, \mathrm{d}x' \, \mathrm{d}y,$$

with $a_k = \left(\frac{k-1}{M}\right)^{\gamma} \mathcal{Y}$ and $b_k = \left(\frac{k+1}{M}\right)^{\gamma} \mathcal{Y}$. We now invoke the local estimate (2.43), as well as the fact that $b_k - a_k \lesssim \left(\frac{k}{M}\right)^{\gamma-1} \frac{\mathcal{Y}}{M}$, to end up with

$$S_3 \lesssim \sum_{k=2}^{M} \mathcal{Y}^{1-\alpha} \frac{k^{\gamma(1-\alpha)-3}}{M^{\gamma(1-\alpha)}} \|f\|^2_{L^2(\Omega)} \lesssim \mathcal{Y}^{1-\alpha} M^{-2} \|f\|^2_{L^2(\Omega)}.$$

We now handle the middle term S_2 with the help of (2.42), which is valid for $b_k \le 1$. This imposes the restriction $k \le k_0 \le M\mathcal{Y}^{-1/\gamma}$, whereas for $k > k_0$ we know that the estimate decays exponentially. We thus have

$$S_2 \lesssim \|f\|^2_{\mathbb{H}^{1-s}(\Omega)} \sum_{k=2}^{k_0} \left(\left(\frac{k}{M}\right)^{\gamma-1} \frac{\mathcal{Y}}{M} \right)^3 \lesssim \frac{\mathcal{Y}^{2/\gamma}}{M^2} \|f\|^2_{\mathbb{H}^{1-s}(\Omega)} \lesssim \frac{\mathcal{Y}^{1-\alpha}}{M^2} \|f\|^2_{\mathbb{H}^{1-s}(\Omega)}.$$

The first term S_1 is easy to estimate. Since $h_K \lesssim M^{-1}$ for all $K \in \mathscr{T}_\Omega$, we get

$$S_1 \lesssim M^{-2} \|\nabla_{x'} \nabla v\|^2_{L^2(\mathcal{C}_{\mathcal{Y}}, y^\alpha)} \lesssim M^{-2} \|f\|^2_{\mathbb{H}^{1-s}(\Omega)} \lesssim \mathcal{Y}^{1-\alpha} M^{-2} \|f\|^2_{\mathbb{H}^{1-s}(\Omega)}.$$

For elements in $\mathcal{T}_1$, we rely on the stability estimates (4.28), (4.29), (4.47) and (4.48) of $\Pi_{\mathscr{T}_{\mathcal{Y}}}$ and thus repeat the arguments used to derive (5.8) and (5.9). Adding the estimates for $\mathcal{T}_0$ and $\mathcal{T}_1$ we obtain the assertion. □

Remark 5.5 (Choice of $\mathcal{Y}$). A natural choice of $\mathcal{Y}$ comes from equilibrating the two terms on the right-hand side of (5.10):

$$\epsilon \approx \#(\mathscr{T}_{\mathcal{Y}})^{-\frac{1}{n+1}} \quad \Leftrightarrow \quad \mathcal{Y} \approx \log(\#(\mathscr{T}_{\mathcal{Y}})).$$

This implies the near-optimal estimate

$$\|\mathfrak{u} - V_{\mathscr{T}_{\mathcal{Y}}}\|_{\mathring{H}^1_L(\mathcal{C}, y^\alpha)} \lesssim |\log(\#\mathscr{T}_{\mathcal{Y}})|^s (\#\mathscr{T}_{\mathcal{Y}})^{-1/(n+1)} \|f\|_{\mathbb{H}^{1-s}(\Omega)}. \tag{5.11}$$

Remark 5.6 (Estimate for u). In view of (4.6), we deduce the energy estimate

$$\|u - U_{\mathscr{T}_\Omega}\|_{\mathbb{H}^s(\Omega)} \lesssim |\log(\#\mathscr{T}_{\mathcal{Y}})|^s \, (\#\mathscr{T}_{\mathcal{Y}})^{-1/(n+1)} \|f\|_{\mathbb{H}^{1-s}(\Omega)}.$$

We can rewrite this estimate in terms of regularity $u \in \mathbb{H}^{1+s}(\Omega)$ and $\#\mathscr{T}_\Omega$ as

$$\|u - U_{\mathscr{T}_\Omega}\|_{\mathbb{H}^s(\Omega)} \lesssim |\log(\#\mathscr{T}_\Omega)|^s \, (\#\mathscr{T}_\Omega)^{-1/n} \|u\|_{\mathbb{H}^{1+s}(\Omega)}.$$

and realize that the order is near-optimal given the regularity shift from left to right. However, our PDE approach does not allow for a larger rate $(\#\mathscr{T}_\Omega)^{(2-s)/n}$ that would still be compatible with piecewise bilinear polynomials but not with (5.11).

