

OPTIMAL CONTROL OF A PARABOLIC FRACTIONAL PDE: ANALYSIS AND DISCRETIZATION

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Abstract. We consider the integral definition of the fractional Laplacian and analyze a linear-quadratic optimal control problem for the so-called fractional heat equation; control constraints are also considered. We derive existence and uniqueness results, first order optimality conditions, and regularity estimates for the optimal variables. To discretize the state equation equation we propose a fully discrete scheme that relies on an implicit finite difference discretization in time combined with a piecewise linear finite element discretization in space. We derive stability results and a novel $L^2(0, T; L^2(\Omega))$ a priori error estimate. On the basis of the aforementioned solution technique, we propose a fully discrete scheme for our optimal control problem that discretizes the control variable with piecewise constant functions and derive a priori error estimates for it. We illustrate the theory with one- and two-dimensional numerical experiments.

Key words. linear-quadratic optimal control problem, fractional diffusion, integral fractional Laplacian, regularity estimates, fully-discrete methods, finite elements, stability, error estimates.

AMS subject classifications. 49J20, 49M25, 65M12, 65M15, 65M60.

1. Introduction. In this work we shall be interested in the design and analysis of solution techniques for a linear-quadratic optimal control problem involving an initial boundary value problem for a fractional parabolic equation. To make matters precise, for $n \geq 1$, we let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with Lipschitz boundary $\partial\Omega$; additional regularity requirements will be imposed in the course of our convergence rate analysis ahead. Given a desired state $\mathbf{u}_d : \Omega \times (0, T) \rightarrow \mathbb{R}$ and a regularization parameter $\mu > 0$, we define the cost functional

$$J(\mathbf{u}, \mathbf{z}) = \frac{1}{2} \int_0^T \left(\|\mathbf{u} - \mathbf{u}_d\|_{L^2(\Omega)}^2 + \mu \|\mathbf{z}\|_{L^2(\Omega)}^2 \right) dt. \quad (1.1)$$

Let $\mathbf{f} : \Omega \times (0, T) \rightarrow \mathbb{R}$ and $\mathbf{u}_0 : \Omega \rightarrow \mathbb{R}$ be fixed functions. We will call them the right-hand side and initial datum, respectively. Let $s \in (0, 1)$. We shall be concerned with the following PDE-constrained optimization problem: Find

$$\min J(\mathbf{u}, \mathbf{z}) \quad (1.2)$$

subject to the *fractional heat equation*

$$\partial_t \mathbf{u} + (-\Delta)^s \mathbf{u} = \mathbf{f} + \mathbf{z} \text{ in } \Omega \times (0, T), \quad \mathbf{u} = 0 \text{ in } \Omega^c \times (0, T), \quad \mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega, \quad (1.3)$$

and the *control constraints*

$$\mathbf{a}(x, t) \leq \mathbf{z}(x, t) \leq \mathbf{b}(x, t) \quad \text{a.e. } (x, t) \in Q := \Omega \times (0, T). \quad (1.4)$$

The functions \mathbf{a} and \mathbf{b} both belong to $L^2(Q)$ and satisfy the property $\mathbf{a}(x, t) \leq \mathbf{b}(x, t)$ for almost every $(x, t) \in Q$. In (1.3), $\Omega^c := \mathbb{R}^n \setminus \Omega$. For convenience, we will refer to the optimal control problem (1.2)–(1.4) as the *parabolic fractional optimal control problem*; see section 4 for its precise description and analysis.

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We adopt the integral definition of the fractional Laplace operator $(-\Delta)^s$, which, from now on, we shall simply refer to as the *integral fractional Laplacian*. For smooth functions $w : \mathbb{R}^n \rightarrow \mathbb{R}$, $(-\Delta)^s$ can be naturally defined via Fourier transform:

$$\mathcal{F}((-\Delta)^s w)(\xi) = |\xi|^{2s} \mathcal{F}(w)(\xi). \quad (1.5)$$

Equivalently, $(-\Delta)^s$ can be defined by means of the following pointwise formula:

$$(-\Delta)^s w(x) = C(n, s) \text{p.v} \int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|x - y|^{n+2s}} dy, \quad C(n, s) = \frac{2^{2s} s \Gamma(s + \frac{n}{2})}{\pi^{n/2} \Gamma(1 - s)}, \quad (1.6)$$

where p.v stands for the Cauchy principal value and $C(n, s)$ is a positive normalization constant that depends only on n and s [17, equation (3.2)]. $C(n, s)$ is introduced to guarantee that the symbol of the resulting operator is $|\xi|^{2s}$. We refer the reader to [25, section 1.1] and [17, Proposition 3.3] for a proof of the equivalence of these two definitions. We must immediately mention that in bounded domains, and in addition to (1.6), there are many, *non-equivalent*, definitions of nonlocal operators related to the fractional Laplacian; for instance, the regional fractional Laplacian and the spectral fractional Laplacian.

Since the seminal work of Caffarelli and Silvestre [10], the analysis of regularity properties of solutions to fractional partial differential equations (PDEs) has received a tremendous attention: fractional diffusion has been one of the most studied topics in the past decade [10, 21, 29, 30]. Such an analysis has been motivated, in part, by the fact that the integral fractional Laplacian of order $2s$ corresponds to the infinitesimal generator of a $2s$ -stable Lévy process. These processes have been widely employed for modeling market fluctuations, both for risk management and option pricing purposes [14]. Further applications of fractional diffusion include nonlocal electrostatics [22], image processing [20, 26], fluids [11, 23], predator search behaviour [31], and many others. It is then only natural that interest in efficient approximation schemes for these problems arises and that one might be interested in their control.

The study of solution techniques for problems involving fractional diffusion is a relatively new but rapidly growing area of research and thus it is impossible to provide a complete overview of the available results and limitations. We restrict ourselves to referring the interested reader to [8] for a survey. In contrast to these advances, the study of solution techniques for PDE-constrained optimization problems involving fractional and nonlocal equations have not been fully developed. To the best of our knowledge, one of the first works in the elliptic setting is [16], where the authors consider an optimal control problem for a general nonlocal diffusion operator with finite range interactions. Later, an *elliptic* optimal control problem for the *spectral* fractional powers of elliptic operators was analyzed in [4]; numerical scheme were also proposed and studied. Recently, a similar PDE-constrained optimization problem, but for the *integral* fractional Laplacian, has been considered in [15]. In this work, the authors analyze the underlying control problem, derive regularity estimates, propose numerical schemes, and derive a priori error estimates. We also mention [6], where an optimal control problem for a fractional semilinear equation is considered. Concerning parabolic optimal control problems, the first work that propose and study numerical schemes when the state equation is the fractional heat equation is [5]. In this work, the authors consider the *spectral* fractional powers of elliptic operators and derive error estimates for a fully discrete scheme that approximates the solution of the underlying optimal control problem. To close this paragraph, we would like to stress that the integral and spectral definitions of the fractional Laplace operator *do not coincide*. In

fact, solutions to linear and elliptic problems associated to these operator behave in an essentially different way in terms of regularity properties.

The outline of this paper is as follows. The notation and functional setting is described in section 2, where we also recall, in section 2.3, regularity results for the elliptic counterpart of (1.3). In section 3, we derive the existence and uniqueness of a weak solution for problem (1.3). In addition, we present energy estimates and review regularity results. In section 4, we study the *parabolic fractional optimal control problem*. We derive existence and uniqueness results together with first order sufficient and necessary optimality conditions. In addition, we derive regularity estimates for the optimal variables. In section 5, we introduce a fully discrete scheme for (1.3): we consider the standard backward Euler scheme for time discretization and a piecewise linear finite element discretization in space. For $s \in (0, 1)$, we derive discrete stability results and a $L^2(Q)$ a priori error estimate. To the best of our knowledge, these results are not available in the literature. Section 6 is devoted to the design and analysis of a numerical scheme to approximate the control problem (1.2)–(1.4). In particular, in section 6.4, we derive a priori error estimates. Finally, section 7 presents one- and two-dimensional numerical experiments that illustrate the theory developed in section 6.4.

2. Notation and preliminaries. In this section we will introduce some notation and the set of assumptions that we shall operate under.

2.1. Notation. Throughout this work Ω is an open and bounded domain with Lipschitz boundary $\partial\Omega$. The complement of Ω will be denoted by Ω^c . If $T > 0$ is a fixed time, we set $Q = \Omega \times (0, T)$. Whenever \mathcal{X} is a normed space we denote by $\|\cdot\|_{\mathcal{X}}$ its norm and by \mathcal{X}' its dual. For normed spaces \mathcal{X} and \mathcal{Y} , we write $\mathcal{X} \hookrightarrow \mathcal{Y}$ to indicate that \mathcal{X} is continuously embedded in \mathcal{Y} .

If $D \subset \mathbb{R}^n$ is open and $\phi : D \times [0, T] \rightarrow \mathbb{R}$, we consider ϕ as a function of t with values in a Banach space \mathcal{X} , i.e., $\phi : [0, T] \ni t \mapsto \phi(t) \equiv \phi(\cdot, t) \in \mathcal{X}$. For $1 \leq p \leq \infty$, $L^p(0, T; \mathcal{X})$ is the space of \mathcal{X} -valued functions whose \mathcal{X} -norm is in $L^p(0, T)$. This is a Banach space for the norm

$$\|\phi\|_{L^p(0, T; \mathcal{X})} = \left(\int_0^T \|\phi(t)\|_{\mathcal{X}}^p dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad \|\phi\|_{L^\infty(0, T; \mathcal{X})} = \operatorname{esssup}_{t \in (0, T)} \|\phi(t)\|_{\mathcal{X}}.$$

The relation $a \lesssim b$ indicates that $a \leq Cb$ with a nonessential constant C that might change at each occurrence.

2.2. Function spaces. For any $s \geq 0$, we define $H^s(\mathbb{R}^n)$, the Sobolev space of order s over \mathbb{R}^n , by [32, Definition 15.7]

$$H^s(\mathbb{R}^n) := \left\{ v \in L^2(\mathbb{R}^n) : (1 + |\xi|^2)^{s/2} \mathcal{F}(v) \in L^2(\mathbb{R}^n) \right\}.$$

With the space $H^s(\mathbb{R}^n)$ at hand, we define $\tilde{H}^s(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $H^s(\mathbb{R}^n)$. This space can be equivalently characterized by [27, Theorem 3.29]

$$\tilde{H}^s(\Omega) = \{v|_\Omega : v \in H^s(\mathbb{R}^n), \operatorname{supp} v \subset \overline{\Omega}\}. \quad (2.1)$$

When $\partial\Omega$ is Lipschitz, $\tilde{H}^s(\Omega)$ is equivalent to $\mathbb{H}^s(\Omega) = [L^2(\Omega), H_0^1(\Omega)]_s$, the real interpolation between $L^2(\Omega)$ and $H_0^1(\Omega)$, for $s \in (0, 1)$ and to $H^s(\Omega) \cap H_0^1(\Omega)$ for

$s \in (1, 3/2)$ [27, Theorem 3.33]. We denote by $H^{-s}(\Omega)$ the dual space of $\tilde{H}^s(\Omega)$ and by $\langle \cdot, \cdot \rangle$ the duality pair between these two spaces. We also define the bilinear form

$$\mathcal{A}(v, w) = \frac{C(n, s)}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{n+2s}} dx dy, \quad (2.2)$$

and denote by $\|\cdot\|_s$ the norm that $\mathcal{A}(\cdot, \cdot)$ induces, which is just a multiple of the $H^s(\mathbb{R}^n)$ -seminorm:

$$\|v\|_s = \mathcal{A}(v, v)^{\frac{1}{2}} = \sqrt{\frac{C(n, s)}{2}} |v|_{H^s(\mathbb{R}^n)}. \quad (2.3)$$

2.3. Elliptic regularity. Let $f \in H^{-s}(\Omega)$. Since the bilinear form \mathcal{A} is continuous and coercive, an application of the Lax-Milgram Lemma immediately yields well-posedness of the following elliptic problem: Find $u \in \tilde{H}^s(\Omega)$ such that

$$\mathcal{A}(u, v) = \langle f, v \rangle \quad \forall v \in \tilde{H}^s(\Omega). \quad (2.4)$$

When $\partial\Omega$ is smooth the following regularity properties for u can be derived.

PROPOSITION 2.1 (Sobolev regularity of u on smooth domains). *Let $s \in (0, 1)$ and Ω be a domain such that $\partial\Omega \in C^\infty$. If $f \in H^r(\Omega)$, for some $r \geq -s$, then the solution u of problem (2.4) belongs to $H^{s+\vartheta}(\Omega)$, where $\vartheta = \min\{s + r, 1/2 - \epsilon\}$ and $\epsilon > 0$ is arbitrarily small. In addition, the following estimate holds:*

$$\|u\|_{H^{s+\vartheta}(\Omega)} \lesssim \|f\|_{H^r(\Omega)}, \quad (2.5)$$

where the hidden constant depends on the domain Ω , n , s , and ϑ .

