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Optimal control of fractional diffusion with Dirac measures

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We dedicate this work to the memory of Ricardo Durán.

Abstract In two and three dimensions, we study an optimal control problem for an elliptic equation involving the spectral fractional Laplace operator and a forcing term given by a linear combination of Dirac measures. The control variable is a vector whose entries are the amplitudes of these singular sources. The singular nature of the forcing imposes a restriction on the order of the fractional operator that ensures that the state equation is well posed, the tracking term of the cost functional is finite, and the point evaluations of the adjoint state entering the optimality conditions are meaningful. We prove existence and uniqueness of an optimal solution and derive first-order optimality conditions. We then propose a discretization of the optimal control problem based on finite elements; since the admissible control set is finite-dimensional, the control variable itself does not require discretization. Finally, we derive error bounds for the approximation of the optimal control variable.

Keywords PDE-constrained optimization · spectral fractional Laplacian · Dirac measures · optimality conditions · finite elements · error bounds.

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1 Introduction

Let $\Omega \subset \mathbb{R}^d$, with $d \in \{2, 3\}$, be an open, bounded, and convex polytope with boundary $\partial\Omega$, and let $s \in (\frac{d}{4}, 1)$. We denote by $(-\Delta)^s$ the spectral

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fractional Laplacian, with homogeneous Dirichlet boundary conditions built into this definition. Let $\mathcal{D}$ be a finite subset of Ω with cardinality $\ell := \#\mathcal{D}$, and let δ_z denote the Dirac measure supported at $z \in \mathcal{D}$. Given a desired state $u_\Omega \in L^2(\Omega)$ and a regularization parameter $\alpha > 0$, we introduce the cost functional

$$J(u, \mathbf{q}) := \frac{1}{2}\|u - u_\Omega\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\mathbf{q}\|^2_{\mathbb{R}^\ell}. \tag{1.1}$$

The PDE-constrained optimization problem we consider is as follows: Minimize $J(u, \mathbf{q})$ subject to the fractional PDE

$$(-\Delta)^s u = \sum_{z\in\mathcal{D}} q_z \delta_z \quad \text{in } \Omega, \tag{1.2}$$

and the control constraints $\mathbf{q} \in \mathbf{Q}_{ad}$, where

$$\mathbf{Q}_{ad} := \left\{\mathbf{q} \in \mathbb{R}^\ell : a_z \le q_z \le b_z \ \ \forall z \in \mathcal{D}\right\}.$$

The amplitudes of the singular sources in (1.2) are collected in the vector $\mathbf{q} = \{q_z\}_{z\in\mathcal{D}}$, which is our control variable. The control bounds $\mathbf{a} = \{a_z\}_{z\in\mathcal{D}}$ and $\mathbf{b} = \{b_z\}_{z\in\mathcal{D}}$ are vectors in $\mathbb{R}^\ell$ for which we assume that $a_z < b_z$ for every $z \in \mathcal{D}$.

Control problems with singular sources arise naturally in applications. In the active control of sound [1], an undesired noise is attenuated by the destructive interference produced by a secondary wave, which is generated by loudspeakers acting as actuators. These are idealized as point sources, so that the corresponding state equation is forced by a linear combination of Dirac measures, and the control variable is a vector whose entries are the amplitudes of these sources. A related structure arises in the active control of vibrations of flexible structures [2]. There, secondary forces modeled by Dirac measures are applied to a beam, and their amplitudes are chosen to reduce the vibration, measured either at finitely many points or over the whole structure. The control problem (1.1)–(1.2) has the same structure — sources on a prescribed set and amplitudes as controls — with the novelty that the state equation is governed by the nonlocal operator $(-\Delta)^s$.

In addition to the nonlocal feature of the spectral fractional Laplace operator $(-\Delta)^s$, the control problem (1.1)–(1.2) presents a further difficulty that affects both the analysis and the discretization: the forcing term of the state equation (1.2) is singular. As a consequence, results based on [3] suggest that solutions to (1.2) cannot be sought in the classical functional framework provided by $\mathbb{H}^s(\Omega)$, which is the setting adopted in [4,5,6,7,8] for a right-hand side in $\mathbb{H}^{-s}(\Omega)$; both spaces are defined in Section 2.1. Instead, we shall resort to a formulation in which trial and test functions are sought in different spaces [9]. These spaces cannot be chosen independently: the test space must be sufficiently smooth for measures to define bounded linear functionals on it, while the trial space, which is correspondingly rougher, must still guarantee some Sobolev regularity of the solution. A compatible pair of spaces can be

constructed provided $s > \frac{d}{4}$. Interestingly, the same condition $s > \frac{d}{4}$ also guarantees that the adjoint state belongs to $\mathbb{H}^{2s}(\Omega) \hookrightarrow C(\bar{\Omega})$, so that the point evaluations entering the optimality conditions are well defined.