Remark 5.7 (Computational complexity). The cost of solving the discrete problem (4.4) is related to $\#\mathscr{T}_{\mathcal{Y}}$, and not to $\#\mathscr{T}_\Omega$, but the resulting system is sparse. The structure of (4.4) is so that fast multilevel solvers can be designed with complexity proportional to $\#\mathscr{T}_{\mathcal{Y}}$. On the other hand, using an integral formulation requires sparsification of an otherwise dense matrix with associated cost $(\#\mathscr{T}_\Omega)^2$.

Remark 5.8 (Fractional regularity). The function $\mathfrak{u}$, solution of the α-harmonic extension problem, may also have singularities in the direction of the x'-variables and thus exhibit fractional regularity. This depends on Ω and the right hand side f (see Remark 2.8). The characterization of such singularities is as yet an open problem to us. The polynomial interpolation theory developed in § 4.2, however, applies to shape-regular but graded mesh $\mathscr{T}_\Omega$, which can resolve such singularities, provided we maintain the Cartesian structure of $\mathscr{T}_\mathcal{Y}$. The corresponding a posteriori error analysis is an entirely different but important direction currently under investigation.

Remark 5.9 (Simplicial elements). The approximation results presented in § 4.2.2, the interpolation theory developed in § 4.2.3 and § 4.2.4 and, consequently, the error estimates of this section hinge solely on the fact that the mesh $\mathscr{T}_\mathcal{Y}$ has a tensor product structure, i.e., it is composed of cells of the form $T = K \times I$. If we consider $\mathscr{T}_\Omega = \{K\}$ to be a mesh of $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ made of simplicial elements, together with the finite element space,

$$\mathbb{V}(\mathscr{T}_\mathcal{Y}) = \left\{ W \in C^0(\bar{\mathcal{C}}_\mathcal{Y}) : W|_T \in \mathbb{P}_1(K) \otimes \mathbb{P}_1(I) \ \forall T \in \mathscr{T}_\mathcal{Y},\ W|_{\Gamma_D} = 0 \right\},$$

we can adapt, without major modifications, all the approximation, interpolation and convergence results of this work.

Remark 5.10 (Hanging nodes). It is important to notice that the assumption that the mesh is conforming was never explicitly used in the results of Section 4 and that, actually, all that was required from the finite element space is the partition of unity property, i.e., (4.40). This observation allows us to generalize the results of Section 4 to meshes that possess hanging nodes, which is important if one desires to use mesh adaptation to resolve possible singularities in the solution.

6. Numerical experiments for the fractional Laplacian. To illustrate the proposed techniques here we present a couple of numerical examples. The implementation has been carried out with the help of the `deal.II` library (see [6, 7]) which, by design, is based on tensor product elements and thus is perfectly suitable for our needs. The main concern while developing the code was correctness and, therefore, integrals are evaluated numerically with Gaussian quadratures of sufficiently high order and linear systems are solved using CG with ILU preconditioner with the exit criterion being that the ℓ^2-norm of the residual is less than 10^{-12}. More efficient techniques for quadrature and preconditioning are currently under investigation.

6.1. A square domain. Let $\Omega = (0,1)^2$. It is common knowledge that

$$\varphi_{m,n}(x_1,x_2) = \sin(m\pi x_1)\sin(n\pi x_2), \quad \lambda_{m,n} = \pi^2\left(m^2+n^2\right), \qquad m,n \in \mathbb{N}.$$

If $f(x_1,x_2) = (2\pi^2)^s \sin(\pi x_1)\sin(\pi x_2)$, by (2.12) we have

$$u(x_1,x_2) = \sin(\pi x_1)\sin(\pi x_2),$$

and, by (2.24),

$$\mathfrak{u}(x_1,x_2,y) = \frac{2^{1-s}}{\Gamma(s)}(2\pi^2)^{s/2}\sin(\pi x_1)\sin(\pi x_2) y^s K_s(\sqrt{2}\pi y).$$