Proof. See [21]. \square

As a consequence of the previous result, it can be observed that smoothness of f does not ensure that solutions are any smoother than $H^{s+1/2-\epsilon}(\Omega)$ with $\epsilon > 0$ being arbitrarily small.

When Ω is a bounded Lipschitz domain satisfying the exterior ball condition, the following regularity estimate can be derived [29]: If $f \in L^\infty(\Omega)$, then $u \in C^s(\mathbb{R}^n)$.

3. The state equation. In this section we derive the existence and uniqueness of a weak solution for the fractional heat equation (1.3). In addition, we present an energy estimate and review regularity results.

3.1. Eigenvalue problem. Let us introduce the eigenvalue problem: Find $(\lambda, \varphi) \in \mathbb{R} \times \tilde{H}^s(\Omega) \setminus \{0\}$ such that

$$\mathcal{A}(\varphi, v) = \lambda(\varphi, v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega). \quad (3.1)$$

Spectral theory yields the existence of a countable collection of solutions $\{\lambda_k, \varphi_k\} \subset \mathbb{R}^+ \times \tilde{H}^s(\Omega)$, with the real eigenvalues enumerated in increasing order, counting multiplicities, and such that $\{\varphi_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $\tilde{H}^s(\Omega)$.

3.2. Solution representation. We invoke the eigenpairs $\{\lambda_k, \varphi_k\}_{k \in \mathbb{N}}$, defined in section 3.1, and formally write the solution to problem (1.3) as

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) \varphi_k(x). \quad (3.2)$$

Since, at this formal stage, we have $u(x, 0) = u_0(x)$, this representation yields the following fractional initial value problem for u_k :

$$\partial_t u_k(t) + \lambda_k u_k(t) = f_k(t) + z_k(t), \quad u_k(0) = u_{0,k}, \quad k \in \mathbb{N}, \quad (3.3)$$

where $u_{0,k} = (u_0, \varphi_k)_{L^2(\Omega)}$, $f_k(t) = (f(\cdot, t), \varphi_k)_{L^2(\Omega)}$, and $z_k(t) = (z(\cdot, t), \varphi_k)_{L^2(\Omega)}$. An explicit representation formula for the solution u_k to problem (3.3) holds:

$$u_k(t) = u_{k,0} e^{-\lambda_k t} + \int_0^t e^{-\lambda_k(t-r)} (f_k(r) + z_k(r)) dr. \quad (3.4)$$

3.3. Well-posedness. A weak formulation for problem (1.3) reads as follows: Find $u \in \mathbb{V}$ such that $u(0) = u_0$ and, for a.e. $t \in (0, T)$,

$$\langle \partial_t u, \phi \rangle + \mathcal{A}(u, \phi) = \langle f + z, \phi \rangle \quad \forall \phi \in \tilde{H}^s(\Omega). \quad (3.5)$$

The space \mathbb{V} is defined as

$$\mathbb{V} := \{v \in L^2(0, T; \tilde{H}^s(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) : \partial_t v \in L^2(0, T; H^{-s}(\Omega))\}. \quad (3.6)$$

To simplify the exposition, we define

$$\Sigma^2(v, g) := \|v\|_{L^2(\Omega)}^2 + \|g\|_{L^2(0, T; H^{-s}(\Omega))}^2. \quad (3.7)$$

The following result provides the existence and uniqueness of a weak solution for problem (3.5).

THEOREM 3.1 (well-posedness of (3.5)). *Given $s \in (0, 1)$, $f \in L^2(0, T; H^{-s}(\Omega))$, $z \in L^2(0, T; H^{-s}(\Omega))$, and $u_0 \in L^2(\Omega)$, problem (3.5) has a unique weak solution. In addition, we have the following energy estimate*

$$\|u\|_{L^\infty(0, T; L^2(\Omega))} + \|u\|_{L^2(0, T; H^s(\mathbb{R}^n))} \lesssim \Sigma(u_0, f + z). \quad (3.8)$$

The hidden constant does not depend on u nor the problem data.

Proof. Existence and uniqueness of a weak solution for problem (1.3) can be obtained in view of a standard spectral decomposition approach based on the solution representation (3.2). The energy estimate (3.8) also follows from such a spectral decomposition approach. \square

Define

$$\Upsilon^2(v, g) := \|v\|_{H^s(\mathbb{R}^n)}^2 + \|g\|_{L^2(0, T; L^2(\Omega))}^2. \quad (3.9)$$

Let $f \in L^2(0, T; L^2(\Omega))$, $z \in L^2(0, T; L^2(\Omega))$, and $u_0 \in H^s(\mathbb{R}^n)$. Standard arguments, which heuristically entail multiplying the state equation (1.3) by the derivative of the solution u , yield the energy estimate

$$\|\partial_t u\|_{L^2(0, T; L^2(\Omega))} + \|u\|_{L^\infty(0, T; H^s(\mathbb{R}^n))} \lesssim \Upsilon(u_0, f + z), \quad (3.10)$$

where the hidden constant does not depend on u nor the problem data.

3.4. Regularity estimates. We present the following regularity result.

THEOREM 3.2 (regularity estimate). *Let $s \in (0, 1)$ and Ω be a domain such that $\partial\Omega \in C^\infty$. If $f + z \in L^\infty(0, T; L^2(\Omega))$ and $u_0 \in \tilde{H}^s(\Omega)$, then*

$$\|u\|_{L^2(0, T; H^{s+\gamma}(\Omega))} + \|\partial_t u\|_{L^2(Q)} \lesssim \|u_0\|_{H^s(\mathbb{R}^n)} + \|f + z\|_{L^\infty(0, T; L^2(\Omega))}, \quad (3.11)$$

where $\gamma = \min\{s, 1/2 - \epsilon\}$. *The hidden constant is independent of u and the problem data.*

Proof. See [1, Theorems 3.1 and 3.2]. \square

4. The fractional control problem. In this section, we study the *parabolic fractional optimal control problem*. We provide existence and uniqueness results together with first order necessary and sufficient optimality conditions.

The parabolic fractional optimal control problem reads: Find $\min J(u, z)$ subject to the state equation (1.3) and the control constraints (1.4). The set of *admissible controls* is defined by

$$Z_{\text{ad}} := \{w \in L^2(Q) : a(x, t) \leq w(x, t) \leq b(x, t) \text{ a.e. } (x, t) \in Q\}. \quad (4.1)$$

Notice that Z_{ad} is a nonempty, bounded, closed, and convex subset of $L^2(Q)$.

To analyze (1.2)–(1.4), we introduce the so-called control to state operator.

DEFINITION 4.1 (control to state operator). *The map $\mathbf{S} : L^2(0, T; H^{-s}(\Omega)) \ni z \mapsto u(z) \in \mathbb{V}$, where $u(z)$ solves (1.3), is called the fractional control to state operator.*

We immediately notice that the control to state operator \mathbf{S} is affine. In fact,

$$\mathbf{S}(z) = \mathbf{S}_0(z) + \psi_0, \quad (4.2)$$

where $\mathbf{S}_0(z)$ denotes the solution to (1.3) with $f = 0$ and $u_0 = 0$, while ψ_0 solves (1.3) with $z = 0$. Notice that \mathbf{S}_0 is linear and continuous. By the estimates of Theorem 3.1, \mathbf{S} is continuous as well. Since $\mathbb{V} \hookrightarrow L^2(Q) \hookrightarrow L^2(0, T; H^{-s}(\Omega))$, we may consider the operator \mathbf{S} as acting from $L^2(Q)$ into itself. For simplicity, we keep the notation \mathbf{S} .

We now define an optimal fractional state-control pair.

DEFINITION 4.2 (optimal fractional state-control pair). *A state-control pair $(\bar{u}(\bar{z}), \bar{z}) \in \mathbb{V} \times Z_{\text{ad}}$ is called optimal for (1.2)–(1.4) if $\bar{u}(\bar{z}) = \mathbf{S}\bar{z}$ and*

$$J(\bar{u}(\bar{z}), \bar{z}) \leq J(u(z), z)$$

for all $(u(z), z) \in \mathbb{V} \times Z_{\text{ad}}$ such that $u(z) = \mathbf{S}z$.

The existence and uniqueness of an optimal state-control pair is as follows.

THEOREM 4.3 (existence and uniqueness). *The optimal control problem (1.2)–(1.4) has a unique solution $(\bar{u}(\bar{z}), \bar{z}) \in \mathbb{V} \times Z_{\text{ad}}$.*

Proof. Invoke \mathbf{S} and reduce the optimal control problem (1.2)–(1.4) to: Minimize

$$f(z) := \frac{1}{2} \|\mathbf{S}z - u_d\|_{L^2(Q)}^2 + \frac{\mu}{2} \|z\|_{L^2(Q)}^2 \quad (4.3)$$

over Z_{ad} . The strict convexity of f is immediate ($\mu > 0$). In addition, f is weakly lower semicontinuous and Z_{ad} is weakly sequentially compact. The direct method of the calculus of variations [13, Theorem 5.51] allows us to conclude. \square

4.1. Optimality conditions. The following result is standard.

LEMMA 4.4 (variational inequality). *$\bar{z} \in Z_{\text{ad}}$ minimizes f over Z_{ad} if and only if it solves the variational inequality*

$$(f'(\bar{z}), z - \bar{z})_{L^2(Q)} \geq 0 \quad (4.4)$$

for every $z \in Z_{\text{ad}}$.

Proof. See [33, Lemma 2.21]. \square

To explore first order optimality conditions, we introduce the adjoint state.

DEFINITION 4.5 (fractional adjoint state). *The solution $p = p(z) \in \mathbb{V}$ of*

$$-\partial_t p + (-\Delta)^s p = u - u_d \text{ in } Q, \quad p = 0 \text{ in } \Omega^c \times (0, T), \quad p(T) = 0 \text{ in } \Omega, \quad (4.5)$$

for $\mathbf{z} \in L^2(0, T; H^{-s}(\Omega))$, is called the fractional adjoint state associated to $\mathbf{u} = \mathbf{u}(\mathbf{z})$.

The following result is instrumental.

LEMMA 4.6 (auxiliary result). *Let $\bar{\mathbf{z}}$ denote the optimal control for problem (1.2)–(1.4) and $\bar{\mathbf{u}} = \mathbf{S}\bar{\mathbf{z}}$. For every $\mathbf{z} \in \mathbf{Z}_{ad}$, we have*

$$(\bar{\mathbf{u}} - \mathbf{u}_d, \mathbf{u} - \bar{\mathbf{u}})_{L^2(Q)} = (\bar{\mathbf{p}}, \mathbf{z} - \bar{\mathbf{z}})_{L^2(Q)}, \quad (4.6)$$

where $\mathbf{u} = \mathbf{S}\mathbf{z} \in \mathbb{V}$ and $\mathbf{p} = \mathbf{p}(\mathbf{z}) \in \mathbb{V}$ solve problems (3.5) and (4.5), respectively.

Proof. Define $\chi := \mathbf{u} - \bar{\mathbf{u}} \in \mathbb{V}$. Since \mathbf{u} solves (3.5) and $\bar{\mathbf{u}} = \mathbf{S}\bar{\mathbf{z}}$, we obtain that $\chi(0) = 0$ in Ω and that, for a.e. $t \in (0, T)$,

$$\langle \partial_t \chi, \phi \rangle + \mathcal{A}(\chi, \phi) = (\mathbf{z} - \bar{\mathbf{z}}, \phi)_{L^2(\Omega)} \quad \forall \phi \in \tilde{H}^s(\Omega). \quad (4.7)$$

Set $\phi = \bar{\mathbf{p}}(t)$ in (4.7) and integrate over time to arrive at the identity

$$\int_0^T [\langle \partial_t \chi, \bar{\mathbf{p}} \rangle + \mathcal{A}(\chi, \bar{\mathbf{p}})] dt = (\mathbf{z} - \bar{\mathbf{z}}, \bar{\mathbf{p}})_{L^2(Q)}.$$

In view of the initial condition $\chi(0) = 0$, the terminal condition $\bar{\mathbf{p}}(T) = 0$, and the symmetry of the bilinear form \mathcal{A} , an integration by parts formula yields

$$\int_0^T [-\langle \partial_t \bar{\mathbf{p}}, \chi \rangle + \mathcal{A}(\bar{\mathbf{p}}, \chi)] dt = (\bar{\mathbf{p}}, \mathbf{z} - \bar{\mathbf{z}})_{L^2(Q)}.$$

Now, set χ as a test function in the weak version of (4.5) and integrate over time. These arguments allow us to arrive at

$$\int_0^T [-\langle \partial_t \bar{\mathbf{p}}, \chi \rangle + \mathcal{A}(\bar{\mathbf{p}}, \chi)] dt = (\bar{\mathbf{u}} - \mathbf{u}_d, \mathbf{u} - \bar{\mathbf{u}})_{L^2(Q)}.$$

The desired identity (4.6) follows immediately from the derived expressions. \square

We now prove necessary and sufficient optimality conditions for (1.2)–(1.4).