We next briefly relate problem (1.1)–(1.2) to optimal control problems with point sources governed by *local* operators. Several related results are available. The articles [10,11,12] consider the Poisson problem with forcing terms of the form appearing in (1.2), whereas [13] addresses the corresponding semilinear setting. Motivated by applications to active sound control, the authors of [1] consider a Helmholtz equation with point sources as controls and a cost functional based on observations at finitely many sensor locations, rather than the distributed tracking functional considered here. We also mention [14], where the authors consider the Navier–Stokes system with point forces acting in the momentum equation, with their amplitudes serving as control variables. These works illustrate the relevance of controlling the amplitudes of point sources for local PDEs. In contrast, the problem considered here involves the nonlocal spectral fractional Laplacian, which requires a different analytical and numerical treatment.

We list what we consider to be the main contributions of this work.

1. *Fractional PDEs with measure-valued right-hand sides.* We extend the analysis of [9], originally developed for $d = 2$, to dimensions $d \in \{2, 3\}$ under the restriction $s > \frac{d}{4}$.
2. *The optimal control problem.* We prove existence and uniqueness of an optimal solution and derive necessary and sufficient first-order optimality conditions.
3. *Discretization and error bounds.* We develop a finite element discretization of the optimal control problem. The state equation is discretized using the scheme of [15], based on the exact diagonalization of a finite element discretization of the Caffarelli–Silvestre extension [16], together with the regularization of the singular forcing introduced in [9]. Since the control space is finite dimensional, the control variable itself does not require discretization. We derive an error estimate for the approximation of the optimal control variable of order $\mathcal{O}(h^{s+\theta-d/2})$, where θ is a parameter introduced in Section 3.

The outline of this paper is as follows. Section 2 introduces the notation and recalls the definition of the spectral fractional Laplace operator. In Section 3 we study a linear problem with a measure-valued right-hand side. The optimal control problem is analyzed in Section 4, where we establish the existence and uniqueness of an optimal solution and derive first-order optimality conditions. Our solution technique, together with the corresponding error bounds, is presented in Section 5. We conclude in Section 6 with some final comments.

2 Notation and preliminaries

We write $A \lesssim B$ whenever $A \leq cB$ with a constant $c > 0$ independent of A, B, and any discretization parameter; $A \gtrsim B$ means $B \lesssim A$, and $A \eqsim B$ stands

for $A \lesssim B \lesssim A$. The Euclidean norm in $\mathbb{R}^\ell$ is denoted by $\|\cdot\|_{\mathbb{R}^\ell}$. Finally, $\mathcal{M}(\Omega)$ denotes the space of finite Radon measures on Ω, which can be identified with the dual of $C_0(\Omega)$, the space of continuous functions on $\bar{\Omega}$ vanishing on $\partial\Omega$. The symbol $\langle\cdot,\cdot\rangle$ denotes the associated duality pairing, so that

$$\langle \mu, v\rangle = \int_\Omega v \,\mathrm{d}\mu.$$

2.1 The spectral fractional Laplacian and fractional Sobolev spaces

Consider the problem: Find $(\lambda,\varphi) \in \mathbb{R}\times(H^1_0(\Omega)\setminus\{0\})$ such that

$$(\nabla\varphi,\nabla v)_{L^2(\Omega)} = \lambda\,(\varphi, v)_{L^2(\Omega)}, \qquad \forall v \in H^1_0(\Omega).$$

This problem has a countable family of eigenpairs $\{(\lambda_k,\varphi_k)\}_{k\in\mathbb{N}} \subset \mathbb{R}_+ \times (H^1_0(\Omega)\setminus\{0\})$ such that $\lambda_k \to \infty$ as $k\to\infty$ and $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H^1_0(\Omega)$. For $r \geq 0$, we define the following fractional Sobolev spaces

$$\mathbb{H}^r(\Omega) \coloneqq \left\{ w \in L^2(\Omega) : w = \sum_{k=1}^\infty w_k\varphi_k,\ \sum_{k=1}^\infty \lambda_k^r |w_k|^2 < \infty \right\}, \tag{2.3}$$

$$\|w\|_{\mathbb{H}^r(\Omega)} \coloneqq \left(\sum_{k=1}^\infty \lambda_k^r |w_k|^2 \right)^{\frac{1}{2}}, \tag{2.4}$$

where, for $k\in\mathbb{N}$,

$$w_k \coloneqq \int_\Omega w\varphi_k \,\mathrm{d}x.$$

In particular, $\mathbb{H}^0(\Omega) = L^2(\Omega)$ and $\mathbb{H}^1(\Omega) = H^1_0(\Omega)$ with

$$\|w\|_{\mathbb{H}^0(\Omega)} = \|w\|_{L^2(\Omega)}, \qquad \|w\|_{\mathbb{H}^1(\Omega)} = \|\nabla w\|_{L^2(\Omega)}.$$

For $r>0$, $\mathbb{H}^{-r}(\Omega)$ denotes the dual space of $\mathbb{H}^r(\Omega)$, and we denote the corresponding duality pairing by ${}_{-r}\langle\cdot,\cdot\rangle_r$. Through this pairing, elements of $\mathbb{H}^{-r}(\Omega)$ can be represented by sequences, and the definition of the norm in (2.4) extends naturally to negative indices.

