SEMILINEAR OPTIMAL CONTROL WITH DIRAC MEASURES*

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Abstract. The purpose of this work is to analyze an optimal control problem for a semilinear elliptic partial differential equation (PDE) involving Dirac measures; the control variable corresponds to the amplitude of forces modeled as point sources. We analyze the existence of optimal solutions and derive first and, necessary and sufficient, second order optimality conditions. We devise a solution technique that discretizes the state and adjoint equations with continuous piecewise linear finite elements; the control variable is already discrete. We analyze convergence properties of discretizations and obtain an a priori error estimate for the underlying approximation of an optimal control variable.

Key words. optimal control problems, semilinear elliptic PDEs, Dirac measures, second-order optimality conditions, finite element approximations, a priori error estimates.

AMS subject classifications. 35J61, 35R06, 49J20, 49M25, 65N15, 65N30.

1. Introduction. In this work we are interested in the analysis and discretization of an optimal control problem for a semilinear elliptic partial differential equation (PDE) involving Dirac measures or point sources. To make matters precise, for $d \in \{2,3\}$, we let $\Omega \subset \mathbb{R}^d$ be an open, bounded, and convex polytope with boundary $\partial\Omega$ and $\mathcal{D}$ be a finite ordered subset of Ω with cardinality $\ell := \#\mathcal{D} < \infty$. Given a desired state $\mathsf{y}_d \in L^2(\Omega)$, a regularization parameter $\alpha > 0$, and the cost functional

$$J(\mathsf{y},\mathbf{u}) := \frac{1}{2}\|\mathsf{y}-\mathsf{y}_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\mathbf{u}\|^2_{\mathbb{R}^\ell}, \tag{1.1}$$

we shall be concerned with the following PDE-constrained optimization problem: Find $\min J(\mathsf{y},\mathbf{u})$ subject to the *monotone, semilinear, and elliptic PDE*

$$-\Delta\mathsf{y} + a(\cdot,\mathsf{y}) = \sum_{\mathsf{z}\in\mathcal{D}} \mathsf{u}_\mathsf{z}\delta_\mathsf{z} \text{ in } \Omega, \qquad \mathsf{y} = 0 \text{ on } \partial\Omega, \tag{1.2}$$

where δ_z corresponds to the Dirac delta supported at the interior point $\mathsf{z} \in \mathcal{D}$, and

$$\mathbf{u} = \{\mathsf{u}_\mathsf{z}\}_{\mathsf{z}\in\mathcal{D}} \in \mathbb{R}^\ell, \qquad \mathsf{a}_\mathsf{z} \leq \mathsf{u}_\mathsf{z} \leq \mathsf{b}_\mathsf{z} \ \forall \mathsf{z} \in \mathcal{D}. \tag{1.3}$$

Here, $\mathbf{u}$ denotes the control variable. The control bounds $\mathbf{a} = \{\mathsf{a}_\mathsf{z}\}_{\mathsf{z}\in\mathcal{D}}$ and $\mathbf{b} = \{\mathsf{b}_\mathsf{z}\}_{\mathsf{z}\in\mathcal{D}}$ both belong to $\mathbb{R}^\ell$ and satisfy that $\mathsf{a}_\mathsf{z} < \mathsf{b}_\mathsf{z}$ for all $\mathsf{z} \in \mathcal{D}$. Assumptions on the nonlinear function a will be deferred until Section 2.1.

PDE-constrained optimization problems involving measures have previously been considered in a number of works. In particular, we mention the work by [7] where the authors consider an optimal control problem where the state variable is governed by the semilinear elliptic equation (1.2) but with a control variable u that is measure valued: $u \in \mathcal{M}(\omega)$, where $\omega \subset \Omega$. The main motivation behind this consideration is what the authors denote *sparsity promoting properties* of the control variable. Within this setting, a complete analysis for the state equation and the optimization problem is provided. In particular, first and second order optimality conditions are derived. The work by [7], however, is not concerned with approximation.

For the particular linear case $a \equiv 0$, there are a few works that consider the numerical approximation of the previously introduced optimal control problem. In

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[17], the authors consider a discretization technique based on the so-called variational discretization approach and perform an a priori error analysis [17, Theorem 3.7]. The authors operate under the fact that the optimal state belongs to $W_0^{1,r}(\Omega)$ for $r < d/(d-1)$. In contrast, an approach involving Muckenhoupt weights, weighted Sobolev spaces, and the corresponding weighted norm inequalities has been explored in [1]. In such a work, the authors obtain the following rates of convergence for the error approximation of the optimal control variable: $\mathcal{O}(h^{2-\epsilon})$ in two dimensions and $\mathcal{O}(h^{1-\epsilon})$ in three dimensions, where $\epsilon > 0$ is arbitrarily small [1, Theorem 5.1]; the error estimate in two dimensions being improved in Theorem 5.8 below to $\mathcal{O}(h^2|\log h|^3)$. The extension of such weighted techniques for when the state equations are the Stokes equations has been recently considered in [15]. Related optimal control problems within the context of the active control of sound and vibrations have been studied in [3] and [18], respectively. Finally, we mention references [6], [21], and [16], where the authors study discretization techniques for a PDE-constrained optimization problem without control constraints, but where the control is a regular Borel measure.

In addition to this exposition being the first one that studies a numerical scheme for the semilinear optimal control problem min $J(\mathsf{y}, \mathbf{u})$ subject to (1.2) and (1.3), the analysis itself comes with its own set of difficulties. In what follows, we list, what we believe are, the main contributions of our work:

(i) *Error estimates for semilinear PDEs with Dirac measures:* For a basic finite element discretization of (1.2), we obtain an L^2-error estimate that is optimal in terms of regularity (Theorem 3.1). We also derive a nearly–optimal, in terms of approximation, L^1-error estimate (Theorem 5.3).

(ii) *Existence of an optimal control:* We show that our optimal control problem admits at least one global solution: see Theorem 4.1.

(iii) *Optimality conditions:* We derive necessary first and second order optimality conditions. We also analyze a sufficient second order optimality condition with a minimal gap with respect to the necessary one.

(iv) *Convergence of discretizations:* We prove that global solutions of discrete optimal control problems converge to a global solution of the continuous one and that strict local continuous solutions can be approximated by local discrete solutions; see Theorems 5.5 and 5.6, respectively.

(v) *Error estimates:* We provide, in Theorem 5.8, an a priori error analysis for the underlying approximation of an optimal control variable; the error estimate in two dimensions being nearly–optimal in terms of approximation.