THEOREM 4.7 (first-order optimality conditions). *$\bar{\mathbf{z}} \in \mathbf{Z}_{ad}$ is the optimal control of problem (1.2)–(1.4) if and only if it solves the variational inequality*

$$(\mu \bar{\mathbf{z}} + \bar{\mathbf{p}}, \mathbf{z} - \bar{\mathbf{z}})_{L^2(Q)} \geq 0 \quad \forall \mathbf{z} \in \mathbf{Z}_{ad}, \quad (4.8)$$

where $\bar{\mathbf{p}} = \bar{\mathbf{p}}(\bar{\mathbf{z}})$ solves (4.5) with \mathbf{u} replaced by $\bar{\mathbf{u}}$.

Proof. We invoke the results of Lemma 4.4 to conclude that $\bar{\mathbf{z}} \in \mathbf{Z}_{ad}$ is optimal for problem (1.2)–(1.4) if and only if

$$(\bar{\mathbf{u}} - \mathbf{u}_d, \mathbf{S}_0(\mathbf{z} - \bar{\mathbf{z}}))_{L^2(Q)} + \mu(\bar{\mathbf{z}}, \mathbf{z} - \bar{\mathbf{z}})_{L^2(Q)} \geq 0.$$

We recall that the control to state map \mathbf{S} is affine (4.2). Notice that $\mathbf{S}_0(\mathbf{z} - \bar{\mathbf{z}}) = \mathbf{S}_0 \mathbf{z} + \psi_0 - (\psi_0 + \mathbf{S}_0 \bar{\mathbf{z}}) = \mathbf{u}(\mathbf{z}) - \bar{\mathbf{u}}$. Consequently,

$$(\bar{\mathbf{u}} - \mathbf{u}_d, \mathbf{u}(\mathbf{z}) - \bar{\mathbf{u}})_{L^2(Q)} + \mu(\bar{\mathbf{z}}, \mathbf{z} - \bar{\mathbf{z}})_{L^2(Q)} \geq 0.$$

Finally, we invoke (4.6) to arrive at the desired variational inequality:

$$(\bar{\mathbf{p}}, \mathbf{z} - \bar{\mathbf{z}})_{L^2(Q)} + \mu(\bar{\mathbf{z}}, \mathbf{z} - \bar{\mathbf{z}})_{L^2(Q)} \geq 0.$$

This concludes the proof. \square

4.2. Regularity of the optimal control. In this section we derive regularity estimates for the optimal control \bar{z} . To accomplish this task, we recall the well-known projection formula

$$\bar{z} = \text{proj}_{[a,b]} \left(-\frac{1}{\mu} \bar{p} \right), \quad \text{proj}_{[a,b]}(v) := \min \{b, \max \{a, v\}\}, \quad (4.9)$$

and refer the reader to [33, section 3.6.3] for a proof of this result.

To simplify the exposition, we define

$$\mathfrak{A} := \Sigma(u_0, f) + \|u_d\|_{L^2(Q)} + \|a\|_{H^1(0,T;L^2(\Omega))} + \|b\|_{H^1(0,T;L^2(\Omega))}, \quad (4.10)$$

where Σ is defined as in (3.7).

We begin by deriving regularity estimates in time.

THEOREM 4.8 (time regularity estimates). *Let $s \in (0, 1)$, $f \in L^2(0, T; H^{-s}(\Omega))$, and $u_0 \in L^2(\Omega)$. If $a, b \in H^1(0, T; L^2(\Omega))$, then*

$$\|\partial_t \bar{z}\|_{L^2(Q)} + \|\partial_t \bar{p}\|_{L^2(Q)} \lesssim \mathfrak{A}, \quad (4.11)$$

where the hidden constant is independent of the problem data and the optimal variables.

Proof. Since $f + \bar{z} \in L^2(0, T; H^{-s}(\Omega))$ and $u_0 \in L^2(\Omega)$, an application of the energy estimate (3.8) yields

$$\|\bar{u}\|_{L^\infty(0,T;L^2(\Omega))} + \|\bar{u}\|_{L^2(0,T;H^s(\mathbb{R}^n))} \lesssim \Sigma(u_0, f + \bar{z}). \quad (4.12)$$

Now, in view of the fact that $\bar{u} - u_d \in L^2(0, T; L^2(\Omega))$, we can apply the energy estimate (3.10) for the problem that \bar{p} solves, i.e., problem (4.5) with u replaced by \bar{u} , to arrive at

$$\|\partial_t \bar{p}\|_{L^2(Q)} + \|\bar{p}\|_{L^\infty(0,T;H^s(\mathbb{R}^n))} \lesssim \|\bar{u} - u_d\|_{L^2(Q)}. \quad (4.13)$$

We invoke (4.12) to conclude that

$$\|\partial_t \bar{p}\|_{L^2(Q)} + \|\bar{p}\|_{L^\infty(0,T;H^s(\mathbb{R}^n))} \lesssim \|u_0\|_{L^2(\Omega)} + \|f + \bar{z}\|_{L^2(0,T;H^{-s}(\Omega))} + \|u_d\|_{L^2(Q)}.$$

On the basis of this estimate, we invoke the projection formula (4.9) and [24, Theorem A.1] to conclude that $\bar{z} \in H^1(0, T; L^2(\Omega))$ together with the estimate (4.11). This concludes the proof. \square

To present regularity estimates in space, we define

$$\mathfrak{B} := \Sigma(u_0, f) + \|u_d\|_{L^\infty(0,T;L^2(\Omega))} + \|a\|_{L^2(0,T;H^1(\Omega))} + \|b\|_{L^2(0,T;H^1(\Omega))} \quad (4.14)$$

and

$$\mathfrak{C} := \|u_0\|_{H^s(\mathbb{R}^n)} + \|f\|_{L^\infty(0,T;L^2(\Omega))} + \|a\|_{L^\infty(0,T;L^2(\Omega))} + \|b\|_{L^\infty(0,T;L^2(\Omega))}, \quad (4.15)$$

where Σ is defined as in (3.7).

THEOREM 4.9 (space regularity estimates: $s \in (0, 1)$). *Let $s \in (0, 1)$ and Ω be a domain such that $\partial\Omega \in C^\infty$. If $u_0 \in L^2(\Omega)$, $f \in L^2(0, T; H^{-s}(\Omega))$, $u_d \in L^\infty(0, T; L^2(\Omega))$, and $a, b \in L^2(0, T; H^1(\Omega))$, then*

$$\|\bar{z}\|_{L^2(0,T;H^1(\Omega))} + \|\bar{p}\|_{L^2(0,T;H^1(\Omega))} \lesssim \mathfrak{B}, \quad s > \frac{1}{2}, \quad (4.16)$$

$$\|\bar{z}\|_{L^2(0,T;H^{1-\epsilon}(\Omega))} + \|\bar{p}\|_{L^2(0,T;H^{1-\epsilon}(\Omega))} \lesssim \mathfrak{B}, \quad s = \frac{1}{2}, \quad \epsilon > 0, \quad (4.17)$$

$$\|\bar{z}\|_{L^2(0,T;H^{2s}(\Omega))} + \|\bar{p}\|_{L^2(0,T;H^{2s}(\Omega))} \lesssim \mathfrak{B}, \quad s < \frac{1}{2}. \quad (4.18)$$

If $u_0 \in \tilde{H}^s(\Omega)$, $f \in L^\infty(0, T; L^2(\Omega))$, and $a, b \in L^\infty(0, T; L^2(\Omega))$, then

$$\|\bar{u}\|_{L^2(0, T; H^1(\Omega))} \lesssim \mathfrak{C}, \quad s > \frac{1}{2}, \quad (4.19)$$

$$\|\bar{u}\|_{L^2(0, T; H^{1-\epsilon}(\Omega))} \lesssim \mathfrak{C}, \quad s = \frac{1}{2}, \quad \epsilon > 0, \quad (4.20)$$

$$\|\bar{u}\|_{L^2(0, T; H^{2s}(\Omega))} \lesssim \mathfrak{C}, \quad s < \frac{1}{2}. \quad (4.21)$$

In all the estimates the hidden constants are independent of the optimal variables and the problem data.

Proof. We consider the following three cases.

[1] $s \in (\frac{1}{2}, 1)$: By assumption, the right-hand side $f \in L^2(0, T; H^{-s}(\Omega))$ and the control bounds $a, b \in L^2(Q)$. Thus, an application of the energy estimate (3.8) implies that $\bar{u} \in L^\infty(0, T; L^2(\Omega))$ and, by assumption, that $\bar{u} - u_d \in L^\infty(0, T; L^2(\Omega))$. We now invoke the regularity estimate (3.11) to conclude that

$$\|\bar{p}\|_{L^2(0, T; H^{s+\gamma}(\Omega))} + \|\partial_t \bar{p}\|_{L^2(Q)} \lesssim \|\bar{u} - u_d\|_{L^\infty(0, T; L^2(\Omega))}, \quad (4.22)$$

where $\gamma = \min\{s, 1/2 - \epsilon\}$ and $\epsilon > 0$. Since $s > 1/2$, this estimate implies that $\bar{p} \in L^2(0, T; H^1(\Omega))$. In view of the projection formula (4.9), we thus conclude that $\bar{z} \in L^2(0, T; H^1(\Omega))$ together with the estimate

$$\begin{aligned} \|\bar{z}\|_{L^2(0, T; H^1(\Omega))} &\lesssim \|\bar{u} - u_d\|_{L^\infty(0, T; L^2(\Omega))} + \|a\|_{L^2(0, T; H^1(\Omega))} + \|b\|_{L^2(0, T; H^1(\Omega))} \\ &\lesssim \Sigma(u_0, f + \bar{z}) + \|u_d\|_{L^\infty(0, T; L^2(\Omega))} + \|a\|_{L^2(0, T; H^1(\Omega))} + \|b\|_{L^2(0, T; H^1(\Omega))}. \end{aligned}$$

To obtain the last inequality, we have used the energy estimate (3.8). The previous estimate combined with (4.22) yield (4.16).

If $u_0 \in \tilde{H}^s(\Omega)$, $f \in L^\infty(0, T; L^2(\Omega))$, and $a, b \in L^\infty(0, T; L^2(\Omega))$, the estimate (4.19) follows from the regularity estimate (3.11).

[2] $s \in (0, \frac{1}{2})$: In view of (4.22), we immediately conclude that the optimal adjoint state $\bar{p} \in L^2(0, T; H^{2s}(\Omega))$. This, on the basis of a nonlinear operator interpolation result, implies that $\bar{z} \in L^2(0, T; H^{2s}(\Omega))$ with the estimate

$$\begin{aligned} \|\bar{z}\|_{L^2(0, T; H^{2s}(\Omega))} &\lesssim \Sigma(u_0, f + \bar{z}) + \|u_d\|_{L^\infty(0, T; L^2(\Omega))} \\ &\quad + \|a\|_{L^2(0, T; H^1(\Omega))} + \|b\|_{L^2(0, T; H^1(\Omega))}. \end{aligned} \quad (4.23)$$

If $u_0 \in \tilde{H}^s(\Omega)$, $f \in L^\infty(0, T; L^2(\Omega))$, and $a, b \in L^\infty(0, T; L^2(\Omega))$, we apply the regularity estimate (3.11) to arrive at

$$\|\bar{u}\|_{L^2(0, T; H^{2s}(\Omega))} \lesssim \|u_0\|_{H^s(\mathbb{R}^n)} + \|f + \bar{z}\|_{L^\infty(0, T; L^2(\Omega))}.$$

[3] $s = \frac{1}{2}$: The proof of the estimates (4.17) and (4.20) follow similar arguments. For brevity, we skip the details. \square

5. Approximation of the state equation. Let us now propose and analyze a fully discrete numerical scheme to solve the state equation (3.5). The space discretization hinges on the standard finite element space of continuous and piecewise linear functions. The discretization in time uses the backward Euler scheme. We derive stability estimates and a $L^2(Q)$ a priori error estimate.