We construct a sequence of meshes $\{\mathscr{T}_{\mathcal{Y}_k}\}_{k\geq 1}$, where the triangulation of Ω is obtained by uniform refinement and the partition of $[0,\mathcal{Y}_k]$ is as in § 5.2, i.e., $[0,\mathcal{Y}_k]$ is divided with mesh points given by (5.6) with the election of the parameter $\gamma > 3/(1-\alpha)$. On the basis of Theorem 3.5, for each mesh the truncation parameter $\mathcal{Y}_k$ is chosen so that $\epsilon \approx (\#\mathscr{T}_{\mathcal{Y}_{k-1}})^{-1/3}$. This can be achieved, for instance, by setting

$$\mathcal{Y}_k \geq \mathcal{Y}_{0,k} = \frac{2}{\sqrt{\lambda_1}}(\log C - \log \epsilon).$$

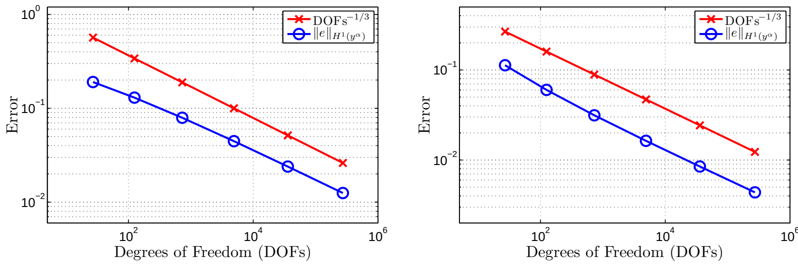

FIG. 6.1. *Computational rate of convergence for the approximate solution of the fractional Laplacian over a square with graded meshes on the extended dimension. The left panel shows the rate for* $s = 0.2$ *and the right one for* $s = 0.8$. *In both cases, the rate is* $\approx (\#\mathscr{T}_{\mathcal{Y}_k})^{-1/3}$ *in agreement with Theorem 5.4 and Remark 5.5*

With this type of meshes,

$$\|u - U_{\mathscr{T}_{\Omega,k}}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\mathfrak{u} - V_{\mathscr{T}_{\mathcal{Y}_k}}\|_{\mathring{H}^1_L(\mathcal{C},y^\alpha)} \lesssim |\log(\#\mathscr{T}_{\mathcal{Y}_k})|^s \cdot (\#\mathscr{T}_{\mathcal{Y}_k})^{-1/3},$$

which is near-optimal in $\mathfrak{u}$ but suboptimal in u, since we should expect (see [17])

$$\|u - U_{\mathscr{T}_{\Omega,k}}\|_{\mathbb{H}^s(\Omega)} \lesssim h^{2-s}_{\mathscr{T}_\Omega} \lesssim (\#\mathscr{T}_{\mathcal{Y}_k})^{-(2-s)/3}.$$

Figure 6.1 shows the rates of convergence for $s = 0.2$ and $s = 0.8$ respectively. In both cases, we obtain the rate given by Theorem 5.4 and Remark 5.5.

6.2. A circular domain. Let $\Omega = \{|x'| \in \mathbb{R}^2 : |x'| < 1\}$. Using polar coordinates it can be shown that

$$\varphi_{m,n}(r,\theta) = J_m(j_{m,n}r)\left(A_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)\right), \tag{6.1}$$

where J_m is the m-th Bessel function of the first kind; $j_{m,n}$ is the n-th zero of J_m and $A_{m,n}$, $B_{m,n}$ are real normalization constants that ensure $\|\varphi_{m,n}\|_{L^2(\Omega)} = 1$ for all $m, n \in \mathbb{N}$. It is also possible to show that $\lambda_{m,n} = (j_{m,n})^2$.

If $f = (\lambda_{1,1})^s \varphi_{1,1}$, then (2.12) and (2.24) show that $u = \varphi_{1,1}$ and

$$\mathfrak{u}(r,\theta,y) = \frac{2^{1-s}}{\Gamma(s)}(\lambda_{1,1})^{s/2}\varphi_{1,1}(r,\theta)y^s K_s(\sqrt{2}\pi y).$$

From [1, Chapter 9], we have that $j_{1,1} \approx 3.8317$.