The outline of this manuscript is as follows. In Section 2 we introduce the notation and functional framework we shall work with. We also briefly review basic results for semilinear elliptic PDEs with singular forcing and derive, in Section 3, an L^2-error estimate for a standard finite element approximation. Section 4 is dedicated to the analysis of the semilinear optimal control problem. In particular, we derive first and, necessary and sufficient, second order optimality conditions. We analyze, in Section 5, a suitable finite element discretization for the semilinear optimal control problem, derive convergence results, and obtain a priori error estimates.

2. Notation, assumptions, and preliminaries.

Let us fix notation and the setting in which we will operate. Throughout this work, $d \in \{2, 3\}$ and Ω is an open, bounded, and convex polytopal domain of $\mathbb{R}^d$. We denote by $\partial\Omega$ the boundary of Ω.

If $\mathscr{X}$ and $\mathscr{Y}$ are Banach function spaces, we write $\mathscr{X} \hookrightarrow \mathscr{Y}$ to denote that $\mathscr{X}$ is continuously embedded in $\mathscr{Y}$. We denote by $\mathscr{X}'$ and $\|\cdot\|_{\mathscr{X}}$ the dual and the norm of $\mathscr{X}$, respectively. We denote by $\langle \cdot, \cdot \rangle_{\mathscr{X}', \mathscr{X}}$ the duality paring between $\mathscr{X}'$ and $\mathscr{X}$; we shall simply denote $\langle \cdot, \cdot \rangle$ whenever the underlying spaces are clear from the context. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{X}$. We will denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong and weak convergence, respectively, of $\{x_n\}_{n\in\mathbb{N}}$ to x. We will use standard notation for Sobolev spaces, norms, and seminorms. We denote by $\mathcal{M}(\Omega)$ the space of Radon measures on Ω – the space of regular Borel measures μ which are such that $\mu(K) < \infty$ for every compact set $K \subset \mathbb{R}^n$ [13] – and recall that $\mathcal{M}(\Omega)$ can be identified with $C_0(\Omega)'$ – the dual of the space of continuous functions on $\bar{\Omega}$ vanishing on $\partial\Omega$ [14, Theorem 7.17].

Given $s \in (1, \infty)$, we denote by s' its Hölder conjugate, i.e., the real number such that $1/s + 1/s' = 1$. The relation $A \lesssim B$ indicates that $A \leq CB$, with a positive constant which is independent of A, B, and the underlying discretization parameters. The value of C might change at each occurrence.

2.1. Assumptions. To analyze the optimal control problem introduced in Section 1, we will assume that a is such that (A.1)–(A.4) hold; see [7]. We must, however, immediately mention that some of the results obtained in this work are valid under less restrictive requirements. When possible we will explicitly mention the assumptions on the nonlinear term a that are needed to obtain a particular result.

(A.1) $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that is monotone increasing in y for a.e. x in Ω and satisfies the growth condition

$$|a(x, y)| \leq |\phi_0(x)| + C_a|y|^r \quad \text{a.e. } x \in \Omega, \ \forall y \in \mathbb{R}. \tag{2.1}$$

Here, $C_a > 0$ is a constant, $\phi_0 \in L^1(\Omega)$, $r < \infty$ if $d = 2$, and $r < 3$ if $d = 3$.

(A.2) $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class C^1 with respect to y for a.e. x in Ω and there exists $\phi_1 \in L^q(\Omega)$, with $q > d/2$, such that

$$0 \leq \frac{\partial a}{\partial y}(x, y) \leq |\phi_1(x)| + C_a|y|^r \quad \text{a.e. } x \in \Omega, \ \forall y \in \mathbb{R}. \tag{2.2}$$

Here, $C_a > 0$ is a constant, $r < \infty$ if $d = 2$, and $r < 2$ if $d = 3$.

(A.3) $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class C^1 with respect to y for a.e. x in Ω and there exists $\phi_1 \in L^q(\Omega)$ such that

$$0 \leq \frac{\partial a}{\partial y}(x, y) \leq |\phi_1(x)| + C_a|y|^r \quad \text{a.e. } x \in \Omega, \ \forall y \in \mathbb{R}. \tag{2.3}$$

Here, $r < \infty$ and $q > 2$ if $d = 2$, $r < 1$ and $q = 3$ if $d = 3$, and $C_a > 0$.

(A.4) $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class C^2 with respect to y for a.e. x in Ω and there exists $\phi_2 \in L^t(\Omega)$ such that

$$\left|\frac{\partial^2 a}{\partial y^2}(x, y)\right| \leq |\phi_2(x)| + C_a|y|^r \quad \text{a.e. } x \in \Omega, \ \forall y \in \mathbb{R}, \tag{2.4}$$

where $t > 1$ if $d = 2$ and $t > 3$ if $d = 3$, $C_a > 0$ is a constant, $r < \infty$ if $d = 2$, and $r < 1$ if $d = 3$.

Further assumptions on a that will be particularly needed for performing an error analysis of a suitable finite element discretization will be deferred until Section 5.

2.2. Semilinear PDEs with singular forcing. Let $x_0 \in \Omega$ and $a : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that (A.1) holds. We consider the following semilinear elliptic PDE: Find y such that

$$-\Delta y + a(\cdot, y) = \delta_{x_0} \text{ in } \Omega, \qquad y = 0 \text{ on } \partial\Omega. \tag{2.5}$$

The following notion of weak solution follows from [7, Section 2]: $y \in L^1(\Omega)$ is a weak solution for problem (2.5) if $a(\cdot, y) \in L^1(\Omega)$ and

$$\int_\Omega [-y\Delta v + a(x,y)v]\, \mathrm{d}x = v(x_0) \qquad \forall v \in \mathcal{Z}(\Omega). \tag{2.6}$$

Here, $\mathcal{Z}(\Omega) := \{v \in H_0^1(\Omega) : \Delta v \in C(\bar{\Omega})\}$. We immediately notice that $\mathcal{Z}(\Omega) \subset C_0(\Omega)$ and observe that this, combined with the definition of $\mathcal{Z}(\Omega)$, imply that all the terms involved in the weak formulation (2.6) are well defined.

The following result states the well-posedness of problem (2.5) and further regularity properties for the solution y [7, Theorem 2.1].

THEOREM 2.1 (well-posedness and regularity). *Let $x_0 \in \Omega$ and let $a : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that (A.1) holds. If Ω denotes an open and bounded domain with Lipschitz boundary, then problem (2.6) admits a unique solution y. In addition, we have $y \in W_0^{1,p}(\Omega)$ and*

$$\|\nabla y\|_{L^p(\Omega)} \lesssim \|a(\cdot, 0)\|_{L^1(\Omega)} + \|\delta_{x_0}\|_{\mathcal{M}(\Omega)}, \qquad p < d/(d-1). \tag{2.7}$$

The hidden constant is independent of y, a, δ_{x_0}, and x_0.