5.1. Time discretization. Let $\mathcal{K} \in \mathbb{N}$ be the number of time steps. Define the uniform time step $\tau = T/\mathcal{K} > 0$ and set $t_k = k\tau$ for $k = 0, \dots, \mathcal{K}$. We denote the time partition by $\mathcal{T} := \{t_k\}_{k=0}^{\mathcal{K}}$. Given a function $\phi \in C([0, T], \mathcal{X})$, we denote $\phi^k = \phi(t_k) \in \mathcal{X}$ and $\phi^\tau = \{\phi^k\}_{k=0}^{\mathcal{K}} \subset \mathcal{X}$. For any sequence $\phi^\tau \subset \mathcal{X}$, we define the piecewise linear interpolant $\hat{\phi}^\tau \in C([0, T]; \mathcal{X})$ by

$$\hat{\phi}^\tau(t) := \frac{t - t_k}{\tau} \phi^{k+1} + \frac{t_{k+1} - t}{\tau} \phi^k, \quad t \in [t_k, t_{k+1}], \quad k = 0, \dots, \mathcal{K} - 1. \quad (5.1)$$

We also define, for any sequence $\phi^\tau \subset \mathcal{X}$, the first order differences operators

$$\mathfrak{d}\phi^{k+1} = \tau^{-1}(\phi^{k+1} - \phi^k), \quad k = 0, \dots, \mathcal{K} - 1, \quad (5.2)$$

and

$$\bar{\mathfrak{d}}\phi^k = -\tau^{-1}(\phi^{k+1} - \phi^k), \quad k = \mathcal{K} - 1, \dots, 0, \quad (5.3)$$

and the norms $\|\phi^\tau\|_{\ell^\infty(\mathcal{X})} = \max\{\|\phi^k\|_{\mathcal{X}} : k = 0, \dots, \mathcal{K}\}$ and

$$\|\phi^\tau\|_{\ell^p(\mathcal{X})} = \left(\sum_{k=1}^{\mathcal{K}} \tau \|\phi^k\|_{\mathcal{X}}^p \right)^{\frac{1}{p}}, \quad p \in [1, \infty).$$

REMARK 5.1 (identification with a piecewise constant function). We note that any sequence $\phi^\tau \subset \mathcal{X}$ can be equivalently understood as a piecewise constant, in time, function $\phi \in L^\infty(0, T; \mathcal{X})$. In fact, consider

$$\phi(t) = \phi^k \quad \forall t \in (t_{k-1}, t_k], \quad k = 1, \dots, \mathcal{K}.$$

In what follows we will use this identification repeatedly and without explicit mention.

5.2. Space discretization. Let $\mathcal{T} = \{K\}$ be a conforming partition of $\bar{\Omega}$ into simplices K with size $h_K = \text{diam}(K)$. Set $h_{\mathcal{T}} = \max_{K \in \mathcal{T}} h_K$. We denote by \mathbb{T} the collection of conforming and shape regular meshes that are refinements of an initial mesh \mathcal{T}_0 . By shape regular we mean that there exists a constant $\sigma > 1$ such that $\max\{\sigma_K : K \in \mathcal{T}\} \leq \sigma$ for all $\mathcal{T} \in \mathbb{T}$. Here, $\sigma_K = h_K/\rho_K$ denotes the shape coefficient of K , where ρ_K is the diameter of the largest ball that can be inscribed in K [9, 12, 19].

Given a mesh $\mathcal{T} \in \mathbb{T}$, we define the finite element space of continuous piecewise polynomials of degree one as

$$\mathbb{V}(\mathcal{T}) = \{V \in C^0(\bar{\Omega}) : V|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}, \quad V = 0 \text{ on } \partial\Omega\}. \quad (5.4)$$

Note that discrete functions are trivially extended by zero to Ω^c and that we enforce a classical homogeneous Dirichlet boundary condition at the degrees of freedom that are located at the boundary of Ω .

5.3. Elliptic projection. In this section, we define an elliptic projector that will be of fundamental importance to derive error estimates. This projector $G_{\mathcal{T}} : \tilde{H}^s(\Omega) \rightarrow \mathbb{V}(\mathcal{T})$ is such that, for $w \in \tilde{H}^s(\Omega)$, it is given by

$$G_{\mathcal{T}}w \in \mathbb{V}(\mathcal{T}) : \quad \mathcal{A}(G_{\mathcal{T}}w, W) = \mathcal{A}(w, W) \quad \forall W \in \mathbb{V}(\mathcal{T}). \quad (5.5)$$

The operator $G_{\mathcal{T}}$ satisfies the following stability and approximation properties.

PROPOSITION 5.2 (elliptic projector). *Let $s \in (0, 1)$. The elliptic projector $G_{\mathcal{T}}$ is stable in $\tilde{H}^s(\Omega)$, i.e.,*

$$\|G_{\mathcal{T}}w\|_s \lesssim \|w\|_s \quad \forall w \in \tilde{H}^s(\Omega). \quad (5.6)$$

If, in addition, $w \in H^\kappa(\Omega)$, for $\kappa \geq s$, then $G_{\mathcal{T}}$ has the following approximation property:

$$\|w - G_{\mathcal{T}}w\|_s \lesssim h_{\mathcal{T}}^{\kappa-s} |w|_{H^\kappa(\Omega)}. \quad (5.7)$$

In both estimates the hidden constants are independent of w and $h_{\mathcal{T}}$.

Proof. To show stability set $W = G_{\mathcal{T}}w$ in (5.5), invoke the definition of the norm $\|\cdot\|_s$ given by (2.3), and utilize the continuity of \mathcal{A} .

Obtaining the estimate (5.7) hinges on Galerkin orthogonality: If $\Pi_{\mathcal{T}}$ denotes the Scott–Zhang quasi-interpolation operator, then

$$\|w - G_{\mathcal{T}}w\|_s^2 = \mathcal{A}(w - G_{\mathcal{T}}w, w - \Pi_{\mathcal{T}}w) \lesssim \|w - G_{\mathcal{T}}w\|_s \|w - \Pi_{\mathcal{T}}w\|_s.$$

The assertion thus follows from an interpolation error estimate for $\Pi_{\mathcal{T}}$:

$$\|w - \Pi_{\mathcal{T}}w\|_s \lesssim h_{\mathcal{T}}^{\kappa-s} |w|_{H^\kappa(\Omega)};$$

see [2, Section 4.2] for details. This concludes the proof. \square

PROPOSITION 5.3 ($L^2(\Omega)$ -error estimate: elliptic projector). *Let $s \in (0, 1)$ and Ω be a domain such that $\partial\Omega \in C^\infty$. If $w \in H^\kappa(\Omega)$, for $\kappa \geq s$, then we have*

$$\|w - G_{\mathcal{T}}w\|_{L^2(\Omega)} \lesssim h_{\mathcal{T}}^{\kappa+\vartheta-s} |w|_{H^\kappa(\Omega)}, \quad (5.8)$$

where $\vartheta = \min\{s, 1/2 - \epsilon\}$. The hidden constant is independent of w and $h_{\mathcal{T}}$.

Proof. To obtain (5.8) we argue by duality. Let $z \in \tilde{H}^s(\Omega)$ be the solution to

$$\mathcal{A}(\phi, z) = \langle w - G_{\mathcal{T}}w, \phi \rangle \quad \forall \phi \in \tilde{H}^s(\Omega).$$

Set $\phi = w - G_{\mathcal{T}}w$ and utilize that $\mathcal{A}(w - G_{\mathcal{T}}w, \Pi_{\mathcal{T}}z) = 0$, where $\Pi_{\mathcal{T}}$ denotes the Scott–Zhang quasi-interpolation operator, to obtain

$$\|w - G_{\mathcal{T}}w\|_{L^2(\Omega)}^2 = \mathcal{A}(w - G_{\mathcal{T}}w, z) \leq \|w - G_{\mathcal{T}}w\|_s \|z - \Pi_{\mathcal{T}}z\|_s.$$

Invoke an interpolation error estimate for $\Pi_{\mathcal{T}}$ combined with the regularity results of Proposition 2.1, with $r = 0$, to obtain

$$\|z - \Pi_{\mathcal{T}}z\|_s \lesssim h_{\mathcal{T}}^\vartheta |z|_{H^{s+\vartheta}(\Omega)} \lesssim h_{\mathcal{T}}^\vartheta \|w - G_{\mathcal{T}}w\|_{L^2(\Omega)},$$

where $\vartheta = \min\{s, 1/2 - \epsilon\}$. The estimate (5.7) allows us to conclude. \square

5.4. A fully discrete scheme. Let us now describe a fully discrete numerical method to solve the state equation (3.5). The discretization in time uses the backward Euler scheme. The space discretization hinges on the finite element space introduced in section 5.2.

Set $\mathbf{z} = 0$. The fully discrete scheme computes the sequence $U_{\mathcal{T}}^\tau \subset \mathbb{V}(\mathcal{T})$, an approximation of the solution to problem (3.5) at each time step. We initialize the scheme by setting

$$U_{\mathcal{T}}^0 = P_{\mathcal{T}}\mathbf{u}_0, \quad (5.9)$$

where $P_{\mathcal{T}}$ denotes the $L^2(\Omega)$ -orthogonal projection onto $\mathbb{V}(\mathcal{T})$. For $k = 0, \dots, \mathcal{K} - 1$, $U_{\mathcal{T}}^{k+1} \in \mathbb{V}(\mathcal{T})$ solves

$$(\mathfrak{d}U_{\mathcal{T}}^{k+1}, W)_{L^2(\Omega)} + \mathcal{A}(U_{\mathcal{T}}^{k+1}, W) = \langle \mathbf{f}^{k+1}, W \rangle \quad \forall W \in \mathbb{V}(\mathcal{T}), \quad (5.10)$$

where $\mathbf{f}^{k+1} = \tau^{-1} \int_{t_k}^{t_{k+1}} \mathbf{f} \, dt$. We recall that \mathfrak{d} is defined by (5.2).

The fully discrete scheme (5.9)–(5.10) is unconditionally stable.

THEOREM 5.4 (unconditional stability). *Let $U_{\mathcal{T}}^{\tau}$ be the solution to the fully discrete scheme (5.9)–(5.10). If $\mathbf{f} \in L^2(0, T; H^{-s}(\Omega))$ and $\mathbf{u}_0 \in L^2(\Omega)$, then*

$$\|U_{\mathcal{T}}^{\tau}\|_{\ell^{\infty}(L^2(\Omega))}^2 + |U_{\mathcal{T}}^{\tau}|_{\ell^2(H^s(\mathbb{R}^n))}^2 \lesssim \|\mathbf{u}_0\|_{L^2(\Omega)}^2 + \|\mathbf{f}^{\tau}\|_{\ell^2(H^{-s}(\Omega))}^2, \quad (5.11)$$

where the hidden constant is independent of the data, the solution $U_{\mathcal{T}}^{\tau}$, and the discretization parameters.

Proof. Set $W = 2\tau U_{\mathcal{T}}^{k+1}$ in (5.10). The relation $2(a-b)a = a^2 - b^2 + (a-b)^2$ and Young's inequality yield

$$\|U_{\mathcal{T}}^{k+1}\|_{L^2(\Omega)}^2 - \|U_{\mathcal{T}}^k\|_{L^2(\Omega)}^2 + \|U_{\mathcal{T}}^{k+1} - U_{\mathcal{T}}^k\|_{L^2(\Omega)}^2 + \tau \|U_{\mathcal{T}}^{k+1}\|_s^2 \lesssim \tau \|\mathbf{f}^{k+1}\|_{H^{-s}(\Omega)}^2.$$

The stability estimate (5.11) follows from adding the previous inequality over k . \square

5.5. $L^2(Q)$ -error estimate. We introduce, as a technical instrument, a semidiscrete approximation of problem (3.5): Set $\mathbf{z} = 0$ and $U^0 = \mathbf{u}_0$. For $k = 0, \dots, \mathcal{K} - 1$, $U^{k+1} \in \tilde{H}^s(\Omega)$ solves

$$(\mathfrak{d}U^{k+1}, \phi)_{L^2(\Omega)} + \mathcal{A}(U^{k+1}, \phi) = \langle \mathbf{f}^{k+1}, \phi \rangle \quad \forall \phi \in \tilde{H}^s(\Omega). \quad (5.12)$$

The scheme (5.12) is unconditionally stable.

THEOREM 5.5 (unconditional stability). *Let U^{τ} be the solution to (5.12). If $\mathbf{f} \in L^2(Q)$ and $\mathbf{u}_0 \in \tilde{H}^s(\Omega)$, then*

$$\|\mathfrak{d}U^{\tau}\|_{\ell^2(L^2(\Omega))}^2 + |U^{\tau}|_{\ell^{\infty}(H^s(\mathbb{R}^n))}^2 \lesssim \|\mathbf{u}_0\|_{H^s(\mathbb{R}^n)}^2 + \|\mathbf{f}^{\tau}\|_{\ell^2(L^2(\Omega))}^2, \quad (5.13)$$

where the hidden constant is independent of the data, the solution U^{τ} , and τ .