We construct a sequence of meshes $\{\mathscr{T}_{\mathcal{Y}_k}\}_{k\geq 1}$, where the triangulation of Ω is obtained by quasi-uniform refinement and the partition of $[0, \mathcal{Y}_k]$ is as in § 5.2. The parameter $\mathcal{Y}_k$ is chosen so that $\epsilon \approx (\#\mathscr{T}_{\mathcal{Y}_{k-1}})^{-1/3}$. With these meshes

$$\|\mathfrak{u} - V_{\mathscr{T}_{\mathcal{Y}_k}}\|_{\mathring{H}^1_L(\mathcal{C},y^\alpha)} \lesssim |\log(\#\mathscr{T}_{\mathcal{Y}_k})|^s (\#\mathscr{T}_{\mathcal{Y}_k})^{-1/3}, \tag{6.2}$$

which is near-optimal.

Figure 6.2 shows the errors of $\|\mathfrak{u} - V_{\mathscr{T}_{k,\mathcal{Y}}}\|_{H^1(y^\alpha,\mathcal{C}_{\mathcal{Y}_k})}$ for $s = 0.3$ and $s = 0.7$. The results, again, are in agreement with Theorem 5.4 and Remark 5.5.

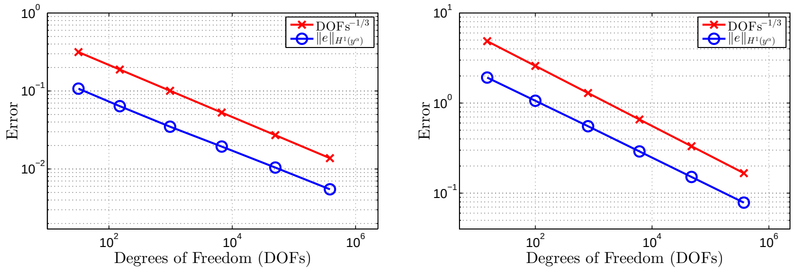

FIG. 6.2. *Computational rate of convergence for the approximate solution of the fractional Laplacian over a circle with graded meshes on the extended dimension. The left panel shows the rate for $s = 0.3$ and the right one for $s = 0.7$. In both cases, the rate is $\approx (\#\mathscr{T}_{\mathcal{Y}_k})^{-1/3}$ in agreement with Theorem 5.4 and Remark 5.5*

6.3. Incompatible data for $s \in (0, 1)$. The computational results of previous paragraphs always entail $f \in \mathbb{H}^{1-s}(\Omega)$ and illustrate the error estimates of Theorem 5.4. Let us now consider a data f smooth but incompatible. Set $\Omega = (0, 1)$ and $f \equiv 1$. Notice that, if $s \leq \frac{1}{2}$ then $f \notin \mathbb{H}^{1-s}(\Omega)$ due to the fact that the function does not vanish at the boundary. In fact, we have that

$$\sum_{k=1}^{\infty} \lambda_k^{\sigma} |f_k|^2 < \infty \quad \Leftrightarrow \quad \sigma < \frac{1}{2},$$

in other words $f \in \mathbb{H}^{\sigma}(\Omega)$ if and only if $\sigma < \frac{1}{2}$. Since, the coefficients of the solution to (1.1) are given by $u_k = \lambda_k^{-s} f_k$, we can only expect that

$$\sum_{k=1}^{\infty} \lambda_k^{\mu} |u_k|^2 = \sum_{k=1}^{\infty} \lambda_k^{\mu-2s} |f_k|^2 < \infty \quad \Leftrightarrow \quad \mu - 2s < \frac{1}{2},$$

namely $u \in \mathbb{H}^{\mu}(\Omega)$ for $\mu < 2s + \frac{1}{2}$. In conclusion, full regularity is not possible but owing to the special character of the data some shift can be expected; see Remark 2.8 and the discussion at the end of § 2.4.