The following remark is in order.

Remark 2.2 (variational formulation). Since the solution y to problem (2.6) belongs to $W_0^{1,p}(\Omega)$, for every $p < d/(d-1)$, the following alternative weak formulation for problem (2.5) can be formulated; see [7, Remark 2.3] for details: Find $y \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \nabla y \cdot \nabla v \mathrm{d}x + \int_\Omega a(x,y) v \mathrm{d}x = v(x_0) \quad \forall v \in W_0^{1,p'}(\Omega). \tag{2.8}$$

Here, $p' > d$ denotes the Hölder conjugate of p. Notice that, within the considered functional space setting, all the terms involved in the weak formulation (2.6) are well defined.

3. Approximation of semilinear PDEs with singular forcing.

We begin this section by introducing the discrete setting in which we will operate. We denote by $\mathscr{T}_h = \{T\}$ a conforming partition, or mesh, of $\bar{\Omega}$ into closed simplices T, with size $h_T = \mathrm{diam}(T)$, and define $h := \max_{T \in \mathscr{T}_h} h_T$. We denote by $\mathbb{T} = \{\mathscr{T}_h\}_{h>0}$ a collection of conforming and quasi-uniform meshes $\mathscr{T}_h$, which are refinements of a common mesh $\mathscr{T}_\star$. Given a mesh $\mathscr{T}_h \in \mathbb{T}$, we define the finite element space of continuous piecewise polynomials of degree one as

$$\mathbb{V}_h := \{v_h \in C(\bar{\Omega}) : v_h|_T \in \mathbb{P}_1(T) \ \forall T \in \mathscr{T}_h\} \cap H_0^1(\Omega). \tag{3.1}$$

We define the Galerkin approximation of the solution y to problem (2.8) by

$$y_h \in \mathbb{V}_h : \quad \int_\Omega \nabla y_h \cdot \nabla v_h \mathrm{d}x + \int_\Omega a(x, y_h) v_h \mathrm{d}x = v_h(x_0) \quad \forall v_h \in \mathbb{V}_h. \tag{3.2}$$

We provide an error estimate in $L^2(\Omega)$ that is optimal in terms of regularity. This estimate, which is of independent interest, extends the linear theory developed in [5] to a semilinear scenario.

THEOREM 3.1 (error estimate). *Let Ω be an open, bounded, and convex polytopal domain. Let $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that is monotone increasing in y a.e. in Ω and satisfies (2.1). Assume, in addition, that*

$$|a(x,y) - a(x,z)| \leq |\psi(x)||y - z| \text{ a.e. } x \in \Omega,\ \forall y, z \in \mathbb{R}, \quad \psi \in L^s(\Omega), \tag{3.3}$$

with $s > d$. Let $y \in W_0^{1,p}(\Omega)$ be the solution to (2.8) and let $y_h \in \mathbb{V}_h$ be its finite element approximation given as in (3.2); $p < d/(d-1)$. If h is sufficiently small, we have the optimal error estimate

$$\|y - y_h\|_{L^2(\Omega)} \lesssim h^{2-\frac{d}{2}}, \tag{3.4}$$

with a hidden constant that is independent of y, y_h, and h.

Proof. We proceed on the basis of a duality argument. We begin the proof by introducing the nonnegative function $\chi \in L^s(\Omega)$ as

$$\chi(x) = \frac{a(x, y(x)) - a(x, y_h(x))}{y(x) - y_h(x)} \text{ if } y(x) \neq y_h(x), \quad \chi(x) = 0 \text{ if } y(x) = y_h(x).$$

Let $\mathfrak{f} \in L^2(\Omega)$. Let $\mathfrak{z}$ be the solution to $(\nabla v, \nabla \mathfrak{z})_{L^2(\Omega)} + (\chi\mathfrak{z}, v)_{L^2(\Omega)} = (\mathfrak{f}, v)_{L^2(\Omega)}$, for all $v \in H_0^1(\Omega)$. Let $\mathfrak{z}_h$ be the finite element approximation of $\mathfrak{z}$ within $\mathbb{V}_h$. Since $\chi\mathfrak{z}, \mathfrak{f} \in L^2(\Omega)$, we have $\mathfrak{z} \in H^2(\Omega) \cap H_0^1(\Omega)$. The fact that $d \in \{2, 3\}$ thus yields the existence of $r > d$ such that $\mathfrak{z} \in W_0^{1,r}(\Omega)$. Set $\mathfrak{f} = y - y_h \in L^2(\Omega)$. A density argument allows us to set $v = y - y_h$ in the problem that $\mathfrak{z}$ solves and obtain

$$\begin{aligned} \|y - y_h\|_{L^2(\Omega)}^2 &= (\nabla(y - y_h), \nabla \mathfrak{z})_{L^2(\Omega)} + (\chi\mathfrak{z}, y - y_h)_{L^2(\Omega)} \\ &= (\nabla(y - y_h), \nabla(\mathfrak{z} - \mathfrak{z}_h))_{L^2(\Omega)} + (a(\cdot, y) - a(\cdot, y_h), \mathfrak{z} - \mathfrak{z}_h)_{L^2(\Omega)}. \end{aligned}$$

We stress that, since $y - y_h \in W_0^{1,p}(\Omega)$, for every $p < d/(d-1)$, and there exists $r > d$ such that $\mathfrak{z} \in W_0^{1,r}(\Omega)$, the term $(\nabla(y - y_h), \nabla\mathfrak{z})_{L^2(\Omega)}$ is well defined. We now invoke the fact that y solves problem (2.8) and, again, the existence of $r > d$ such that $\mathfrak{z} \in W_0^{1,r}(\Omega) \hookrightarrow C(\bar{\Omega})$ to obtain

$$\begin{aligned} \|y - y_h\|_{L^2(\Omega)}^2 &= (\nabla y, \nabla(\mathfrak{z} - \mathfrak{z}_h))_{L^2(\Omega)} + (a(\cdot, y), \mathfrak{z} - \mathfrak{z}_h) - (\nabla y_h, \nabla(\mathfrak{z} - \mathfrak{z}_h))_{L^2(\Omega)} \\ &\quad - (a(\cdot, y_h), \mathfrak{z} - \mathfrak{z}_h) = \mathfrak{z}(x_0) - \mathfrak{z}_h(x_0) - (\nabla y_h, \nabla(\mathfrak{z} - \mathfrak{z}_h))_{L^2(\Omega)} - (a(\cdot, y_h), \mathfrak{z} - \mathfrak{z}_h). \end{aligned}$$