Proof. Set $W = U^{k+1} - U^k$ in (5.10), use the relation $2(a-b)a = a^2 - b^2 + (a-b)^2$, and add over k . \square

Define the piecewise linear function $\hat{U} \in C^{0,1}([0, T]; \tilde{H}^s(\Omega))$ by

$$\hat{U}(0) = U^0, \quad \hat{U}(t) = U^k + (t - t_k)\mathfrak{d}U^{k+1}, \quad t \in (t_k, t_{k+1}], \quad (5.14)$$

for $k = 0, \dots, \mathcal{K} - 1$. An important observation is that, for $t \in (t_k, t_{k+1}]$, $\partial_t \hat{U}(t) = \mathfrak{d}U^{k+1}$. We can thus rewrite the semidiscrete scheme (5.12), for a.e. $t \in (0, T)$, as

$$(\partial_t \hat{U}(t), \phi)_{L^2(\Omega)} + \mathcal{A}(U^{\tau}(t), \phi) = \langle \mathbf{f}^{\tau}(t), \phi \rangle \quad \forall \phi \in \tilde{H}^s(\Omega). \quad (5.15)$$

Define $\hat{e} := \mathbf{u} - \hat{U}$ and $\bar{e} := \mathbf{u} - U^{\tau}$. We observe that $\hat{e}(0) = \bar{e}(0) = 0$. In addition, since the form \mathcal{A} is bilinear and continuous, basic computations reveal that

$$\frac{d}{dt} \mathcal{A} \left(\int_0^t \bar{e}(\xi) \, d\xi, \int_0^t \bar{e}(\xi) \, d\xi \right) = 2\mathcal{A} \left(\int_0^t \bar{e}(\xi) \, d\xi, \bar{e}(t) \right).$$

Consequently,

$$\int_0^T \mathcal{A} \left(\int_0^t \bar{e}(\xi) \, d\xi, \bar{e}(t) \right) dt = \frac{1}{2} \mathcal{A} \left(\int_0^T \bar{e}(t) \, dt, \int_0^T \bar{e}(t) \, dt \right) \geq 0. \quad (5.16)$$

We now derive an error estimate for the semidiscrete scheme (5.12).

THEOREM 5.6 (semi-discrete error estimate). *Let \mathbf{u} and U^τ be the solutions to (3.5) and (5.12), respectively. If $\mathbf{u}_0 \in \tilde{H}^s(\Omega)$ and $\mathbf{f} \in L^\infty(0, T; L^2(\Omega))$, then*

$$\|\mathbf{u} - U^\tau\|_{L^2(0, T; L^2(\Omega))} \lesssim \tau \left(|\mathbf{u}_0|_{H^s(\mathbb{R}^n)} + \|\mathbf{f}\|_{L^\infty(0, T; L^2(\Omega))} \right). \quad (5.17)$$

The hidden constant is independent of the data, the solutions \mathbf{u} and U^τ , and the discretization parameter τ .

Proof. We begin by recalling that, we have set $\mathbf{z} = 0$ in (3.5). Subtract from it (5.15) and integrate the resulting expression with respect to time. This yields

$$\begin{aligned} (\bar{e}(t), \phi)_{L^2(\Omega)} + \mathcal{A} \left(\int_0^t \bar{e}(\xi) d\xi, \phi \right) &= \left\langle \int_0^t (\mathbf{f}(\xi) - \mathbf{f}^\tau(\xi)) d\xi, \phi \right\rangle \\ &+ (\bar{e}(t) - \hat{e}(t), \phi)_{L^2(\Omega)} \quad \forall \phi \in \tilde{H}^s(\Omega), \quad \text{a.e. } t \in (0, T). \end{aligned}$$

Set, for a.e. $t \in (0, T)$, $\phi = \bar{e}(t) \in \tilde{H}^s(\Omega)$. Integrate with respect to time, again, and invoke the identity (5.16), to arrive at

$$\begin{aligned} \int_0^T \|\bar{e}(t)\|_{L^2(\Omega)}^2 dt &\leq \left| \int_0^T \left\langle \int_0^t [\mathbf{f}(\xi) - \mathbf{f}^\tau(\xi)] d\xi, \bar{e}(t) \right\rangle dt \right| \\ &+ \left| \int_0^T (\bar{e}(t) - \hat{e}(t), \bar{e}(t))_{L^2(\Omega)} dt \right| =: \text{I} + \text{II}. \quad (5.18) \end{aligned}$$

It thus suffices to estimate I and II. To control the term I, we first notice that, since $\mathbf{f}^{k+1} = \tau^{-1} \int_{t_k}^{t_{k+1}} \mathbf{f}(t) dt$, we have, for $\ell \in \{1, \dots, \mathcal{K}\}$,

$$\int_0^{t_\ell} (\mathbf{f}(\xi) - \mathbf{f}^\tau(\xi)) d\xi = \sum_{k=1}^{\ell} \int_{t_{k-1}}^{t_k} (\mathbf{f}(\xi) - \mathbf{f}^k) d\xi = 0.$$

Consequently, if $t_\ell < t < t_{\ell+1}$, then

$$\int_0^t (\mathbf{f}(\xi) - \mathbf{f}^\tau(\xi)) d\xi = \int_{t_\ell}^t (\mathbf{f}(\xi) - \mathbf{f}^\tau(\xi)) d\xi \lesssim \tau \|\mathbf{f}\|_{L^\infty(0, T)}.$$

We can thus apply Cauchy-Schwarz and Young's inequalities to arrive at

$$\text{I} \leq \int_0^T \left\| \int_0^t [\mathbf{f}(\xi) - \mathbf{f}^\tau(\xi)] d\xi \right\|_{L^2(\Omega)} \|\bar{e}(t)\|_{L^2(\Omega)} dt \lesssim \tau^2 \|\mathbf{f}\|_{L^\infty(0, T; L^2(\Omega))}^2 + \frac{1}{4} \|\bar{e}\|_{L^2(Q)}^2.$$

We now focus on estimating the term II. Since, on $(t_k, t_{k+1}]$, we have that $|\bar{e}(t) - \hat{e}(t)| \lesssim \tau |\mathfrak{D}U^{k+1}|$, we invoke the stability estimate (5.13) to conclude that

$$\int_0^T \|\hat{e}(t) - \bar{e}(t)\|_{L^2(\Omega)}^2 dt \lesssim \tau^2 \|\mathfrak{D}U^\tau\|_{\ell^2(L^2(\Omega))}^2 \lesssim \tau^2 \left(|\mathbf{u}_0|_{H^s(\mathbb{R}^n)}^2 + \|\mathbf{f}^\tau\|_{\ell^2(L^2(\Omega))}^2 \right).$$

This yields the following bound for the term II:

$$\text{II} \leq \frac{1}{4} \|\bar{e}\|_{L^2(0, T; L^2(\Omega))}^2 + C\tau^2 \left(|\mathbf{u}_0|_{H^s(\mathbb{R}^n)}^2 + \|\mathbf{f}^\tau\|_{\ell^2(L^2(\Omega))}^2 \right).$$

The desired estimate (5.17) follows from replacing the estimates for I and II into (5.18). This concludes the proof. \square

We now control the difference between the fully and the semidiscrete problems.

THEOREM 5.7 (auxiliary error estimate). *Let U^τ and $U_{\mathcal{T}}^\tau$ be the solutions to problems (5.12) and (5.10), respectively. Let $\gamma = \min\{s, 1/2 - \epsilon\}$ with $\epsilon > 0$. If $u_0 \in \tilde{H}^{2\gamma}(\Omega)$ and $f \in L^\infty(0, T; L^2(\Omega))$, then*

$$\|U^\tau - U_{\mathcal{T}}^\tau\|_{\ell^2(L^2(\Omega))} \lesssim h_{\mathcal{T}}^{2\gamma} (|u_0|_{H^{2\gamma}(\mathbb{R}^n)} + \|f + z\|_{L^\infty(0, T; L^2(\Omega))}).$$

The hidden constant does not depend on U^τ , $U_{\mathcal{T}}^\tau$, or the problem data.

Proof. As it is customary, we split the error into the so-called interpolation and approximation errors:

$$E^\tau = (U^\tau - G_{\mathcal{T}}U^\tau) + (G_{\mathcal{T}}U^\tau - U_{\mathcal{T}}^\tau) =: \theta^\tau + \rho_{\mathcal{T}}^\tau; \quad (5.19)$$

$G_{\mathcal{T}}$ denotes the elliptic projector defined in (5.5). We estimate θ^τ in view of the estimate (5.8) and the regularity results of Theorem 3.2. In fact,

$$\|\theta^\tau\|_{\ell^2(L^2(\Omega))} \lesssim h_{\mathcal{T}}^{\vartheta+\gamma} \|U^\tau\|_{\ell^2(H^{s+\gamma}(\Omega))} \lesssim h_{\mathcal{T}}^{\vartheta+\gamma} (\|u_0\|_{H^s(\mathbb{R}^n)} + \|f\|_{L^\infty(0, T; L^2(\Omega))}),$$

where $\vartheta = \gamma = \min\{s, 1/2 - \epsilon\}$. The estimate of the term $\rho_{\mathcal{T}}^\tau$ follows along the same lines of [7, Lemma 5.6]. For brevity, we skip the details. \square

We collect the estimates of Theorems 5.6 and 5.7 to derive a $L^2(Q)$ -error estimate for the fully discrete scheme (5.9)–(5.10).

THEOREM 5.8 (error estimate for fully discrete scheme). *Let u and $U_{\mathcal{T}}^\tau$ solve (3.5) and (5.10), respectively. If $f \in L^\infty(0, T; L^2(\Omega))$ and $u_0 \in \tilde{H}^{2\gamma}(\Omega)$, then*

$$\|u - U_{\mathcal{T}}^\tau\|_{L^2(Q)} \lesssim (\tau + h_{\mathcal{T}}^{2\gamma}) (|u_0|_{H^{2\gamma}(\mathbb{R}^n)} + \|f + z\|_{L^\infty(0, T; L^2(\Omega))}).$$

The hidden constant does not depend on $h_{\mathcal{T}}$, τ , u , $U_{\mathcal{T}}^\tau$, or the problem data.

6. Approximation of parabolic fractional control problem. In this section, we introduce an implicit fully-discrete scheme to approximate the solution of the fractional control problem (1.2)–(1.4). The scheme discretizes the control variable with piecewise constant functions. The state variable is discretized with standard piecewise linear finite elements in space, as detailed in section 5.2, and with the backward Euler scheme in time, as described in section 5.1.

To simplify the exposition, in what follows we assume that a and b are constants.

6.1. An implicit fully discrete-scheme. To discretize the control variable, we introduce the finite element space of piecewise constant functions over \mathcal{T} ,

$$\mathbb{Z}(\mathcal{T}) = \{Z \in L^\infty(\Omega) : Z|_K \in \mathbb{P}_0(K) \ \forall K \in \mathcal{T}\},$$

and the space of piecewise constant functions in time and space,

$$\mathbb{Z}(\mathcal{T}, \mathcal{T}) = \{Z^\tau \subset L^\infty(Q) : Z^k \in \mathbb{Z}(\mathcal{T})\}. \quad (6.1)$$

The space of discrete admissible controls is defined as

$$Z_{\text{ad}}(\mathcal{T}, \mathcal{T}) = Z_{\text{ad}} \cap \mathbb{Z}(\mathcal{T}, \mathcal{T}), \quad (6.2)$$

where Z_{ad} is defined in (4.1).

To perform an a priori error analysis, it is useful to introduce the $L^2(Q)$ -orthogonal projection onto $\mathbb{Z}(\mathcal{T}, \mathcal{S})$. This operator, $\Pi_{\mathcal{T}}^{\mathcal{T}} : L^2(Q) \rightarrow \mathbb{Z}(\mathcal{T}, \mathcal{S})$, is defined by

$$z \in L^2(Q) : \quad (z - \Pi_{\mathcal{T}}^{\mathcal{T}} z, Z)_{L^2(Q)} = 0 \quad \forall Z \in \mathbb{Z}(\mathcal{T}, \mathcal{S}). \quad (6.3)$$

If $z \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^\kappa(\Omega))$, with $\kappa \in (0, 1)$, we have the error estimate

$$\|z - \Pi_{\mathcal{T}}^{\mathcal{T}} z\|_{L^2(Q)} \lesssim h_{\mathcal{T}}^\kappa \|z\|_{L^2(0, T; H^\kappa(\Omega))} + \tau \|\partial_t z\|_{L^2(Q)}. \quad (6.4)$$

An important observation in favor of $\Pi_{\mathcal{T}}^{\mathcal{T}}$ is that $\Pi_{\mathcal{T}}^{\mathcal{T}} \mathbf{Z}_{\text{ad}} \subset \mathbf{Z}_{\text{ad}}(\mathcal{T}, \mathcal{S})$. We recall that \mathbf{a} and \mathbf{b} , that define the set (4.1), are constant.