Given $s\in(0,1)$ and $w\in C_0^\infty(\Omega)$, we define the spectral fractional Laplacian [4,5,6] by

$$(-\Delta)^s w \coloneqq \sum_{k=1}^\infty \lambda_k^s w_k \varphi_k. \tag{2.5}$$

By density, $(-\Delta)^s$ extends to an isomorphism from $\mathbb{H}^s(\Omega)$ onto $\mathbb{H}^{-s}(\Omega)$.

The spaces $\mathbb{H}^r(\Omega)$ are the natural ones in this setting. However, in the analysis that follows we need to identify these with the classical fractional

Sobolev spaces [17,18,19,20]. For the range of indices that is relevant to our analysis, the following characterizatios hold with equivalent norms:

$$\mathbb{H}^r(\Omega) = \begin{cases} H^r(\Omega), & r \in \left(0, \frac{1}{2}\right), \\ H^{1/2}_{00}(\Omega), & r = \frac{1}{2}, \\ H^r_0(\Omega), & r \in \left(\frac{1}{2}, 1\right), \\ H^1_0(\Omega) \cap H^r(\Omega), & r \in (1,2). \end{cases} \tag{2.6}$$

The last identification holds because Ω is convex.

3 Fractional diffusion under a measure-valued right-hand side

We consider the following fractional PDE with a measure-valued right-hand side:

$$(-\Delta)^s \mathfrak{u} = \mu \quad \text{in } \Omega, \qquad \mu \in \mathcal{M}(\Omega). \tag{3.7}$$

Given $s \in (\frac{d}{4}, 1)$, we choose $\theta \in (\frac{d}{2} - s, s)$, an interval that is nonempty because $s > \frac{d}{4}$. The following inequalities will be fundamental in the analysis below:

$$0 < s - \theta < 2s - \frac{d}{2} < 2 - \frac{d}{2}, \qquad \frac{d}{2} < s + \theta < 2s < 2. \tag{3.8}$$

We now introduce the bilinear form

$$\mathcal{A} : \mathbb{H}^{s-\theta}(\Omega) \times \mathbb{H}^{s+\theta}(\Omega) \to \mathbb{R},$$
$$\mathcal{A}(v, w) \coloneqq \sum_{k=1}^{\infty} \lambda_k^s v_k w_k, \quad v = \sum_{k=1}^{\infty} v_k \varphi_k, \quad w = \sum_{k=1}^{\infty} w_k \varphi_k.$$

Definition 1 (weak solution) A function $\mathfrak{u} \in \mathbb{H}^{s-\theta}(\Omega)$ is a weak solution of (3.7) if

$$\mathcal{A}(\mathfrak{u}, v) = \langle \mu, v \rangle \qquad \forall v \in \mathbb{H}^{s+\theta}(\Omega). \tag{3.9}$$

Since $s + \theta > \frac{d}{2}$, [21, Theorem 6.7] shows that $H^{s+\theta-1}(\Omega) \hookrightarrow L^p(\Omega)$ for every $p \le p^\star \coloneqq 2d/(d - 2(s + \theta - 1))$. Moreover, we have $p^\star > d$. Thus, the space characterizations in Section 2.1 and [22, Theorem 4.12, Part II] yield

$$\mathbb{H}^{s+\theta}(\Omega) = H^{s+\theta}(\Omega) \cap H^1_0(\Omega) \hookrightarrow W^{1,p^\star}_0(\Omega) \hookrightarrow C_0(\Omega). \tag{3.10}$$

Consequently, every $\mu \in \mathcal{M}(\Omega)$ defines a bounded linear functional on the space $\mathbb{H}^{s+\theta}(\Omega)$, so the right-hand side of (3.9) is well-defined.

Theorem 1 (well-posedness) *Given $\mu \in \mathcal{M}(\Omega)$, problem* (3.9) *has a unique solution $\mathfrak{u} \in \mathbb{H}^{s-\theta}(\Omega)$ satisfying*

$$\|\mathfrak{u}\|_{\mathbb{H}^{s-\theta}(\Omega)} \lesssim \|\mu\|_{\mathbb{H}^{-s-\theta}(\Omega)} \lesssim \|\mu\|_{\mathcal{M}(\Omega)}. \tag{3.11}$$

Proof The two-dimensional argument of [9, Theorem 3.2] extends verbatim to $d = 3$: the bilinear form $\mathcal{A}$ is continuous and satisfies the two inf-sup conditions stated therein, both with constant one. Existence, uniqueness, and the first bound in (3.11) thus follow from the BNB theorem [23, Theorem 2.2]. The second bound is a consequence of (3.10).