Observe that $\mathfrak{z} - \mathfrak{z}_h$ satisfies

$$(\nabla v_h, \nabla(\mathfrak{z} - \mathfrak{z}_h))_{L^2(\Omega)} + (\chi(\mathfrak{z} - \mathfrak{z}_h), v_h)_{L^2(\Omega)} = 0 \quad \forall v_h \in \mathbb{V}_h.$$

We thus set $v_h = y_h$ in the previous relation and obtain

$$\|y - y_h\|_{L^2(\Omega)}^2 = \mathfrak{z}(x_0) - \mathfrak{z}_h(x_0) + (\chi(\mathfrak{z} - \mathfrak{z}_h), y_h)_{L^2(\Omega)} - (a(\cdot, y_h), \mathfrak{z} - \mathfrak{z}_h).$$

A basic error estimate yields $\|\mathfrak{z} - \mathfrak{z}_h\|_{L^\infty(\Omega)} \lesssim h^\sigma \|\mathfrak{f}\|_{L^2(\Omega)} = h^\sigma \|y - y_h\|_{L^2(\Omega)}$, where $\sigma = 2 - d/2$. This bound implies the estimate

$$\|y - y_h\|_{L^2(\Omega)}^2 \lesssim h^\sigma \left(1 + \|\chi y_h\|_{L^1(\Omega)} + \|a(\cdot, y_h)\|_{L^1(\Omega)}\right) \|y - y_h\|_{L^2(\Omega)}. \tag{3.5}$$

Invoke (2.1) to obtain $\|a(\cdot, y_h)\|_{L^1(\Omega)} \leq \|\phi_0\|_{L^1(\Omega)} + C_a\|y_h\|_{L^r(\Omega)} \lesssim \|\phi_0\|_{L^1(\Omega)} + C_a\|\nabla y_h\|_{L^p(\Omega)}$, where r is as in (2.1) and $p < d/(d-1)$. The control of the term $\|\chi y_h\|_{L^1(\Omega)}$ follows from Hölder and Poincaré inequalities. In fact, we have the bound

$\|\chi y_h\|_{L^1(\Omega)} \lesssim \|\chi\|_{L^s(\Omega)}\|\nabla y_h\|_{L^p(\Omega)}$. We recall that $s > d$. It thus suffices to control $\|\nabla y_h\|_{L^p(\Omega)}$. To accomplish this task, we invoke a discrete inf-sup condition, that follows from [4, Theorem 8.5.3] and holds for h sufficiently small, and use that

$$(\nabla(y - y_h), \nabla v_h)_{L^2(\Omega)} + (a(\cdot, y) - a(\cdot, y_h), v_h)_{L^2(\Omega)} = 0 \qquad \forall v_h \in \mathbb{V}_h,$$

to arrive at

$$\|\nabla y_h\|_{L^p(\Omega)} \lesssim \sup_{v_h \in \mathbb{V}_h} \frac{(\nabla y_h, \nabla v_h)_{L^2(\Omega)}}{\|\nabla v_h\|_{L^{p'}(\Omega)}} \lesssim \sup_{v_h \in \mathbb{V}_h} \left[\frac{\|\nabla y\|_{L^p(\Omega)}\|\nabla v_h\|_{L^{p'}(\Omega)}}{\|\nabla v_h\|_{L^{p'}(\Omega)}} + \frac{\|a(\cdot, y) - a(\cdot, y_h)\|_{L^1(\Omega)}\|v_h\|_{L^\infty(\Omega)}}{\|\nabla v_h\|_{L^{p'}(\Omega)}} \right].$$

We now utilize the Sobolev embedding $W^{1,p'}(\Omega) \hookrightarrow L^\infty(\Omega)$ ($p' > d$) and the Lipschitz property (3.3) of $a(\cdot, y)$ to obtain $\|\nabla y_h\|_{L^p(\Omega)} \lesssim \|\nabla y\|_{L^p(\Omega)} + \|y - y_h\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}$. The stability estimate (2.7) allows us to conclude the bound

$$\|\nabla y_h\|_{L^p(\Omega)} \lesssim \|a(\cdot, 0)\|_{L^1(\Omega)} + \|\delta_{x_0}\|_{\mathcal{M}(\Omega)} + \|y - y_h\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}. \tag{3.6}$$

Replace this estimate, and the obtained ones for $\|\chi y_h\|_{L^1(\Omega)}$ and $\|a(\cdot, y_h)\|_{L^1(\Omega)}$, into (3.5) and utilize the assumption that h is sufficiently small so that the term $\|y - y_h\|_{L^2(\Omega)}$ in (3.6) can be absorbed in the left hand side of (3.5). These arguments yield the desired estimate (3.4) and conclude the proof. ∎

4. The semilinear optimal control problem with singular sources. Let us precisely describe and analyze the optimal control problem with point sources introduced in Section 1. We begin our studies by defining the set of admissible controls

$$\mathbb{U}_{ad} := \left\{ \mathbf{u} = \{\mathsf{u}_\mathsf{z}\}_{\mathsf{z}\in\mathcal{D}} \in \mathbb{R}^\ell : \ \mathsf{a}_\mathsf{z} \leq \mathsf{u}_\mathsf{z} \leq \mathsf{b}_\mathsf{z} \ \forall \mathsf{z} \in \mathcal{D} \right\}, \tag{4.1}$$

where the bounds $\mathbf{a} = \{\mathsf{a}_\mathsf{z}\}_{\mathsf{z}\in\mathcal{D}}$ and $\mathbf{b} = \{\mathsf{b}_\mathsf{z}\}_{\mathsf{z}\in\mathcal{D}}$ both belong to $\mathbb{R}^\ell$ and satisfy that $\mathsf{a}_\mathsf{z} < \mathsf{b}_\mathsf{z}$ for all $\mathsf{z} \in \mathcal{D}$. Notice that $\mathbb{U}_{ad} \neq \emptyset$ is a closed and bounded subset of $\mathbb{R}^\ell$. We recall that $\mathcal{D}$ denotes a finite ordered subset of Ω with cardinality $\#\mathcal{D} = \ell$.