We define the discrete functional $J_{\mathcal{T}}^{\mathcal{T}} : \mathbb{V}(\mathcal{T})^{\mathcal{K}} \times \mathbb{Z}(\mathcal{T}, \mathcal{S}) \rightarrow \mathbb{R}$ by

$$J_{\mathcal{T}}^{\mathcal{T}}(U_{\mathcal{T}}^{\tau}, Z_{\mathcal{T}}^{\tau}) = \frac{1}{2} \|U_{\mathcal{T}}^{\tau} - \mathbf{u}_d^{\tau}\|_{\ell^2(L^2(\Omega))}^2 + \frac{\mu}{2} \|Z_{\mathcal{T}}^{\tau}\|_{\ell^2(L^2(\Omega))}^2.$$

Notice that, if $\mathbf{u}_d^{\tau} = \mathbf{u}_d$ we would have that $J_{\mathcal{T}}^{\mathcal{T}}(w, r) = J(w, r)$ whenever $w^{\tau} = w$ and $r^{\tau} = r$; see Remark 5.1.

With this notation at hand, we introduce the following fully discrete scheme for our parabolic fractional optimal control problem: Find

$$\min J_{\mathcal{T}}^{\mathcal{T}}(U_{\mathcal{T}}^{\tau}, Z_{\mathcal{T}}^{\tau})$$

subject to the discrete equation: initialize as in (5.9) and, for $k = 0, \dots, \mathcal{K} - 1$, let $U_{\mathcal{T}}^{k+1} \in \mathbb{V}(\mathcal{T})$ solve

$$(\partial U_{\mathcal{T}}^{k+1}, W)_{L^2(\Omega)} + \mathcal{A}(U_{\mathcal{T}}^{k+1}, W) = \langle \mathbf{f}^{k+1} + Z_{\mathcal{T}}^{k+1}, W \rangle \quad (6.5)$$

for all $W \in \mathbb{V}(\mathcal{T})$, and the control constraints $Z_{\mathcal{T}}^{\tau} \in \mathbf{Z}_{\text{ad}}(\mathcal{T}, \mathcal{S})$.

6.2. First order optimality conditions. We provide first order necessary and sufficient optimality conditions for the fully discrete optimal control problem. To accomplish this task, we first define the following discrete adjoint problem: Find $P_{\mathcal{T}}^{\tau} \subset \mathbb{V}(\mathcal{T})$ such that $P_{\mathcal{T}}^{\mathcal{K}} = 0$, and for $k = \mathcal{K} - 1, \dots, 0$, $P_{\mathcal{T}}^k \in \mathbb{V}(\mathcal{T})$ solves

$$(\bar{\partial} P_{\mathcal{T}}^k, W)_{L^2(\Omega)} + \mathcal{A}(P_{\mathcal{T}}^k, W) = \langle U_{\mathcal{T}}^{k+1} - \mathbf{u}_d^{k+1}, W \rangle \quad (6.6)$$

for all $W \in \mathbb{V}(\mathcal{T})$. The difference operator $\bar{\partial}$ is defined in (5.3).

The optimality condition reads: $(\bar{U}_{\mathcal{T}}^{\tau}, \bar{Z}_{\mathcal{T}}^{\tau})$ is optimal for the scheme of section 6.1 if and only if $\bar{U}_{\mathcal{T}}^0 = P_{\mathcal{T}}^0 \mathbf{u}_0$, for $k = 0, \dots, \mathcal{K} - 1$, $\bar{U}_{\mathcal{T}}^{k+1} \in \mathbb{V}(\mathcal{T})$ solves (6.5), and

$$(\mu \bar{Z}_{\mathcal{T}}^{\tau} + \bar{P}_{\mathcal{T}}^{\tau}, Z - \bar{Z}_{\mathcal{T}}^{\tau})_{L^2(Q)} \geq 0 \quad \forall Z \in \mathbf{Z}_{\text{ad}}(\mathcal{T}, \mathcal{S}), \quad (6.7)$$

where $\bar{P}_{\mathcal{T}}^{\tau}$ solves (6.6). Set $Z^{\tau} = Z \chi_{(t_{k-1}, t_k]}$ with $Z \in \mathbb{Z}(\mathcal{T}_{\Omega})$ and $\mathbf{a} \leq Z \leq \mathbf{b}$ in (6.7). We thus obtain that (6.7) can be equivalently written as

$$(\bar{P}_{\mathcal{T}}^k + \mu \bar{Z}_{\mathcal{T}}^k, Z - \bar{Z}_{\mathcal{T}}^k)_{L^2(\Omega)} \geq 0 \quad \forall Z \in \mathbb{Z}(\mathcal{T}), \quad \mathbf{a} \leq Z \leq \mathbf{b}, \quad \forall k = 1, \dots, \mathcal{K}.$$

6.3. Auxiliary problems. We introduce two auxiliary problems that will be instrumental to derive error estimates for the fully discrete scheme of section 6.1.

The first problem reads as follows: Find $Q_{\mathcal{T}}^{\tau} \subset \mathbb{V}(\mathcal{T})$ such that $Q_{\mathcal{T}}^{\mathcal{K}} = 0$ and, for $k = \mathcal{K} - 1, \dots, 0$, $Q_{\mathcal{T}}^k \in \mathbb{V}(\mathcal{T})$ solves

$$(\bar{\partial} Q_{\mathcal{T}}^k, W)_{L^2(\Omega)} + \mathcal{A}(Q_{\mathcal{T}}^k, W) = \langle \bar{\mathbf{u}}^{k+1} - \mathbf{u}_d^{k+1}, \text{tr}_{\Omega} W \rangle \quad (6.8)$$

for all $W \in \mathbb{V}(\mathcal{T})$; $\bar{u} = \bar{u}(\bar{z})$ denotes the solution to (3.5) with z replaced by \bar{z} .

The second auxiliary problem is: Find $R_{\mathcal{T}}^{\tau} \subset \mathbb{V}(\mathcal{T})$ such that $R_{\mathcal{T}}^{\mathcal{K}} = 0$ and, for $k = \mathcal{K} - 1, \dots, 0$, $R_{\mathcal{T}}^k \in \mathbb{V}(\mathcal{T})$ solves

$$(\bar{\delta} R_{\mathcal{T}}^k, W)_{L^2(\Omega)} + \mathcal{A}(R_{\mathcal{T}}^k, W) = \langle U_{\mathcal{T}}^{k+1}(\bar{z}) - u_d^{k+1}, W \rangle \quad (6.9)$$

for all $W \in \mathbb{V}(\mathcal{T})$; $U_{\mathcal{T}}^{k+1}(\bar{z})$ denotes the solution to (6.5) with $Z_{\mathcal{T}}^{k+1}$ replaced by \bar{z}^{k+1} .

6.4. A priori error analysis: $s \in (0, 1)$. We derive the following a priori error estimate.

THEOREM 6.1 (error estimate for the control: $s \in (0, 1)$). *Let \bar{z} be the optimal control for problem (1.2)–(1.4) and let $\bar{Z}_{\mathcal{T}}$ be the optimal control for the fully discrete scheme of section 6.1. If $u_0 \in \tilde{H}^{2\gamma}(\Omega)$ and $f \in L^\infty(0, T; L^2(\Omega))$, then*

$$\|\bar{z} - \bar{Z}_{\mathcal{T}}^{\tau}\|_{L^2(Q)} \lesssim \tau + h_{\mathcal{T}}^{2\gamma},$$

where $\gamma = \min\{s, 1/2 - \epsilon\}$ and $\epsilon > 0$. The hidden constant is independent of \bar{z} , $\bar{Z}_{\mathcal{T}}$, and the discretization parameters, but depends on the problem data.

Proof. We proceed in several steps.

Step 1. Set $z = \bar{Z}_{\mathcal{T}}^{\tau}$ in (4.8) and $Z = \Pi_{\mathcal{T}}^{\tau} \bar{z}$ in (6.7), where $\Pi_{\mathcal{T}}^{\tau}$ denotes the $L^2(Q)$ -orthogonal projection onto $\mathbb{Z}(\mathcal{T}, \mathcal{T})$. Add the obtained inequalities to arrive at

$$\mu \|\bar{z} - \bar{Z}_{\mathcal{T}}^{\tau}\|_{L^2(Q)}^2 \leq (\bar{p} - \bar{P}_{\mathcal{T}}^{\tau}, \bar{Z}_{\mathcal{T}}^{\tau} - \bar{z})_{L^2(Q)} + (\bar{P}_{\mathcal{T}}^{\tau} + \mu \bar{Z}_{\mathcal{T}}^{\tau}, \Pi_{\mathcal{T}}^{\tau} \bar{z} - \bar{z})_{L^2(Q)}.$$

We recall that the adjoint state \bar{p} solves (4.5) with u replaced by \bar{u} and its fully discrete counterpart $\bar{P}_{\mathcal{T}}^{\tau}$ is defined as the solution to (6.6) with $U_{\mathcal{T}}^{k+1}$ replaced by $\bar{U}_{\mathcal{T}}^{k+1}$.

Step 2. We invoke the solutions to the auxiliary problems (6.8) and (6.9) to write $\bar{p} - \bar{P}_{\mathcal{T}}^{\tau} = (\bar{p} - Q_{\mathcal{T}}^{\tau}) + (Q_{\mathcal{T}}^{\tau} - R_{\mathcal{T}}^{\tau}) + (R_{\mathcal{T}}^{\tau} - \bar{P}_{\mathcal{T}}^{\tau})$. Since $Q_{\mathcal{T}}^{\tau}$ solves (6.8), the estimate for the term $\bar{p} - Q_{\mathcal{T}}^{\tau}$ follows immediately from Theorem 5.8:

$$\|\bar{p} - Q_{\mathcal{T}}^{\tau}\|_{L^2(Q)} \lesssim (\tau + h_{\mathcal{T}}^{2\gamma}) (\|\bar{u}\|_{L^\infty(0, T; L^2(\Omega))} + \|u_d\|_{L^\infty(0, T; L^2(\Omega))}).$$

We invoke the energy estimate (3.8) and complete the previous error estimate:

$$\|\bar{p} - Q_{\mathcal{T}}^{\tau}\|_{L^2(Q)} \lesssim (\tau + h_{\mathcal{T}}^{2\gamma}) (\Sigma(u_0, f + \bar{z}) + \|u_d\|_{L^\infty(0, T; L^2(\Omega))}), \quad (6.10)$$

where Σ is defined in (3.7).

Step 3. The goal of this step is to control the difference $Q_{\mathcal{T}}^{\tau} - R_{\mathcal{T}}^{\tau}$. To accomplish this task, we first invoke the stability result of Theorem 5.4 and then the error estimate of Theorem 5.8. These arguments yield the estimates

$$\|Q_{\mathcal{T}}^{\tau} - R_{\mathcal{T}}^{\tau}\|_{L^2(Q)} \lesssim \|\bar{u} - U_{\mathcal{T}}^{\tau}(\bar{z})\|_{L^2(Q)} \lesssim (\tau + h_{\mathcal{T}}^{2\gamma}) \Sigma(u_0, f + \bar{z}), \quad (6.11)$$

where the hidden constant is independent of $h_{\mathcal{T}}$ and τ .

Step 4. We handle the term $R_{\mathcal{T}}^{\tau} - \bar{P}_{\mathcal{T}}^{\tau}$ in view of an argument based on summation by parts. First, we define

$$\Psi^k := \bar{P}_{\mathcal{T}}^k - R_{\mathcal{T}}^k, \quad \Phi^{k+1} := \bar{U}_{\mathcal{T}}^{k+1} - U_{\mathcal{T}}^{k+1}(\bar{z}).$$

Now, set Ψ^k and Φ^{k+1} in the problems that $\bar{U}_{\mathcal{T}}^\tau - U_{\mathcal{T}}^\tau(\bar{z})$ and $\bar{P}_{\mathcal{T}}^\tau - \bar{R}_{\mathcal{T}}^\tau$ solve, respectively. In view of the fact that $\psi^K = 0 = \Phi^0$, invoke the discrete summation by parts formula

$$\sum_{k=0}^{K-1} \tau(\mathfrak{d}\Phi^{k+1}, \psi^k) = - \sum_{k=0}^{K-1} \tau(\Phi^{k+1}, \mathfrak{d}\psi^{k+1}) = \sum_{k=0}^{K-1} \tau(\Phi^{k+1}, \bar{\mathfrak{d}}\psi^k)$$

to conclude that

$$(R_{\mathcal{T}}^\tau - \bar{P}_{\mathcal{T}}^\tau, \bar{Z}_{\mathcal{T}}^\tau - \bar{z})_{L^2(Q)} \leq 0.$$

Step 5. The goal of this step is to control the term $(\bar{P}_{\mathcal{T}}^\tau + \mu\bar{Z}_{\mathcal{T}}^\tau, \Pi_{\mathcal{T}}^\tau \bar{z} - \bar{z})_{L^2(Q)}$. To accomplish this task, we write

$$\begin{aligned} (\bar{P}_{\mathcal{T}}^\tau + \mu\bar{Z}_{\mathcal{T}}^\tau, \Pi_{\mathcal{T}}^\tau \bar{z} - \bar{z})_{L^2(Q)} &= (\bar{\mathfrak{p}} + \mu\bar{z}, \Pi_{\mathcal{T}}^\tau \bar{z} - \bar{z})_{L^2(Q)} + (\bar{P}_{\mathcal{T}}^\tau - Q_{\mathcal{T}}^\tau, \Pi_{\mathcal{T}}^\tau \bar{z} - \bar{z})_{L^2(Q)} \\ &+ (Q_{\mathcal{T}}^\tau - \bar{\mathfrak{p}}, \Pi_{\mathcal{T}}^\tau \bar{z} - \bar{z})_{L^2(Q)} + \mu(\bar{Z}_{\mathcal{T}}^\tau - \bar{z}, \Pi_{\mathcal{T}}^\tau \bar{z} - \bar{z})_{L^2(Q)} = \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned} \quad (6.12)$$

We recall that the auxiliary variable $Q_{\mathcal{T}}^\tau$ is defined as the solution to (6.8).