4 The optimal control problem

The optimal control problem introduced in Section 1 reads as follows:

$$\min\{J(u,\mathbf{q}) : (u,\mathbf{q}) \in \mathbb{H}^{s-\theta}(\Omega) \times \mathbf{Q}_{ad}\}, \tag{4.12}$$

subject to the weak formulation of the state equation: Find $u \in \mathbb{H}^{s-\theta}(\Omega)$ such that

$$\mathcal{A}(u,v) = \sum_{z\in\mathcal{D}} q_z v(z) \quad \forall v \in \mathbb{H}^{s+\theta}(\Omega). \tag{4.13}$$

By Theorem 1, the state equation is well-posed. Define the control-to-state map $\mathbf{S} : \mathbb{R}^\ell \to \mathbb{H}^{s-\theta}(\Omega)$, which maps $\mathbf{q}$ to the unique solution $u = \mathbf{Sq}$ of (4.13); $\mathbf{S}$ is linear and bounded, since (3.11) yields

$$\|\mathbf{Sq}\|_{\mathbb{H}^{s-\theta}(\Omega)} \lesssim \|\mathbf{q}\|_{\mathbb{R}^\ell}.$$

Moreover, $s-\theta > 0$ gives $\mathbf{Sq} \in L^2(\Omega)$, so that the reduced functional $j : \mathbb{R}^\ell \to \mathbb{R}$, defined by $j(\mathbf{q}) \coloneqq J(\mathbf{Sq},\mathbf{q})$, is well defined.

4.1 Existence and uniqueness of an optimal control

Theorem 2 (well-posedness) *The optimal control problem* (4.12)–(4.13) *has a unique solution* $(\bar{u},\bar{\mathbf{q}}) \in \mathbb{H}^{s-\theta}(\Omega) \times \mathbf{Q}_{ad}$.

Proof Using $\mathbf{S}$ and j, the control problem reduces to minimizing j over $\mathbf{Q}_{ad}$. The set $\mathbf{Q}_{ad}$ is closed and bounded in $\mathbb{R}^\ell$, and hence compact, while j is continuous, because $\mathbf{S}$ is linear and bounded, and strictly convex, because $\alpha > 0$. Existence follows from the Weierstrass theorem, and uniqueness is guaranteed by the strict convexity of j.

4.2 Optimality conditions

To derive optimality conditions, we introduce the *adjoint problem*:

$$p \in \mathbb{H}^s(\Omega) : \qquad \mathcal{B}(p,w) = (u - u_\Omega, w)_{L^2(\Omega)} \qquad \forall w \in \mathbb{H}^s(\Omega), \tag{4.14}$$

where the bilinear form $\mathcal{B} : \mathbb{H}^s(\Omega) \times \mathbb{H}^s(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{B}(v,w) \coloneqq \sum_{k=1}^{\infty} \lambda_k^s v_k w_k.$$

The Lax–Milgram lemma immediately yields the well-posedness of (4.14). Since $u - u_\Omega \in L^2(\Omega)$, the definitions of $(-\Delta)^s$ and the spaces $\mathbb{H}^r(\Omega)$ imply that $p \in \mathbb{H}^{2s}(\Omega)$. Moreover, since $2s > \frac{d}{2}$, the characterization from Section 2.1 and [22, Theorem 7.34(c)] yield $\mathbb{H}^{2s}(\Omega) = H^{2s}(\Omega) \cap H^1_0(\Omega) \hookrightarrow C_0(\Omega)$. Consequently, the point value $p(z)$ is well defined for every $z \in \mathcal{D}$.

Theorem 3 (optimality conditions) $\bar{\mathbf{q}} \in \mathbf{Q}_{ad}$ *is optimal for* (4.12)–(4.13) *if and only if*

$$\sum_{z\in\mathcal{D}} \left(\bar{p}(z) + \alpha\bar{q}_z\right)(q_z - \bar{q}_z) \geq 0, \qquad \forall \mathbf{q} = \{q_z\}_{z\in\mathcal{D}} \in \mathbf{Q}_{ad}, \tag{4.15}$$

where $\bar{p}$ *solves* (4.14) *with* u *replaced by* $\bar{u} = \mathbf{S}\bar{\mathbf{q}}$.

Proof Since $\mathbf{S}$ is linear and bounded, it is Fréchet differentiable. The chain rule then shows that j is differentiable and that, for $\mathbf{q}, \mathbf{r} \in \mathbb{R}^\ell$,

$$j'(\mathbf{q})\mathbf{r} = (\mathbf{S}\mathbf{q} - u_\Omega, \mathbf{S}\mathbf{r})_{L^2(\Omega)} + \alpha \sum_{z\in\mathcal{D}} q_z r_z.$$

Thus, since j is convex and $\mathbf{Q}_{ad}$ is convex, $\bar{\mathbf{q}} \in \mathbf{Q}_{ad}$ is optimal if and only if, for all $\mathbf{q} = \{q_z\}_{z\in\mathcal{D}} \in \mathbf{Q}_{ad}$,

$$j'(\bar{\mathbf{q}})(\mathbf{q} - \bar{\mathbf{q}}) = (\bar{u} - u_\Omega, u - \bar{u})_{L^2(\Omega)} + \alpha \sum_{z\in\mathcal{D}} \bar{q}_z(q_z - \bar{q}_z) \geq 0, \tag{4.16}$$

where $u = \mathbf{S}\mathbf{q}$ and $\bar{u} = \mathbf{S}\bar{\mathbf{q}}$. It remains to rewrite $(\bar{u} - u_\Omega, u - \bar{u})_{L^2(\Omega)}$ in (4.16). By linearity, $u - \bar{u}$ solves (4.13) with $\mathbf{q}$ replaced by $\mathbf{q} - \bar{\mathbf{q}}$. Let $\bar{p}$ solve (4.14) with u replaced by $\bar{u} = \mathbf{S}\bar{\mathbf{q}}$. Since $s + \theta < 2s$, we have $\bar{p} \in \mathbb{H}^{2s}(\Omega) \hookrightarrow \mathbb{H}^{s+\theta}(\Omega)$. Setting $v = \bar{p}$ in the equation satisfied by $u - \bar{u}$ yields

$$\mathcal{A}(u - \bar{u}, \bar{p}) = \sum_{z\in\mathcal{D}} (q_z - \bar{q}_z)\bar{p}(z). \tag{4.17}$$