We define the *semilinear optimal control problem with point sources* as follows: Find

$$\min\{J(\mathsf{y}, \mathbf{u}) : (\mathsf{y}, \mathbf{u}) \in W_0^{1,p}(\Omega) \times \mathbb{U}_{ad}\} \tag{4.2}$$

subject to the following weak formulation of the state equation: Find $\mathsf{y} \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \nabla \mathsf{y} \cdot \nabla v \mathrm{d}x + \int_\Omega a(x, \mathsf{y}) v \mathrm{d}x = \sum_{\mathsf{z}\in\mathcal{D}} \mathsf{u}_\mathsf{z} v(\mathsf{z}) \qquad \forall v \in W_0^{1,p'}(\Omega). \tag{4.3}$$

Here, $p < d/(d-1)$ and $p' > d$ denotes the Hölder conjugate of p. We assume that $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that is monotone increasing in y for a.e. x in Ω and satisfies (2.1). In view of this assumption, an application of Theorem 2.1 immediately yields the existence of a unique $\mathsf{y} \in W_0^{1,p}(\Omega)$ solving (4.3).

To analyze the previously defined control problem, we introduce $\mathcal{S} : \mathbb{U}_{ad} \to W_0^{1,p}(\Omega)$, the so-called control to state map, which, given a control $\mathbf{u}$, associates to it the unique state y that solves (4.3). Notice that the map $\mathcal{S}$ is bounded. In fact, Theorem 2.1 immediately yields the bound

$$\|\nabla(\mathcal{S}\mathbf{u})\|_{L^p(\Omega)} \lesssim \|a(\cdot,0)\|_{L^1(\Omega)} + \sum_{\mathsf{z}\in\mathcal{D}} |\mathsf{u}_\mathsf{z}|\|\delta_\mathsf{z}\|_{\mathcal{M}(\Omega)}, \qquad p < d/(d-1).$$

With $\mathcal{S}$ at hand, we also introduce the reduced cost functional $j : \mathbb{U}_{ad} \to \mathbb{R}$ by $j(\mathbf{u}) := J(\mathcal{S}\mathbf{u}, \mathbf{u})$.

4.1. Existence of optimal controls. The existence of an optimal state-control pair $(\bar{\mathsf{y}}, \bar{\mathbf{u}})$ is as follows.

THEOREM 4.1 (existence of an optimal pair). *Let Ω be an open and bounded domain with Lipschitz boundary. Let $a = a(x,y) : \Omega\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function that is monotone increasing in y for a.e. x in Ω and satisfies (2.1). Thus, problem (4.2)–(4.3) admits at least one global solution $(\bar{\mathsf{y}}, \bar{\mathbf{u}}) \in W_0^{1,p}(\Omega) \times \mathbb{U}_{ad}$.*

Proof. Let $\{(\mathsf{y}_k, \mathbf{u}_k)\}_{k\in\mathbb{N}}$ be a minimizing sequence, i.e., for $k \in \mathbb{N}$, $\mathbf{u}_k \in \mathbb{U}_{ad}$ and $\mathsf{y}_k = \mathcal{S}\mathbf{u}_k$ are such that $J(\mathsf{y}_k, \mathbf{u}_k) \to \mathsf{j} := \{\inf J(\mathsf{y}(\mathbf{u}), \mathbf{u}) : \mathbf{u} \in \mathbb{U}_{ad}\}$ as $k \uparrow \infty$. For $k \in \mathbb{N}$, we denote $\mathbf{u}_k = \{\mathsf{u}_\mathsf{z}^k\}_{\mathsf{z}\in\mathcal{D}}$. Observe that y_k is such that

$$\mathsf{y}_k \in W_0^{1,p}(\Omega) : \quad (\nabla \mathsf{y}_k, \nabla v)_{L^2(\Omega)} + (a(\cdot, \mathsf{y}_k), v)_{L^2(\Omega)} = \sum_{\mathsf{z}\in\mathcal{D}} \mathsf{u}_\mathsf{z}^k \langle \delta_\mathsf{z}, v\rangle \tag{4.4}$$

for all $v \in W_0^{1,p'}(\Omega)$. Here, $p < d/(d-1)$ and p' is the Hölder conjugate of p ($p' > d$).

Since $\mathbb{U}_{ad}$ is closed and bounded in $\mathbb{R}^\ell$, there exists a nonrelabeled subsequence $\{\mathbf{u}_k\}_{k\in\mathbb{N}}$ such that $\mathbf{u}_k \to \bar{\mathbf{u}}$ in $\mathbb{R}^\ell$. We now notice that, since y_k solves (4.4), estimate (2.1) reveals that, for every $k \in \mathbb{N}$, $a(\cdot, \mathsf{y}_k) \in L^1(\Omega)$. We can thus conclude that, up to a subsequence, if necessary, $a(\cdot, \mathsf{y}_k) \rightharpoonup \mathfrak{a}$ in $\mathcal{M}(\Omega)$ as $k \uparrow \infty$ [13, Chapter 1.9, Theorem 2]. The compact embedding $\mathcal{M}(\Omega) \hookrightarrow W^{-1,p}(\Omega)$, with $p < d/(d-1)$, yields the strong convergence $a(\cdot, \mathsf{y}_k) \to \mathfrak{a}$ in $W^{-1,p}(\Omega)$ as $k \uparrow \infty$ [12, Section 1, Theorem 6]. On the other hand, since $\mathbf{u}_k \to \bar{\mathbf{u}}$ in $\mathbb{R}^\ell$, we have

$$\left|\sum_{\mathsf{z}\in\mathcal{D}} (\mathsf{u}_\mathsf{z} - \mathsf{u}_\mathsf{z}^k)\langle \delta_\mathsf{z}, v\rangle\right| \leq \sum_{\mathsf{z}\in\mathcal{D}} |\mathsf{u}_\mathsf{z} - \mathsf{u}_\mathsf{z}^k| \|\delta_\mathsf{z}\|_{W^{-1,p}(\Omega)} \|\nabla v\|_{L^{p'}(\Omega)} \to 0, \tag{4.5}$$

as $k \uparrow \infty$, for every $v \in W_0^{1,p'}(\Omega)$. Consequently, $\sum_\mathcal{D} \mathsf{u}_\mathsf{z}^k \delta_\mathsf{z} - a(\cdot, \mathsf{y}_k) \to \sum_\mathcal{D} \mathsf{u}_\mathsf{z}\delta_\mathsf{z} - \mathfrak{a}$ in $W^{-1,p}(\Omega)$ as $k \uparrow \infty$. We are thus in position to invoke [19, Theorem 0.5] to conclude that $\mathsf{y}_k \to \bar{\mathsf{y}}$ in $W_0^{1,p}(\Omega)$ as $k \uparrow \infty$; $\bar{\mathsf{y}}$ being the natural candidate for an optimal state.