This heuristic argument is rather illuminating as it tells us that the best rate of convergence we can expect is

$$\|u - U_{\mathscr{T}_\Omega}\|_{\mathbb{H}^s(\Omega)} \leq (\#\mathscr{T}_\Omega)^{-r} \|u\|_{\mathbb{H}^\mu(\Omega)},$$

with $r = \mu - s < s + \frac{1}{2}$. Since we are dealing with a one dimensional problem, the extension has two dimensions and, consequently, we expect

$$\|\mathfrak{u} - V_{\mathscr{T}_{\mathcal{Y}}}\|_{\mathring{H}_L^1(\mathcal{C}, y^\alpha)} \lesssim \begin{cases} (\#\mathscr{T}_{\mathcal{Y}})^{-\left(\frac{s}{2}+\frac{1}{4}\right)}, & s < \dfrac{1}{2}, \\ (\#\mathscr{T}_{\mathcal{Y}})^{-\frac{1}{2}}, & s > \dfrac{1}{2}. \end{cases} \tag{6.3}$$

Since $\lambda_k = \pi^2 k^2$ and $\varphi_k = \sqrt{2}\sin(\sqrt{\lambda_k} x')$, it is not difficult to show that $f_k = \sqrt{2}(1 - (-1)^k)/\sqrt{\lambda_k}$, whence we can obtain an approximate solution $u_N = \sum_{k=1}^N \lambda_k^{-s} f_k \varphi_k$ with N sufficiently large. Figure 6.3 shows the norm of the difference between $V_{\mathscr{T}_{\mathcal{Y}}}$ and the α-harmonic extension of u_N for different values of s. The experimental rates of convergence seem to agree with (6.3): they are suboptimal for $s < \frac{1}{2}$.

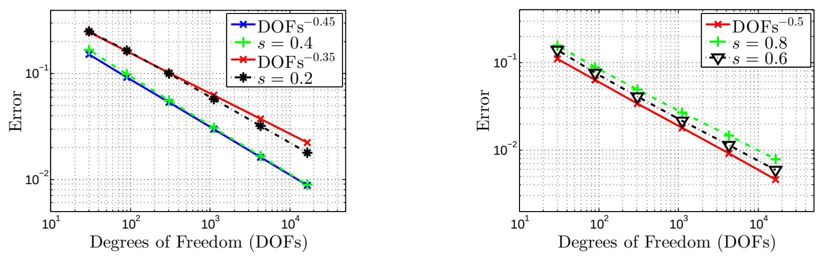

FIG. 6.3. *Computational rate of convergence for the approximate solution of the fractional Laplacian with incompatible data $f \equiv 1$. The domain Ω is the unit interval and the mesh is graded in the extended dimension. We show the $H^1(\mathcal{C}_\mathcal{Y}, y^\alpha)$ norm of the difference between $V_{\mathscr{T}_\mathcal{Y}}$ and the harmonic extension of u_N with $N = 5 \cdot 10^4$. The left panel shows the rate for $s = 0.2$, 0.4 and the right one for $s = 0.6$, 0.8. As expected, the rate of convergence is optimal for values larger than $\frac{1}{2}$. On the other hand, if $s < \frac{1}{2}$ we see a reduction on the rate of convergence in accordance with* (6.3).

To recover the optimal decay rate, we explore the a priori design of graded meshes in the x'-direction, which is within our theory of §4 and §5 (see Remark 5.8). Since $u \in \mathbb{H}^\mu(\Omega)$ with $\mu < 2s + \frac{1}{2}$, we expect that $u \approx r^{2s}$ as $r \to 0$, where r denotes the distance to the boundary. This, at least heuristically, can be figured out as follows: if $\partial_r^\mu r^{2s} \approx r^{2s-\mu}$, then

$$\|u\|^2_{\mathbb{H}^\mu(\Omega)} \approx \int_0^\varepsilon |\partial_r^\mu r^{2s}|^2 \, \mathrm{d}r < \infty \quad \Leftrightarrow \quad \mu < 2s + \frac{1}{2},$$

and $r^{2s} \in \mathbb{H}^\mu(\Omega)$ only for $\mu < 2s + \frac{1}{2}$.