Let $\{\mathtt{u}_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ be such that $\mathtt{u}_n \to u - \bar{u}$ in $\mathbb{H}^{s-\theta}(\Omega)$ as $n \uparrow \infty$. Using $\mathtt{u}_n$ as a test function in (4.14), we obtain

$$\mathcal{B}(\bar{p}, \mathtt{u}_n) = (\bar{u} - u_\Omega, \mathtt{u}_n)_{L^2(\Omega)}$$

for every $n \in \mathbb{N}$. The convergence $\mathtt{u}_n \to u - \bar{u}$ in $\mathbb{H}^{s-\theta}(\Omega)$, and hence in $L^2(\Omega)$, yields

$$(\bar{u} - u_\Omega, \mathtt{u}_n)_{L^2(\Omega)} \to (\bar{u} - u_\Omega, u - \bar{u})_{L^2(\Omega)}$$

as $n \uparrow \infty$. On the other hand, since $\bar{p} \in \mathbb{H}^{s+\theta}(\Omega)$,

$$\mathcal{B}(\bar{p}, \mathtt{u}_n) = \mathcal{A}(\mathtt{u}_n, \bar{p}) \to \mathcal{A}(u - \bar{u}, \bar{p})$$

as $n \uparrow \infty$. Combining these limits, we obtain

$$\mathcal{A}(u - \bar{u}, \bar{p}) = (\bar{u} - u_\Omega, u - \bar{u})_{L^2(\Omega)}.$$

We finally invoke (4.17) to deduce that

$$(\bar{u} - u_\Omega, u - \bar{u})_{L^2(\Omega)} = \sum_{z\in\mathcal{D}} (q_z - \bar{q}_z)\bar{p}(z).$$

Substituting this identity into (4.16) yields (4.15) and concludes the proof.

5 Discretization

In this section, we develop a finite element discretization of the optimal control problem (4.12)–(4.13) and derive error bounds.

5.1 A finite element method for fractional PDEs

We let $\mathbb{T} = \{\mathcal{T}_h\}_{h>0}$ be a quasiuniform family of conforming simplicial meshes of $\bar{\Omega}$ and denote by $\mathbb{V}(\mathcal{T}_h)$ the space of continuous piecewise linear functions on $\mathcal{T}_h$ that vanish on $\partial\Omega$. We follow [15], choose $\mathcal{Y} > 0$ and $K \in \mathbb{N}$, and define the parameters

$$\Upsilon_k \coloneqq \left(\frac{\eta_k}{\mathcal{Y}}\right)^2, \qquad \psi_k \coloneqq \frac{4\sin(\pi s)}{\Upsilon_k^s \mathcal{Y}^2 \pi J_{1-s}(\eta_k)^2}, \qquad k = 1, \dots, K, \tag{5.18}$$

where J_ν denotes the Bessel function of the first kind of order ν and η_k is the k-th positive root of J_{-s}. Given $\mathfrak{f} \in L^2(\Omega)$, for each $k \in \{1, \dots, K\}$, we let $\Psi_k \in \mathbb{V}(\mathcal{T}_h)$ solve

$$\int_\Omega \nabla\Psi_k \cdot \nabla v_h \,\mathrm{d}x + \Upsilon_k \int_\Omega \Psi_k v_h \,\mathrm{d}x = \int_\Omega \mathfrak{f} v_h \,\mathrm{d}x \qquad \forall v_h \in \mathbb{V}(\mathcal{T}_h), \tag{5.19}$$

and define

$$\Psi_{h,\mathcal{Y}}^K \coloneqq \sum_{k=1}^K \psi_k \Psi_k \in \mathbb{V}(\mathcal{T}_h).$$

The following is the main result of [15].