To estimate the term I we invoke the property (6.3), that defines $\Pi_{\mathcal{T}}^\tau$, and the error estimate (6.4). We can thus obtain

$$\begin{aligned} \text{I} &= (\bar{\mathfrak{p}} + \mu\bar{z} - \Pi_{\mathcal{T}}^\tau(\bar{\mathfrak{p}} + \mu\bar{z}), \Pi_{\mathcal{T}}^\tau \bar{z} - \bar{z})_{L^2(Q)} \\ &\lesssim (h_{\mathcal{T}}^\gamma \|\bar{\mathfrak{p}} + \mu\bar{z}\|_{L^2(0,T;H^\gamma(\Omega))} + \tau \|\partial_t(\bar{\mathfrak{p}} + \mu\bar{z})\|_{L^2(Q)}) \\ &\quad \cdot (h_{\mathcal{T}}^\gamma \|\bar{z}\|_{L^2(0,T;H^\gamma(\Omega))} + \tau \|\partial_t \bar{z}\|_{L^2(Q)}). \end{aligned}$$

Notice that the norms $\|\bar{\mathfrak{p}} + \mu\bar{z}\|_{L^2(0,T;H^\gamma(\Omega))}$ and $\|\partial_t(\bar{\mathfrak{p}} + \mu\bar{z})\|_{L^2(Q)}$ are uniformly controlled by the problem data; see the regularity estimates of Theorems 4.8 and 4.9.

In what follows we control II. To accomplish this task, we first notice that

$$\|\bar{P}_{\mathcal{T}}^\tau - Q_{\mathcal{T}}^\tau\|_{L^2(Q)} \leq \|\bar{P}_{\mathcal{T}}^\tau - R_{\mathcal{T}}^\tau\|_{L^2(Q)} + \|R_{\mathcal{T}}^\tau - Q_{\mathcal{T}}^\tau\|_{L^2(Q)},$$

where the auxiliary variable $R_{\mathcal{T}}^\tau$ is defined as the solution to (6.9). The term $\|R_{\mathcal{T}}^\tau - Q_{\mathcal{T}}^\tau\|_{L^2(Q)}$ is bounded as in (6.11). It thus suffices to bound $\|\bar{P}_{\mathcal{T}}^\tau - R_{\mathcal{T}}^\tau\|_{L^2(Q)}$. To do this, we invoke the stability estimate (5.11), twice, to arrive at

$$\|\bar{P}_{\mathcal{T}}^\tau - R_{\mathcal{T}}^\tau\|_{L^2(Q)} \lesssim \|\bar{U}_{\mathcal{T}}^\tau - U_{\mathcal{T}}^\tau(\bar{z})\|_{L^2(Q)} \lesssim \|\bar{Z}_{\mathcal{T}}^\tau - \bar{z}\|_{L^2(Q)}.$$

We thus obtain that $\|\bar{P}_{\mathcal{T}}^\tau - Q_{\mathcal{T}}^\tau\|_{L^2(Q)} \lesssim (\tau + h^{2\gamma})\Sigma(u_0, f + \bar{z}) + \|\bar{Z}_{\mathcal{T}}^\tau - \bar{z}\|_{L^2(Q)}$. We now invoke the Cauchy-Schwarz inequality, the previous estimate for $\bar{P}_{\mathcal{T}}^\tau - Q_{\mathcal{T}}^\tau$, the error estimate (6.4), and Young's inequality to arrive at

$$\begin{aligned} \text{II} &\leq \|\bar{P}_{\mathcal{T}}^\tau - Q_{\mathcal{T}}^\tau\|_{L^2(Q)} \|\Pi_{\mathcal{T}}^\tau \bar{z} - \bar{z}\|_{L^2(Q)} \leq \frac{\mu}{4} \|\bar{Z}_{\mathcal{T}}^\tau - \bar{z}\|_{L^2(Q)}^2 + C \left(\tau^2 \Sigma^2(u_0, f + \bar{z}) \right. \\ &\quad \left. + h^{4\gamma} \Sigma^2(u_0, f + \bar{z}) + h^{2\gamma} \|\bar{z}\|_{L^2(0,T;H^\gamma(\Omega))}^2 + \tau^2 \|\partial_t \bar{z}\|_{L^2(Q)}^2 \right), \end{aligned}$$

where $C > 0$.

The control of the term III follows from (6.10) and (6.4). In fact,

$$\begin{aligned} \text{III} &\lesssim (\tau + h^{2\gamma}) (\Sigma(u_0, f + \bar{z}) + \|u_d\|_{L^\infty(0,T;L^2(\Omega))}) \\ &\quad \cdot (h_{\mathcal{T}}^\gamma \|\bar{z}\|_{L^2(0,T;H^\gamma(\Omega))} + \tau \|\partial_t \bar{z}\|_{L^2(Q)}). \end{aligned}$$

The term IV can be bounded in view of similar arguments.

Step 6. The assertion follows from collecting all the estimates we obtained in previous steps. This concludes the proof. \square

7. Numerical examples. We present a series of numerical examples that illustrate the performance of the fully discrete scheme proposed in section 6.1 when solving the optimal control problem (1.2)–(1.4). We consider one- and two-dimensional numerical experiments posed on the domain $B(0, 1) \times (0, T)$, where $B(0, 1)$ denotes the interval $(0, 1)$, when $n = 1$, and the circle of radius 1 centered at $(0, 0)$, when $n = 2$.

7.1. Exact solutions. We let $n \in \{1, 2\}$, $\Omega = B(0, 1)$, and $s \in (0, 1)$. We consider the fractional Poisson problem: Find u such that

$$(-\Delta)^s u = f \text{ in } \Omega, \quad u = 0 \text{ in } \Omega^c. \quad (7.1)$$

If $n = 1$ and the right-hand side is

$$f_{k,0}^{1D}(x) = 2^{2s} \Gamma(1+s)^2 \binom{s+k-1/2}{s} \binom{s+k}{s} P_k^{(s,-1/2)}(2x^2-1), \quad k \geq 0,$$

then, the solution u of problem (7.1) is given by

$$u_{k,0}^{1D}(x) = P_k^{(s,-1/2)}(2x^2-1) (1-x^2)_+^s.$$

Here, $P_k^{(\alpha,\beta)}$ denote the Jacobi polynomials, $x_+ = \max\{0, x\}$, and

$$\binom{x}{y} = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}$$

correspond to the generalized binomial coefficients. On the other hand, when the right-hand side is

$$f_{k,1}^{1D}(x) = 2^{2s} \Gamma(1+s)^2 \binom{s+k+1/2}{s} \binom{s+k}{s} x P_k^{(s,1/2)}(2x^2-1), \quad k \geq 0,$$

then, the solution u of problem (7.1) is given by

$$u_{k,1}^{1D}(x) = x P_k^{(s,1/2)}(2x^2-1) (1-x^2)_+^s.$$

If $n = 2$ and the right-hand side reads, in polar coordinates,

$$f_{k,\ell}^{2D}(r, \theta) = 2^{2s} \Gamma(1+s)^2 \binom{s+k+\ell}{s} \binom{s+k}{s} r^\ell \cos(\ell\theta) P_k^{(s,\ell)}(2r^2-1), \quad \ell, k \geq 0$$

then, the solution of problem (7.1) is given, in polar coordinates, by

$$u_{k,\ell}^{2D}(r, \theta) = r^\ell \cos(\ell\theta) P_k^{(s,\ell)}(2r^2-1) (1-r^2)_+^s.$$

We refer the reader to [18] for details on how these analytical solutions are determined.

In what follows we construct analytic solutions to the parabolic fractional optimal control problem. Let $\psi, \phi \in C^\infty((0, T))$ be such that $\psi(0) = 1$ and $\phi(T) = 0$. Let $f, g \in C^\infty(\Omega)$ and u and v be the solutions to the fractional Poisson problem (7.1) with right-hand sides f and g , respectively. Set

$$\begin{aligned} \mathbf{f}(t, x) &= \psi'(t)u + \psi(t)f(x) - \text{proj}_{[a,b]}(\phi(t)v(x)), \\ \mathbf{u}_d(t, x) &= \psi(t)u(x) + \mu\phi'(t)v(x) + \mu\phi(t)g(x), \\ \mathbf{u}_0(x) &= u(x). \end{aligned}$$

The exact solution to the parabolic fractional optimal control problem is given by

$$\begin{aligned}\bar{u}(t, x) &= \psi(t)u(x), \\ \bar{p}(t, x) &= -\mu\phi(t)v(x), \\ \bar{z}(t, x) &= \text{proj}_{[a,b]}(\phi(t)v(x)).\end{aligned}$$

Since $f, g \in C^\infty(\Omega)$ and $\partial\Omega \in C^\infty$, we can apply the results of Proposition 2.1 to conclude that $u, v \in H^{s+1/2-\epsilon}(\Omega)$ with $\epsilon > 0$ being arbitrarily small. Consequently,

$$u_0 \in H^{s+1/2-\epsilon}(\Omega), \quad u_d \in L^\infty(0, T; H^{s+1/2-\epsilon}(\Omega)), \quad f \in L^\infty(0, T; H^\xi(\Omega)),$$

where $\xi = \min\{1, s + 1/2 - \epsilon\}$; the last regularity result follows from [33, Lemma 2.21]. Notice that, under this *particular scenario*, we have that $\bar{z} \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^{s+1/2-\epsilon}(\Omega))$, with $\epsilon > 0$ being arbitrarily small. With this regularity result at hand, the techniques developed in the proof of Theorems 5.8 and 6.1 yield the improved error estimates

$$\|\bar{z} - \bar{Z}_{\mathcal{T}}^\tau\|_{L^2(Q)} \lesssim \tau + h_{\mathcal{T}}^{1/2+\gamma}, \quad \gamma = \min\{s, 1/2 - \epsilon\}, \quad \epsilon > 0, \quad (7.2)$$

$$\|\bar{u} - \bar{U}_{\mathcal{T}}^\tau\|_{L^2(Q)} \lesssim \tau + h_{\mathcal{T}}^{1/2+\gamma}. \quad (7.3)$$

Notice that the error estimates obtained in Theorems 5.8 and 6.1 rely on the *general* regularity estimates of Theorem 3.2.

7.2. Implementation details. In what follows, we employ the panel clustering approach described in [3] to obtain a sparse approximation of the integral fractional Laplacian $(-\Delta)^s$. For the minimization problem we use the BFGS algorithm [28]. The linear systems of equations arising from the fully discrete scheme from section 6.1 are solved using conjugate gradient preconditioned by geometric multigrid.

The $L^2(Q)$ -error of approximating the variable \bar{w} with the discrete function $\bar{W}_{\mathcal{T}}^\tau$ is approximated as follows:

$$\begin{aligned}\|\bar{w} - \bar{W}_{\mathcal{T}}^\tau\|_{L^2(Q)}^2 &= \int_0^T \int_\Omega \bar{w}^2 - 2\bar{w}\bar{W}_{\mathcal{T}}^\tau + (\bar{W}_{\mathcal{T}}^\tau)^2 \, dx \, dt \\ &\approx \int_0^T \int_\Omega \bar{w}^2 \, dx \, dt + \sum_{k=0}^{\mathcal{K}} \bar{W}_{\mathcal{T}}^\tau (\bar{W}_{\mathcal{T}}^\tau - 2\bar{w}).\end{aligned}$$

The first term can be evaluated analytically.