We now show that $\bar{\mathsf{y}} = \mathcal{S}\bar{\mathbf{u}}$. Let $\mathfrak{q} \geq 1$ be such that $\mathfrak{q} < \infty$ if $d = 2$ and $\mathfrak{q} < 3$ if $d = 3$. Since $\mathsf{y}_k \to \bar{\mathsf{y}}$ in $W_0^{1,p}(\Omega)$, as $k \uparrow \infty$, we have that $\mathsf{y}_k \to \bar{\mathsf{y}}$ in $L^\mathfrak{q}(\Omega)$, as $k \uparrow \infty$. Consequently, the Lebesgue dominated convergence theorem and (2.1) yield

$$\|a(\cdot, \mathsf{y}_k) - a(\cdot, \mathsf{y})\|_{L^1(\Omega)} \to 0, \qquad k \uparrow \infty.$$

This, (4.5), and $\mathsf{y}_k \to \bar{\mathsf{y}}$ in $W_0^{1,p}(\Omega)$ as $k \uparrow \infty$, for $p < d/(d-1)$, allow us to pass to the limit in problem (4.4) to obtain $\bar{\mathsf{y}} = \mathcal{S}\bar{\mathbf{u}}$.

The local optimality of $\bar{\mathbf{u}}$ thus follows from $\mathbf{u}_k \to \bar{\mathbf{u}}$ in $\mathbb{R}^\ell$, which implies that $\mathsf{y}_k \to \bar{\mathsf{y}}$ in $W_0^{1,p}(\Omega)$, as $k \uparrow \infty$, and the continuity of j as a map from $\mathbb{U}_{ad}$ to $\mathbb{R}$. This concludes the proof. ∎

4.2. First order necessary optimality conditions. In this section, we analyze differentiability properties of $\mathcal{S}$ and derive first order necessary optimality conditions for (4.2)–(4.3). In doing so, we will assume that (A.1) and (A.2) holds. Since the control problem (4.2)–(4.3) is not convex, we discuss optimality conditions in the context of local solutions.

To simplify the presentation of the material, we define for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^\ell$ and $v \in C(\bar{\Omega})$,

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{D}} := \sum_{\mathsf{z} \in \mathcal{D}} \mathsf{u}_\mathsf{z} \mathsf{v}_\mathsf{z}, \qquad \langle \mathbf{u}, v \rangle_{\mathcal{D}} := \sum_{\mathsf{z} \in \mathcal{D}} \mathsf{u}_\mathsf{z} v(\mathsf{z}). \tag{4.6}$$

4.2.1. Differentiability of $\mathcal{S}$. The first order Fréchet differentiability of $\mathcal{S}$ is as follows.

THEOREM 4.2 (first order differentiability of $\mathcal{S}$). *Let Ω be an open and bounded domain with Lipschitz boundary. If $a = a(x,y) : \Omega \times \mathbb{R} \to \mathbb{R}$ denotes a Carathéodory function that satisfies* (A.1) and (A.2)*, then the control to state map $\mathcal{S} : \mathbb{U}_{ad} \to W_0^{1,p}(\Omega)$ is of class C^1 for $p < d/(d-1)$. In addition, if $\mathbf{u}, \mathbf{w} \in \mathbb{U}_{ad}$, then $\phi = \mathcal{S}'(\mathbf{u})\mathbf{w} \in W_0^{1,p}(\Omega)$ corresponds to the unique solution to*

$$(\nabla \phi, \nabla v)_{L^2(\Omega)} + \left(\tfrac{\partial a}{\partial y}(\cdot, \mathsf{y})\phi, v \right)_{L^2(\Omega)} = \sum_{\mathsf{z} \in \mathcal{D}} \mathsf{w}_\mathsf{z} \langle \delta_\mathsf{z}, v \rangle \quad \forall v \in W_0^{1,p'}(\Omega), \tag{4.7}$$

where $\mathsf{y} = \mathcal{S}\mathbf{u}$.

Proof. In view of the fact that $\partial a/\partial y$ satisfies (2.2), an adaption of the arguments elaborated in the proof of [24, Lemma 4.12] reveals that $\mathsf{y} \mapsto a(\cdot, \mathsf{y})$ is C^1 as a map from $W_0^{1,p}(\Omega)$ into $L^1(\Omega)$ for $p < d/(d-1)$; see also [24, §4.3.3]. We now follow the arguments in [24, Theorem 4.17] and prove the Fréchet differentiability of $\mathcal{S}$. Let $\mathbf{u}, \mathbf{w} \in \mathbb{U}_{ad}$. Define $\mathsf{y} := \mathcal{S}\mathbf{u}$ and $\tilde{\mathsf{y}} := \mathcal{S}(\mathbf{u} + \mathbf{w})$. Observe that $\tilde{\mathsf{y}} - \mathsf{y} = 0$ on $\partial\Omega$ and

$$(\nabla(\tilde{\mathsf{y}} - \mathsf{y}), \nabla v)_{L^2(\Omega)} + (a(\cdot, \tilde{\mathsf{y}}) - a(\cdot, \mathsf{y}), v)_{L^2(\Omega)} = \sum_{\mathsf{z} \in \mathcal{D}} \mathsf{w}_\mathsf{z} \langle \delta_\mathsf{z}, v \rangle \tag{4.8}$$

for all $v \in W_0^{1,p'}(\Omega)$. Since $W_0^{1,p}(\Omega) \ni \mathsf{y} \mapsto a(\cdot, \mathsf{y}) \in L^1(\Omega)$ is of class C^1, we have $a(\cdot, \tilde{\mathsf{y}}) - a(\cdot, \mathsf{y}) = \partial a/\partial y(\cdot, \mathsf{y})(\tilde{\mathsf{y}} - \mathsf{y}) + \mathfrak{r}$, where $\mathfrak{r}$ is such that $\|\mathfrak{r}\|_{L^1(\Omega)}/\|\nabla(\tilde{\mathsf{y}} - \mathsf{y})\|_{L^p(\Omega)} \to 0$ as $\|\nabla(\tilde{\mathsf{y}} - \mathsf{y})\|_{L^p(\Omega)} \to 0$. We now define $\mathfrak{y} := \tilde{\mathsf{y}} - \mathsf{y} - \phi$ and observe that $\mathfrak{y}$ solves

$$-\Delta \mathfrak{y} + \tfrac{\partial a}{\partial y}(\cdot, \mathsf{y})\mathfrak{y} = -\mathfrak{r} \text{ in } \Omega, \qquad \mathfrak{y} = 0 \text{ on } \partial\Omega.$$