Proposition 1 (L^2-right-hand side) *Let $\mathfrak{f} \in L^2(\Omega)$. If $\mathcal{Y} \eqsim 2s|\log h|$ and $K \eqsim \mathcal{Y}/h$, then*

$$\|(-\Delta)^{-s}\mathfrak{f} - \Psi_{h,\mathcal{Y}}^K\|_{L^2(\Omega)} \lesssim h^{2s}\|\mathfrak{f}\|_{L^2(\Omega)}.$$

When $\mathfrak{f} = \mu \in \mathcal{M}(\Omega)$, we apply the scheme of [15] to a suitably regularized right-hand side. Given $\varepsilon > 0$, we let $\mu_\varepsilon \coloneqq \mu \star \rho_\varepsilon$, where $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$ and $\rho \in C_0^\infty(\mathbb{R}^d)$ is a standard mollifier satisfying $\rho \geq 0$, $\operatorname{supp}\rho \subset B_1(0)$, and

$$\int_{\mathbb{R}^d} \rho \,\mathrm{d}x = 1.$$

For $d \in \{2, 3\}$, we have the following crucial estimates:

$$\|\mu_\varepsilon\|_{L^2(\Omega)} \leq \|\rho_\varepsilon\|_{L^2(\mathbb{R}^d)}\|\mu\|_{\mathcal{M}(\Omega)} = \varepsilon^{-d/2}\|\rho\|_{L^2(\mathbb{R}^d)}\|\mu\|_{\mathcal{M}(\Omega)}, \tag{5.20}$$

$$\|\mu - \mu_\varepsilon\|_{\mathbb{H}^{-s-\theta}(\Omega)} \lesssim \varepsilon^\gamma \|\mu\|_{\mathcal{M}(\Omega)}, \tag{5.21}$$

where

$$\gamma \coloneqq s + \theta - \frac{d}{2} > 0.$$

The bound in (5.20) follows from [24, Proposition 8.49] and a change of variables, while (5.21) follows upon adapting the arguments in [9, Proposition 7.4] to $d \in \{2, 3\}$.

Define

$$\mathfrak{u}_{h,\mathcal{Y}}^{K,\varepsilon} \coloneqq \sum_{k=1}^{K} \psi_k \mathfrak{u}_k^\varepsilon \in \mathbb{V}(\mathcal{T}_h),$$

where $\mathfrak{u}_k^\varepsilon$ solves (5.19) with $\mathfrak{f}$ replaced by μ_ε.

Theorem 4 (measure-valued right-hand side) *Let $\mu \in \mathcal{M}(\Omega)$, and let $\mathfrak{u}$ solve (3.9). If $\mathcal{Y} \eqsim 2s|\log h|$, $K \eqsim \mathcal{Y}/h$ and $\varepsilon \eqsim h$, then, for $h \leq 1$, we have*

$$\|\mathfrak{u} - \mathfrak{u}_{h,\mathcal{Y}}^{K,\varepsilon}\|_{L^2(\Omega)} \lesssim h^{s+\theta-\frac{d}{2}} \|\mu\|_{\mathcal{M}(\Omega)}. \tag{5.22}$$

Proof Let $\mathfrak{u}_\varepsilon$ solve $(-\Delta)^s \mathfrak{u}_\varepsilon = \mu_\varepsilon$ in Ω. Proposition 1 gives

$$\|\mathfrak{u}_\varepsilon - \mathfrak{u}_{h,\mathcal{Y}}^{K,\varepsilon}\|_{L^2(\Omega)} \lesssim h^{2s} \|\mu_\varepsilon\|_{L^2(\Omega)} \lesssim h^{2s} \varepsilon^{-d/2} \|\mu\|_{\mathcal{M}(\Omega)},$$

where we also used (5.20). Since $\mathfrak{u} - \mathfrak{u}_\varepsilon$ solves problem (3.9) with μ replaced by $\mu - \mu_\varepsilon$, the continuous embedding $\mathbb{H}^{s-\theta}(\Omega) \hookrightarrow L^2(\Omega)$, together with (3.11) and (5.21), yields

$$\|\mathfrak{u} - \mathfrak{u}_\varepsilon\|_{L^2(\Omega)} \lesssim \|\mathfrak{u} - \mathfrak{u}_\varepsilon\|_{\mathbb{H}^{s-\theta}(\Omega)} \lesssim \varepsilon^\gamma \|\mu\|_{\mathcal{M}(\Omega)}.$$

Setting $\varepsilon \eqsim h$ and using $\theta < s$ and $h \leq 1$, we arrive at $h^{2s-d/2} \leq h^\gamma$. The triangle inequality then yields (5.22).

5.2 A finite element method for the optimal control problem

Recall that, for the optimal control problem, the state u solves (3.7) with right-hand side given by

$$\mu = \sum_{z \in \mathcal{D}} q_z \delta_z.$$

For each $z \in \mathcal{D}$ and $\varepsilon > 0$, we define $\delta_{z,\varepsilon} \coloneqq \delta_z \star \rho_\varepsilon$. We then set

$$\mu_\varepsilon \coloneqq \sum_{z \in \mathcal{D}} q_z \delta_{z,\varepsilon}.$$

Next, following Theorem 4, we choose the discretization and regularization parameters so that $\mathcal{Y} \eqsim 2s|\log h|$, $K \eqsim \mathcal{Y}/h$, and $\varepsilon \eqsim h$. We define the discrete control-to-state map

$$\mathbf{S}_h : \mathbb{R}^\ell \to \mathbb{V}(\mathcal{T}_h), \qquad \mathbf{S}_h \mathbf{q} \coloneqq u_{h,\mathcal{Y}}^{K,\varepsilon}. \tag{5.23}$$