7.3. Examples in 1D. We set $\Omega = (0, 1) \subset \mathbb{R}$, $T = 1$, $a = -0.5$, $b = 0.5$, $\mu = 0.1$, $u = u_0^{1D}$, $v = u_0^{1D}$, $\psi(t) = \cos(t)$, and $\phi(t) = \sin(T - t)$. The exact solution is then given by

$$\begin{aligned}\bar{u}(t, x) &= \cos(t)u_0^{1D}(x), \\ \bar{p}(t, x) &= -\mu \sin(T - t)u_0^{1D}(x), \\ \bar{z}(t, x) &= \text{proj}_{[a,b]}(\sin(T - t)u_0^{1D}(x)) = \begin{cases} b & \text{if } |x| < r_o(t), \\ (1 - x^2)^s & \text{if } |x| \geq r_o(t), \end{cases}\end{aligned}$$

where

$$r_o(t) = \begin{cases} 0 & \text{if } \sin(T - t) < b, \\ \sqrt{1 - \left(\frac{b}{\sin(T-t)}\right)^{1/s}} & \text{if } \sin(T - t) \geq b. \end{cases}$$

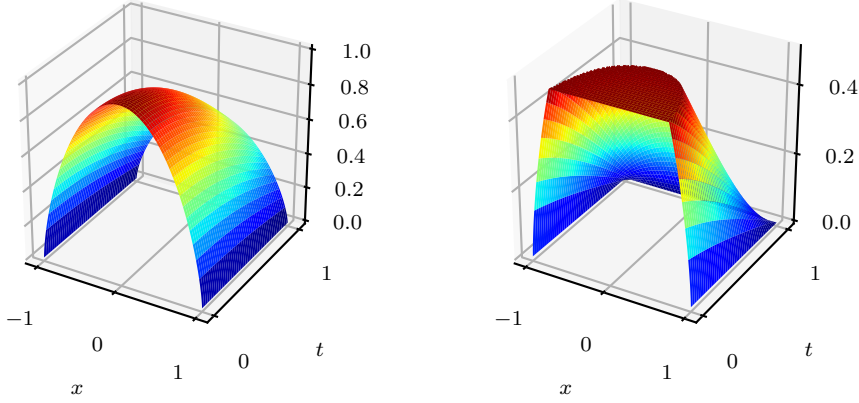


FIG. 7.1. 1D example: Finite element solutions for the optimal state $\bar{U}_{\mathcal{T}}^{\tau}$ (left) and the optimal control $\bar{Z}_{\mathcal{T}}^{\tau}$ (right) for $s = 0.7$. We notice that the upper bound on the control is active.

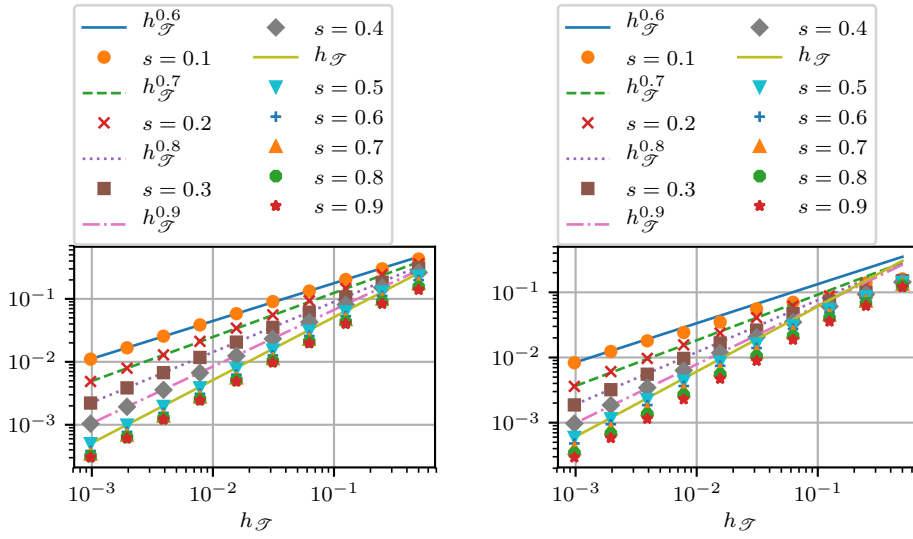


FIG. 7.2. 1D example: Experimental rates of convergence for $\|\bar{u} - \bar{U}_{\mathcal{T}}^{\tau}\|_{L^2(Q)}$ (left) and $\|\bar{z} - \bar{Z}_{\mathcal{T}}^{\tau}\|_{L^2(Q)}$ (right) for $\Omega = (0, 1)$ and $s \in \{0.1, 0.2, \dots, 0.9\}$.

We also set $\tau = h_{\mathcal{T}}^{1/2+\gamma}$, where $\gamma = \min\{s, 1/2 - \epsilon\}$ and $\epsilon > 0$.

In Figure 7.1 we present the finite element solutions for the optimal state and control, on a suitable mesh, for $s = 0.7$. We observe that the constraint $\mathbf{b} = 0.5$ is indeed active for $|x| < r_o(t)$. In Figure 7.2 we display the experimental rates of convergence for the $L^2(Q)$ -errors of the state and control variables. We consider different values for the fractional order $s \in \{0.1, 0.2, \dots, 0.9\}$ and different mesh sizes $h_{\mathcal{T}}$. We observe the predicted rate of convergence (7.2) for the error approximation of the control variable.

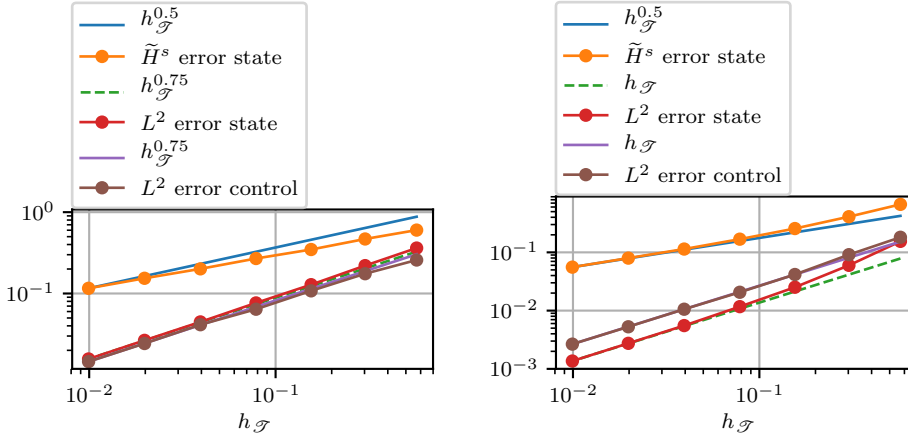


FIG. 7.3. 2D example: Experimental rates of convergence for $\|\bar{u} - \bar{U}_{\mathcal{T}}^{\tau}\|_{L^2((0,T),H^s(\Omega))}$, $\|\bar{u} - \bar{U}_{\mathcal{T}}^{\tau}\|_{L^2(Q)}$, and $\|\bar{z} - \bar{Z}_{\mathcal{T}}^{\tau}\|_{L^2(Q)}$ for $s = 0.25$ (left) and $s = 0.75$ (right) on quasi-uniform meshes.

7.4. Examples in 2D. We set $\Omega = B(0,1) \subset \mathbb{R}^2$, $T = 1$, $\mathbf{a} = -0.5$, $\mathbf{b} = 0.5$, $\mu = 0.1$, $u = u_{0,1}^{2D}$, $v = u_{0,0}^{2D}$, $\psi(t) = \cos(t)$, and $\phi(t) = \sin(T-t)$. The exact solution is then given by

$$\begin{aligned}\bar{u}(t, x) &= \cos(t)u_{0,1}^{2D}(x), \\ \bar{p}(t, x) &= -\mu \sin(T-t)u_{0,0}^{2D}(x), \\ \bar{z}(t, x) &= \text{proj}_{[a,b]}(\sin(T-t)u_{0,0}^{2D}(x)).\end{aligned}$$

7.4.1. Quasi-uniform mesh. We use quasi-uniform meshes with mesh size $h_{\mathcal{T}}$ and a time step of size $\tau = h_{\mathcal{T}}^{1/2+\gamma}$, where $\gamma = \min\{s, 1/2 - \epsilon\}$ and $\epsilon > 0$.

In Figure 7.3 we present, for $s = 0.25$ and $s = 0.75$, the experimental rates of convergence for the $L^2(Q)$ -errors of the state and control variable as well as the $L^2((0,T), H^s(\Omega))$ -error of the state variable. We observe the predicted rates of convergence for the $L^2(Q)$ -error of the control variable.

7.4.2. Graded mesh. In this section we explore the computational performance of the fully discrete scheme, that we have devised, when is used to approximate the solution of the fractional optimal control problem on the basis of finite elements over graded meshes on Ω . When $s \in (1/2, 1)$ and $n = 2$, the singular behavior of the solution to the elliptic fractional Poisson problem (7.1) can be compensated by using a priori adapted meshes; see [2]. These graded meshes allow for an improvement on the priori error estimate obtained in the resolution of the elliptic problem by using quasiuniform meshes and are constructed as follows. In addition to shape regularity, we assume that the meshes \mathcal{T} have the following property: Given a mesh parameter $h > 0$ and $\kappa \in [1, 2]$ every element $T \in \mathcal{T}$ satisfies

$$h_T \approx C(\sigma)h^{\kappa} \text{ if } T \cap \partial\Omega \neq \emptyset, \quad h_T \approx C(\sigma)h \text{dist}(T, \partial\Omega)^{(\kappa-1)/\kappa} \text{ if } T \cap \partial\Omega = \emptyset, \quad (7.4)$$

where $C(\sigma)$ depends only on the shape regularity constant σ of the mesh \mathcal{T} . κ relates the mesh parameter h to the number of degrees of freedom, N , as follows:

$$N \approx h_{\mathcal{T}}^{-2} \text{ if } \kappa \in (1, 2), \quad N \approx h_{\mathcal{T}}^{-2} |\log h_{\mathcal{T}}| \text{ if } \kappa = 2. \quad (7.5)$$

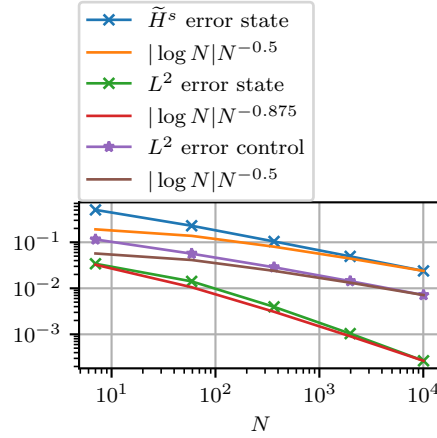


FIG. 7.4. 2D example on graded meshes: Experimental rates of convergence for $\|\bar{u} - \bar{U}_{\mathcal{T}}^{\tau}\|_{L^2(Q)}$, $\|\bar{u} - \bar{U}_{\mathcal{T}}^{\tau}\|_{L^2((0,T),H^s(\Omega))}$, and $\|\bar{z} - \bar{Z}_{\mathcal{T}}^{\tau}\|_{L^2(Q)}$ for $s = 0.75$.

The optimal choice is $\kappa = 2$.

In Figure 7.4 we present the experimental rates of convergence for $\|\bar{u} - \bar{U}_{\mathcal{T}}^{\tau}\|_{L^2(Q)}$, $\|\bar{u} - \bar{U}_{\mathcal{T}}^{\tau}\|_{L^2((0,T),H^s(\Omega))}$ and $\|\bar{z} - \bar{Z}_{\mathcal{T}}^{\tau}\|_{L^2(Q)}$ obtained by using graded meshes on Ω with grading parameter $\kappa = 2$. We observe improved rates of convergence for the error approximation of the state variable in both $L^2(Q)$ - and $L^2((0,T),H^s(\Omega))$ -norms. We note that this setting is not covered by the analysis developed in the previous sections; the main missing ingredient being regularity estimates for the solution of (1.3) over bounded and Lipschitz domains $\Omega \times (0,T)$.

8. Conclusions. We have analyzed a control-constrained linear-quadratic optimal control problem for the fractional heat equation and derived existence and uniqueness results, first order optimality conditions, and regularity estimates for the optimal variables. We have proposed a fully discrete scheme to discretize the state equation equation that relies on an implicit finite difference discretization in time combined with a piecewise linear finite element discretization in space. We have derived stability results and a $L^2(0,T;L^2(\Omega))$ a priori error estimate. Furthermore, we have proposed a fully discrete scheme for the optimal control problem that discretizes the control variable with piecewise constant functions, and derived a priori error estimates for it. Finally, we have illustrated the theory with one- and two-dimensional numerical experiments.

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