Having guessed the nature of the singularity, we can apply the principle of error equidistribution as in § 5.2 to design an optimal graded mesh as x' approaches either 0 or 1, with a grading parameter $\gamma > \frac{3}{2(1+s)}$ (compare with (5.6)). We proceed as follows: construct a quasi-uniform mesh of the interval $\Omega = (0, 1)$ by bisection, and next transform the nodes $\mathtt{v}$ by the rule $\mathtt{v} \leftarrow \psi(\mathtt{v})$, where

$$\psi(\mathtt{v}) = \begin{cases} \dfrac{1}{4}\left(4\mathtt{v}\right)^\gamma, & \mathtt{v} \leq \dfrac{1}{4}, \\ \mathtt{v}, & \dfrac{1}{4} \leq \mathtt{v} \leq \dfrac{3}{4}, \\ 1 - \dfrac{1}{4}\left(4(1-\mathtt{v})\right)^\gamma, & \mathtt{v} \geq \dfrac{3}{4}. \end{cases} \tag{6.4}$$

We display in Figure 6.4 convergence plots for $s = 0.2$ and $s = 0.4$ over graded meshes in Ω which restore the optimal decay rate. The construction requires a priori knowledge of the solution, which is not obvious in higher dimensions. Adaptivity might provide a way to recover an optimal rate without such a knowledge (see Remark 5.10 about hanging nodes).

7. Fractional powers of general second order elliptic operators. Let us now discuss how the methodology developed in previous sections extends to a general second order, symmetric and uniformly elliptic operator. This is an important property of our PDE approach. Recall that, in § 2.4, we discussed how the fractional Laplace operator can be realized as a Dirichlet to Neumann map via an extension problem posed on the semi-infinite cylinder $\mathcal{C}$. In the work of Stinga and Torrea [60],

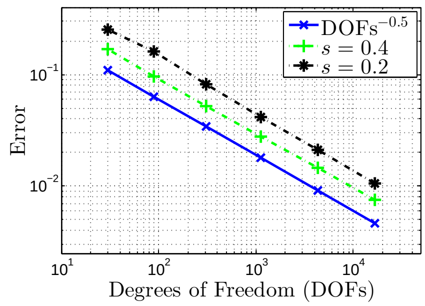

Fig. 6.4. *Computational rate of convergence for the approximate solution of the fractional Laplacian with incompatible data $f \equiv 1$ over meshes that are graded both in the x'- and y-directions. The domain Ω is the unit interval. The grading in the extended dimension obeys* (5.6)*, whereas the one on the x'-direction is constructed using* (6.4)*. We show the $H^1(\mathcal{C}_\mathscr{Y}, y^\alpha)$ norm of the difference between $V_{\mathscr{T}_\mathscr{Y}}$ and the harmonic extension of u_N with $N = 5 \cdot 10^4$. An optimal rate of convergence can be recover irrespective of the fact that the solution does not possess full regularity.*

the same type of characterization has been developed for the fractional powers of second order elliptic operators.

Let $\mathcal{L}$ be a second order symmetric differential operator of the form

$$\mathcal{L}w = -\mathrm{div}_{x'}(A\nabla_{x'}w) + cw, \tag{7.1}$$

where $c \in L^\infty(\Omega)$ with $c \geq 0$ almost everywhere, $A \in \mathcal{C}^{0,1}(\Omega, \mathsf{GL}(n,\mathbb{R}))$ is symmetric and positive definite, and Ω is Lipschitz. Given $f \in L^2(\Omega)$, the Lax-Milgram lemma shows that there is a unique $w \in H_0^1(\Omega)$ that solves

$$\mathcal{L}w = f \text{ in } \Omega, \qquad w = 0 \text{ on } \partial\Omega.$$

The operator $\mathcal{L}^{-1} : L^2(\Omega) \to L^2(\Omega)$ is positive, compact and symmetric, whence its spectrum is discrete, positive and accumulates at zero. Moreover, there exists $\{\lambda_k, \varphi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+ \times H_0^1(\Omega)$ such that $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and for, $k \in \mathbb{N}$,

$$\mathcal{L}\varphi_k = \lambda_k\varphi_k \text{ in } \Omega, \qquad \varphi_k = 0 \text{ on } \partial\Omega, \tag{7.2}$$

and $\lambda_k \to \infty$ as $k \to \infty$. For $u \in C_0^\infty(\Omega)$ we then define the fractional powers of $\mathcal{L}$ as