Since there exists $q > d/2$ such that $\partial a/\partial y(\cdot, \mathsf{y}) \in L^q(\Omega)$ and $\mathfrak{r} \in L^1(\Omega)$, we invoke [23, Theorem 9.1] to obtain $\mathfrak{y} \in W_0^{1,p}(\Omega)$ and $\|\nabla \mathfrak{y}\|_{L^p(\Omega)} \lesssim \|\mathfrak{r}\|_{L^1(\Omega)}$; $p < d/(d-1)$. Define $\mathfrak{D} := \| \sum_{\mathsf{z} \in \mathcal{D}} \mathsf{w}_\mathsf{z} \delta_\mathsf{z} \|_{\mathcal{M}(\Omega)}$ and observe that the solution to problem (4.8) satisfies the stability estimate $\|\nabla(\tilde{\mathsf{y}} - \mathsf{y})\|_{L^p(\Omega)} \lesssim \mathfrak{D}$ [23, Theorem 9.1]. Thus $\|\mathfrak{r}\|_{L^1(\Omega)}/\mathfrak{D} \lesssim \|\mathfrak{r}\|_{L^1(\Omega)}/\|\nabla(\tilde{\mathsf{y}} - \mathsf{y})\|_{L^p(\Omega)}$, which implies that $\|\mathfrak{r}\|_{L^1(\Omega)}/\mathfrak{D} \to 0$ as $\|\nabla(\tilde{\mathsf{y}} - \mathsf{y})\|_{L^p(\Omega)} \to 0$. With all these ingredients at hand, we can thus deduce that

$$\mathcal{S}(\mathbf{u} + \mathbf{w}) - \mathcal{S}(\mathbf{u}) = \tilde{\mathsf{y}} - \mathsf{y} = \phi + \mathfrak{y} = \mathcal{L}\mathbf{w} + \mathfrak{y},$$

where $\mathcal{L}$ denotes the map $\mathbf{w} \mapsto \phi$, with ϕ being the solution to (4.7). Since $\mathcal{L}$ is linear and continuous as a map from $\mathbb{R}^\ell$ into $W_0^{1,p}(\Omega)$ and $\|\nabla \mathfrak{y}\|_{L^p(\Omega)}/\mathfrak{D} \lesssim \|\mathfrak{r}\|_{L^1(\Omega)}/\mathfrak{D} \to 0$ as $\|\mathbf{w}\|_{\mathbb{R}^\ell} \to 0$ we conclude that $\mathcal{S}$ is first order Fréchet differentiable. ∎

4.2.2. The adjoint equation. Before deriving first order optimality conditions, we present a classical result: If $\bar{\mathbf{u}} \in \mathbb{U}_{ad}$ denotes a locally optimal control for (4.2)–(4.3), then we have the variational inequality [24, Lemma 4.18]

$$j'(\bar{\mathbf{u}})(\mathbf{u} - \bar{\mathbf{u}}) \geq 0 \quad \forall \mathbf{u} \in \mathbb{U}_{ad}. \tag{4.9}$$

Here, $j'(\bar{\mathbf{u}})$ denotes the Gateâux derivative of j at $\bar{\mathbf{u}}$. To explore (4.9), we introduce the adjoint variable $\mathsf{p} \in H_0^1(\Omega)$ as the unique solution to the *adjoint equation*

$$(\nabla w, \nabla \mathsf{p})_{L^2(\Omega)} + \left(\frac{\partial a}{\partial y}(\cdot, \mathsf{y})\mathsf{p}, w\right)_{L^2(\Omega)} = (\mathsf{y} - \mathsf{y}_d, w)_{L^2(\Omega)} \quad \forall w \in H_0^1(\Omega), \tag{4.10}$$

where $\mathsf{y} = \mathcal{S}\mathbf{u}$. Further regularity properties for p can be obtained under the assumption that Ω is convex.

Proposition 4.3 (regularity properties of p). *Let Ω be an open, bounded, and convex polytopal domain. Let $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies* (A.1) and (A.3). *Then we have $\mathsf{p} \in H^2(\Omega)$. Let $\Omega_1 \Subset \Omega_0 \Subset \Omega$, with Ω_0 smooth. If, in addition, $\mathsf{y}_d \in L^t(\Omega_0)$ and $\partial a/\partial y(\cdot, y) \in L^t(\Omega_0)$, for every $y \in \mathbb{R}$, where $t < \infty$ if $d = 2$ and $t < 3$ if $d = 3$, then $\mathsf{p} \in W^{2,t}(\Omega_1)$.*

Proof. In view of (2.3), we immediately conclude that $\partial a/\partial y(\cdot, \mathsf{y})\mathsf{p} \in L^2(\Omega)$. We can thus invoke the convexity of Ω and the fact that $\mathsf{y} - \mathsf{y}_d$ also belongs to $L^2(\Omega)$ to conclude the $H^2(\Omega)$-regularity of p. Let us now write the adjoint equation as follows:

$$(\nabla w, \nabla \mathsf{p})_{L^2(\Omega)} = (\mathsf{y} - \mathsf{y}_d, w)_{L^2(\Omega)} - \left(\frac{\partial a}{\partial y}(\cdot, \mathsf{y})\mathsf{p}, w\right)_{L^2(\Omega)} \quad \forall w \in H_0^1(\Omega).$$

By assumption $\mathsf{y} - \mathsf{y}_d \in L^t(\Omega_0)$, where t is as in the statement of the theorem. We can thus invoke [2, Lemma 4.2] to conclude that

$$\|\mathsf{p}\|_{W^{2,t}(\Omega_1)} \leq C_t \left(\left\| \mathsf{y} - \mathsf{y}_d - \frac{\partial a}{\partial y}(\cdot, \mathsf{y})\mathsf{p} \right\|_{L^t(\Omega_0)} + \left\| \mathsf{y} - \mathsf{y}_d - \frac{\partial a}{\partial y}(\cdot, \mathsf{y})\mathsf{p} \right\|_{L^2(\Omega)} \right), \tag{4.11}$$

where C_t behaves as Ct, with $C > 0$, as $t \uparrow \infty$. The assumptions on $\partial a/\partial y$ guarantee that $\|\mathsf{y} - \mathsf{y}_d - \partial a/\partial y(\cdot, \mathsf{y})\mathsf{p}\|_{L^t(\Omega_0)} \leq \|\mathsf{y} - \mathsf{y}_d\|_{L^t(\Omega_0)} + \|\mathsf{p}\|_{L^\infty(\Omega)} \|\partial a/\partial y(\cdot, \mathsf{y})\|_{L^t(\Omega_0)} < \infty$. ∎

4.3. First order optimality conditions. We are now in position to present first order necessary optimality conditions.

Theorem 4.4 (first order necessary optimality conditions). *Let Ω be an open, bounded, and convex polytopal domain. Let $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies* (A.1) and (A.3). *Then every locally optimal control $\bar{\mathbf{u}} = \{\bar{\mathsf{u}}_{\mathsf{z}}\}_{\mathsf{z} \in \mathcal{D}} \in \mathbb{U}_{ad}$ for problem (4.2)–(4.3) satisfies the variational inequality*

$$\sum_{\mathsf{z} \in \mathcal{D}} (\bar{\mathsf{p}}(\mathsf{z}) + \alpha \bar{\mathsf{u}}_{\mathsf{z}})(\mathsf{u}_{\mathsf{z}} - \bar{\mathsf{u}}_{\mathsf{z}}) \geq 0 \quad \forall \mathbf{u} = \{\mathsf{u}_{\mathsf{z}}\}_{\mathsf{z} \in \mathcal{D}} \in \mathbb{U}_{ad}, \tag{4.12}$$

where the optimal adjoint state $\bar{\mathsf{p}}$ solves (4.10) with y being replaced by $\bar{\mathsf{y}} = \mathcal{S}\bar{\mathbf{u}}$.