Since μ_ε depends linearly on $\mathbf{q}$, each problem (5.19) is linear with respect to its right-hand side, and the weights ψ_k in (5.18) are independent of $\mathbf{q}$, the map $\mathbf{S}_h$ is linear. Moreover, since

$$\|\mu\|_{\mathcal{M}(\Omega)} \lesssim \|\mathbf{q}\|_{\mathbb{R}^\ell},$$

Theorem 4 yields

$$\|(\mathbf{S}-\mathbf{S}_h)\mathbf{q}\|_{L^2(\Omega)} \lesssim h^{s+\theta-\frac{d}{2}}\|\mathbf{q}\|_{\mathbb{R}^\ell}, \qquad h \leq 1. \tag{5.24}$$

In addition, the family $\{\mathbf{S}_h\}_{0<h\leq 1}$ is uniformly bounded in h as a family of operators from $\mathbb{R}^\ell$ into $L^2(\Omega)$. Indeed,

$$\|\mathbf{S}_h\mathbf{q}\|_{L^2(\Omega)} \leq \|(\mathbf{S}-\mathbf{S}_h)\mathbf{q}\|_{L^2(\Omega)} + \|\mathbf{S}\mathbf{q}\|_{L^2(\Omega)} \lesssim \|\mathbf{q}\|_{\mathbb{R}^\ell}.$$

We thus propose the following discretization of the optimal control problem (4.12)–(4.13): Find $\bar{\mathbf{q}}_h \in \mathbf{Q}_{ad}$ such that

$$j_h(\bar{\mathbf{q}}_h) = \min\left\{j_h(\mathbf{q}) : \mathbf{q} \in \mathbf{Q}_{ad}\right\}, \tag{5.25}$$

where the discrete reduced cost is defined as

$$j_h(\mathbf{q}) \coloneqq \frac{1}{2}\|\mathbf{S}_h\mathbf{q} - u_\Omega\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\mathbf{q}\|^2_{\mathbb{R}^\ell},$$

and $\mathbf{S}_h$ is the discrete control-to-state map defined in (5.23). Since j_h is continuous and strictly convex and $\mathbf{Q}_{ad}$ is compact and convex, problem (5.25) admits a unique optimal solution $\bar{\mathbf{q}}_h \in \mathbf{Q}_{ad}$. Moreover, $\bar{\mathbf{q}}_h \in \mathbf{Q}_{ad}$ is optimal for (5.25) if and only if

$$j_h'(\bar{\mathbf{q}}_h)(\mathbf{q}-\bar{\mathbf{q}}_h) \geq 0 \qquad \forall \mathbf{q} \in \mathbf{Q}_{ad}. \tag{5.26}$$

We now present the main result of this section.

Theorem 5 (error bounds) *Let $\bar{\mathbf{q}}$ be the optimal solution of the optimal control problem* (4.12)–(4.13)*, and let $\bar{\mathbf{q}}_h$ be the optimal solution of the discrete optimal control problem* (5.25)*. Then, for $h \leq 1$, we have*

$$\|\bar{\mathbf{q}}-\bar{\mathbf{q}}_h\|_{\mathbb{R}^\ell} \lesssim h^{s+\theta-\frac{d}{2}}, \qquad \|\bar{u}-\mathbf{S}_h\bar{\mathbf{q}}_h\|_{L^2(\Omega)} \lesssim h^{s+\theta-\frac{d}{2}}, \tag{5.27}$$

where the hidden constants are independent of h and may depend, in particular, on ℓ, α, $\mathbf{a}$, $\mathbf{b}$, and $\|u_\Omega\|_{L^2(\Omega)}$.

Proof Basic computations show the following identity:

$$(j'(\bar{\mathbf{q}}) - j'(\bar{\mathbf{q}}_h))(\bar{\mathbf{q}}-\bar{\mathbf{q}}_h) = \alpha\|\bar{\mathbf{q}}-\bar{\mathbf{q}}_h\|^2_{\mathbb{R}^\ell} + \|\mathbf{S}(\bar{\mathbf{q}}-\bar{\mathbf{q}}_h)\|^2_{L^2(\Omega)}.$$

As a result, we obtain the bound

$$\alpha\|\bar{\mathbf{q}}-\bar{\mathbf{q}}_h\|^2_{\mathbb{R}^\ell} \leq (j'(\bar{\mathbf{q}}) - j'(\bar{\mathbf{q}}_h))(\bar{\mathbf{q}}-\bar{\mathbf{q}}_h).$$

On the other hand, if we set $\mathbf{q} = \bar{\mathbf{q}}_h$ in (4.16) and $\mathbf{q} = \bar{\mathbf{q}}$ in (5.26), we obtain

$$j'(\bar{\mathbf{q}})(\bar{\mathbf{q}}-\bar{\mathbf{q}}_h) \leq 0$$

and

$$j_h'(\bar{\mathbf{q}}_h)(\bar{\mathbf{q}}-\bar{\mathbf{q}}_h) \geq 0,$$

respectively. Combining these inequalities with the previous bound, we obtain

$$\alpha\|\bar{\mathbf{q}}-\bar{\mathbf{q}}_h\|^2_{\mathbb{R}^\ell} \le \big(j_h'(\bar{\mathbf{q}}_h)-j'(\bar{\mathbf{q}}_h)\big)(\bar{\mathbf{q}}-\bar{\mathbf{q}}_h). \tag{5.28}$$