$$\mathcal{L}^s u = \sum_{k=1}^\infty u_k \lambda_k^s \varphi_k, \tag{7.3}$$

where $u_k = \int_\Omega u\varphi_k$. By density the operator $\mathcal{L}^s$ can be extended again to $\mathbb{H}^s(\Omega)$. This discussion shows that it is legitimate to study the following problem: given $s \in (0,1)$ and $f \in \mathbb{H}^{-s}(\Omega)$, find $u \in \mathbb{H}^s(\Omega)$ such that

$$\mathcal{L}^s u = f \text{ in } \Omega. \tag{7.4}$$

To realize the operator $\mathcal{L}^s$ as the Dirichlet to Neumann map of an extension problem we use the generalization of the result by Caffarelli and Silvestre presented in [60]. We seek a function $\mathfrak{u} : \mathcal{C} \to \mathbb{R}$ that solves

$$\begin{cases} -\mathcal{L}\mathfrak{u} + \dfrac{\alpha}{y}\partial_y \mathfrak{u} + \partial_{yy}\mathfrak{u} = 0, & \text{in } \mathcal{C}, \\ \mathfrak{u} = 0, & \text{on } \partial_L \mathcal{C}, \\ \dfrac{\partial \mathfrak{u}}{\partial \nu^\alpha} = d_s f, & \text{on } \Omega \times \{0\}, \end{cases} \tag{7.5}$$

where the constant d_s is as in (2.23). In complete analogy to § 2.4 it is possible to show that

$$d_s \mathcal{L}^s u = \frac{\partial \mathfrak{u}}{\partial \nu^\alpha} : \mathbb{H}^s(\Omega) \longmapsto \mathbb{H}^{-s}(\Omega).$$

Notice that the differential operator in (7.5) is

$$\operatorname{div}(y^\alpha \mathbf{A} \nabla \mathfrak{u}) + y^\alpha c \mathfrak{u},$$

where, for all $x \in \mathcal{C}$, $\mathbf{A}(x) = \operatorname{diag}\{A(x'), 1\} \in \mathsf{GL}(n+1, \mathbb{R})$.

It suffices now to notice that both $y^\alpha c$ and $y^\alpha \mathbf{A}$ are in $A_2(\mathbb{R}^{n+1}_+)$, to conclude that, given $f \in \mathbb{H}^{-s}(\Omega)$, there is a unique $\mathfrak{u} \in \mathring{H}^1_L(\mathcal{C}, y^\alpha)$ that solves (7.5), [36]. In addition, $u = \mathfrak{u}(\cdot, 0) \in \mathbb{H}^s(\Omega)$ solves (7.4) and we have the stability estimate

$$\|u\|_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla \mathfrak{u}\|_{L^2(\mathcal{C}, y^\alpha)} \lesssim \|f\|_{\mathbb{H}^{-s}(\Omega)}, \tag{7.6}$$

where the hidden constants depend on A, c, C_{2,y^α} and Ω.

The representation (2.24) of $\mathfrak{u}$ in terms of the Bessel functions is still valid. Consequently, we can show $\mathfrak{u}_{yy} \in L^2(\mathcal{C}, y^\beta)$. We can also repeat the arguments in the proof of Theorem 3.5 to conclude that

$$\|\nabla \mathfrak{u}\|_{L^2(\Omega \times (\mathcal{Y}, \infty), y^\alpha)} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/2} \|f\|_{\mathbb{H}^{-s}(\Omega)},$$

and introduce $v \in \mathring{H}^1_L(\mathcal{C}_\mathcal{Y}, y^\alpha)$ — solution of a truncated version of (7.5) — and show that

$$\|\nabla(\mathfrak{u} - v)\|_{L^2(\mathcal{C}, y^\alpha)} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/4} \|f\|_{\mathbb{H}^{-s}(\Omega)}. \tag{7.7}$$

Next, we define the finite element approximation of the solution to (7.5) as the unique function $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ that solves

$$\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \mathbf{A}(x) \nabla V_{\mathscr{T}_\mathcal{Y}} \cdot \nabla W + y^\alpha c(x') V_{\mathscr{T}_\mathcal{Y}} W \, \mathrm{d}x' \, \mathrm{d}y = d_s \langle f, \operatorname{tr}_\Omega W \rangle, \quad \forall W \in \mathbb{V}(\mathscr{T}_\mathcal{Y}). \tag{7.8}$$

We construct, as in § 5.2, a shape regular triangulation $\mathscr{T}_\Omega$ of Ω, which we extend to $\mathscr{T}_\mathcal{Y} \in \mathbb{T}$ with the partition given in (5.6), with $\gamma > 3/(1-\alpha)$. Following the proof of Theorem 5.4 we can also show the following error estimate.