We now add and subtract to the right-hand side of the previous bound the term $(\mathbf{S}\bar{\mathbf{q}}_h-u_\Omega,\mathbf{S}_h(\bar{\mathbf{q}}-\bar{\mathbf{q}}_h))_{L^2(\Omega)}$ to obtain

$$\alpha\|\bar{\mathbf{q}}-\bar{\mathbf{q}}_h\|^2_{\mathbb{R}^\ell} \le ((\mathbf{S}_h-\mathbf{S})\bar{\mathbf{q}}_h,\mathbf{S}_h(\bar{\mathbf{q}}-\bar{\mathbf{q}}_h))_{L^2(\Omega)}+(\mathbf{S}\bar{\mathbf{q}}_h-u_\Omega,(\mathbf{S}_h-\mathbf{S})(\bar{\mathbf{q}}-\bar{\mathbf{q}}_h))_{L^2(\Omega)}.$$

Since $\{\mathbf{S}_h\}_{0<h\le 1}$ is uniformly bounded in h, and the quantities $\|\bar{\mathbf{q}}_h\|_{\mathbb{R}^\ell}$ and $\|\mathbf{S}\bar{\mathbf{q}}_h-u_\Omega\|_{L^2(\Omega)}$ are bounded uniformly in h as well because $\bar{\mathbf{q}}_h\in\mathbf{Q}_{ad}$, we obtain

$$\begin{aligned}\alpha\|\bar{\mathbf{q}}-\bar{\mathbf{q}}_h\|^2_{\mathbb{R}^\ell} &\lesssim \|(\mathbf{S}_h-\mathbf{S})\bar{\mathbf{q}}_h\|_{L^2(\Omega)}\|\bar{\mathbf{q}}-\bar{\mathbf{q}}_h\|_{\mathbb{R}^\ell}+\|(\mathbf{S}_h-\mathbf{S})(\bar{\mathbf{q}}-\bar{\mathbf{q}}_h)\|_{L^2(\Omega)}\\ &\lesssim h^\gamma\|\bar{\mathbf{q}}-\bar{\mathbf{q}}_h\|_{\mathbb{R}^\ell},\end{aligned}$$

where we have also used (5.24). This yields the first estimate in (5.27). The second one follows from

$$\begin{aligned}\|\bar{u}-\mathbf{S}_h\bar{\mathbf{q}}_h\|_{L^2(\Omega)} &\le \|\mathbf{S}(\bar{\mathbf{q}}-\bar{\mathbf{q}}_h)\|_{L^2(\Omega)}+\|(\mathbf{S}-\mathbf{S}_h)\bar{\mathbf{q}}_h\|_{L^2(\Omega)}\\ &\lesssim \|\bar{\mathbf{q}}-\bar{\mathbf{q}}_h\|_{\mathbb{R}^\ell}+h^\gamma\lesssim h^\gamma,\end{aligned}$$

where we have used the first estimate in (5.27).

Remark 1 (on the rate) An appropriate choice of θ in (5.27) yields the rate $\mathcal{O}(h^{2s-\frac{d}{2}-\delta})$, with $\delta>0$ arbitrarily small, which is not optimal. For $s\uparrow 1$ and $d=2$ it approaches $\mathcal{O}(h^{1-\delta})$, whereas the corresponding local problem admits $\mathcal{O}(h^2|\log h|^3)$ [13, Theorem 5.8]. Our argument only exploits the $L^2(\Omega)$-error of the state approximation, while the arguments therein rely on local L^∞ estimates. To the best of our knowledge, local L^∞-error bounds are not available for FE approximations of $(-\Delta)^s\phi=g$, and obtaining them appears to be an open problem of independent interest.

6 Conclusions

We have analyzed an optimal control problem for a linear elliptic PDE involving the spectral fractional Laplace operator, in which the forcing term is a linear combination of Dirac measures and the control variable is the vector of amplitudes of these singular sources. The restriction $s>\frac{d}{4}$ on the order of the fractional operator is intrinsic to our formulation: it guarantees that the singular forcing defines a bounded functional on the test space, that the tracking term is well defined, and that the adjoint state is continuous, so that the point evaluations entering the optimality conditions are meaningful.

We have also proposed a solution technique based on an exact diagonalization of a discretization of the Caffarelli–Silvestre extension; the singular forcing is treated by regularization, with the resulting error balanced against the discretization error. Since the admissible control set is finite-dimensional,

the control variable itself requires no discretization. The resulting rates are positive for every admissible order but are not optimal. Improving them appears to require local L^∞-error estimates for finite element approximations of the underlying linear problem for the spectral fractional Laplace operator.

Semilinear state equations and the optimization of the locations of the sources rather than of their amplitudes are natural directions for further work.

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Competing interests

The authors declare that they have no conflict of interest.

Data availability

No data was used for the research described in the article.

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