Theorem 7.1 (Error estimate for general operators). *Let $V_\mathscr{T} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ be the solution of* (7.8) *and $U_{\mathscr{T}_\Omega} \in \mathbb{U}(\mathscr{T}_\Omega)$ be defined as in* (4.5)*. If $\mathfrak{u}$, solution of* (7.5)*, is such that $\mathcal{L}\mathfrak{u}$, $\partial_y \nabla \mathfrak{u} \in L^2(\mathcal{C}, y^\alpha)$, then we have*

$$\|u - U_{\mathscr{T}_\Omega}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\mathfrak{u} - V_{\mathscr{T}_\mathcal{Y}}\|_{\mathring{H}^1_L(\mathcal{C}, y^\alpha)} \lesssim |\log(\#\mathscr{T}_\mathcal{Y})|^s (\#\mathscr{T}_\mathcal{Y})^{-1/(n+1)} \|f\|_{\mathbb{H}^{1-s}(\Omega)}.$$

8. Conclusions. We develop PDE solution techniques for problems involving fractional powers $(-\Delta)^s u = f$ of the Laplace operator in a bounded domain Ω with Dirichlet boundary conditions. To overcome the inherent difficulty of nonlocality, we exploit the *cylindrical* extension proposed and investigated by X. Cabré and J. Tan [20], which is in turn inspired in the breakthrough by L. Caffarelli and L. Silvestre [21]. This leads to the (local) elliptic PDE (1.2) in one higher dimension y, with variable coefficient y^α, $\alpha = 1 - 2s$, which either degenerates ($s < 1/2$) or blows up ($s > 1/2$). Several remarks and comparisons with recent literature are now in order:

- *Regularity.* In § 2.6 we derive global and local regularity estimates for the solution of problem (1.2) in weighted Sobolev spaces.
- *Truncation.* In § 3 we propose the truncated problem (3.2), and show exponential convergence in the extended variable y to the solution of problem (1.2).
- *Tensor product meshes.* In § 4.1 we study a finite element strategy to approximate problem (1.2) which allows anisotropic elements in the extended dimension y.
- *Anisotropic interpolation theory.* In § 4.2 we extend the anisotropic interpolation estimates of [30] to the weighted Sobolev space $H^1(y^\alpha)$. This hinges on $y^\alpha \in A_2(\mathbb{R}^{n+1})$ and gives rise to a theory in Muckenhoupt weighted Sobolev spaces with a general weight in the class A_p $(1 < p < \infty)$ along with applications [54].
- *Error analysis.* In § 5.1 we derive a priori error estimates for quasi-uniform meshes which exhibit optimal regularity, according to § 2.6, but suboptimal order. In § 5.2 we restore the optimal decay rate upon constructing suitably graded meshes in the extended variable y and applying the interpolation theory of § 4.2.
- *Assumptions on f and Ω.* We assume the regularity conditions of Remark 2.10 throughout solely for convenience. We could in fact compensate the lack of such regularity via graded but shape regular meshes in Ω, as illustrated in § 6.3, which are within our theory.
- *General operators.* In § 7 we extend our FEM and supporting theory to general linear second order, symmetric and uniformly elliptic operators.
- *Comparisons.* Inspired in our work, and while this paper was under review, Bonito and Pasciak developed in [15] an alternative approach, which is based on the integral formulation of fractional powers of self-adjoint operators [13, Chapter 10.4]. This yields a sequence of easily parallelizable uncoupled elliptics PDEs, and leads to quasi-optimal error estimates in the L^2-norm instead on the energy norm provided Ω is convex and $f \in \mathbb{H}^{2-2s}(\Omega)$. Note that we only require $f \in \mathbb{H}^{1-s}(\Omega)$.
- *Parabolic problems.* In [53] we exploit the flexibility of the Caffarelli-Silvestre extension by applying it to the numerical treatment of linear parabolic equations with fractional diffusion and fractional time derivative. In contrast, the extension of [15] to the heat equation with fractional diffusion is not completely evident.

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