Proof. Basic computations reveal that the variational inequality (4.9) can be rewritten as follows:

$$0 \leq j'(\bar{\mathbf{u}})(\mathbf{u} - \bar{\mathbf{u}}) = (\mathcal{S}\bar{\mathbf{u}} - \mathsf{y}_d, \mathcal{S}'(\bar{\mathbf{u}})(\mathbf{u} - \bar{\mathbf{u}}))_{L^2(\Omega)} + \sum_{\mathsf{z} \in \mathcal{D}} \alpha \bar{\mathsf{u}}_{\mathsf{z}}(\mathsf{u}_{\mathsf{z}} - \bar{\mathsf{u}}_{\mathsf{z}}).$$

Since the second term on the right hand side of the previous expression is already present in the desired variational inequality (4.12), we concentrate on the first term.

Define $\chi := \mathcal{S}'(\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}})$ and observe that χ solves (4.7) with y and $\mathsf{w_z}$ being replaced by $\bar{\mathsf{y}} = \mathcal{S}\bar{\mathbf{u}}$ and $\mathsf{u_z} - \bar{\mathsf{u}}_\mathsf{z}$, respectively. Since there exists $q > d$ such that $\bar{\mathsf{p}} \in W_0^{1,q}(\Omega)$, we are allowed to set $v = \bar{\mathsf{p}}$ in the problem that χ solves. This yields

$$(4.13)\qquad (\nabla \chi, \nabla \bar{\mathsf{p}})_{L^2(\Omega)} + \left(\tfrac{\partial a}{\partial y}(\cdot,\bar{\mathsf{y}})\chi, \bar{\mathsf{p}}\right)_{L^2(\Omega)} = \sum_{\mathsf{z}\in\mathcal{D}} \bar{\mathsf{p}}(\mathsf{z})(\mathsf{u_z} - \bar{\mathsf{u}}_\mathsf{z}).$$

Now, we would like to set $w = \chi$ in problem (4.10) to conclude that

$$(4.14)\qquad (\nabla \chi, \nabla \bar{\mathsf{p}})_{L^2(\Omega)} + \left(\tfrac{\partial a}{\partial y}(\cdot,\bar{\mathsf{y}})\bar{\mathsf{p}}, \chi\right)_{L^2(\Omega)} = (\bar{\mathsf{y}} - \mathsf{y}_d, \chi)_{L^2(\Omega)}.$$

Unfortunately, $\chi \notin H_0^1(\Omega)$ and thus we need to justify (4.14) with a different argument. Let $\{\sigma_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ be such that $\sigma_n \to \chi$ in $W_0^{1,p}(\Omega)$, as $n \uparrow \infty$, for $p < d/(d-1)$. Setting $w = \sigma_n$ in (4.10) yields

$$(\nabla \sigma_n, \nabla \bar{\mathsf{p}})_{L^2(\Omega)} + \left(\tfrac{\partial a}{\partial y}(\cdot,\bar{\mathsf{y}})\bar{\mathsf{p}}, \sigma_n\right)_{L^2(\Omega)} = (\bar{\mathsf{y}} - \mathsf{y}_d, \sigma_n)_{L^2(\Omega)}.$$

Observe that $|(\nabla(\chi-\sigma_n), \nabla \bar{\mathsf{p}})_{L^2(\Omega)}| \leq \|\nabla(\chi-\sigma_n)\|_{L^p(\Omega)}\|\nabla\bar{\mathsf{p}}\|_{L^{p'}(\Omega)} \to 0$ as $n\uparrow\infty$; p' being the Hölder conjugate of p. Similarly, $|(\bar{\mathsf{y}} - \mathsf{y}_d, \sigma_n)_{L^2(\Omega)} - (\bar{\mathsf{y}} - \mathsf{y}_d, \chi)_{L^2(\Omega)}| \to 0$ as $n \uparrow \infty$. It thus suffices to analyze

$$\mathsf{I} := \left| \left(\tfrac{\partial a}{\partial y}(\cdot,\bar{\mathsf{y}})\bar{\mathsf{p}}, \chi\right)_{L^2(\Omega)} - \left(\tfrac{\partial a}{\partial y}(\cdot,\bar{\mathsf{y}})\bar{\mathsf{p}}, \sigma_n\right)_{L^2(\Omega)} \right| \leq \|\chi - \sigma_n\|_{L^{\mathfrak{q}}(\Omega)} \left\| \tfrac{\partial a}{\partial y}(\cdot,\bar{\mathsf{y}})\bar{\mathsf{p}} \right\|_{L^{\mathfrak{p}}(\Omega)},$$

where $\mathfrak{q} \in (1,\infty)$ is such that $\mathfrak{q} < \infty$ if $d = 2$ and $\mathfrak{q} < 3$ if $d = 3$ and $\mathfrak{p}$ is such that $\mathfrak{p}^{-1} + \mathfrak{q}^{-1} = 1$. Since there exists $q > d$ such that $\bar{\mathsf{p}} \in W_0^{1,q}(\Omega) \hookrightarrow C(\bar{\Omega})$, we have

$$\left\| \tfrac{\partial a}{\partial y}(\cdot,\bar{\mathsf{y}})\bar{\mathsf{p}} \right\|_{L^{\mathfrak{p}}(\Omega)} \leq \|\bar{\mathsf{p}}\|_{L^\infty(\Omega)} \left\| \tfrac{\partial a}{\partial y}(\cdot,\bar{\mathsf{y}}) \right\|_{L^{\mathfrak{p}}(\Omega)} < \infty.$$

Notice that, in view of assumption (A.2), we conclude the existence of $s > d/2$ such that $\frac{\partial a}{\partial y}(\cdot,\bar{\mathsf{y}}) \in L^s(\Omega)$. We thus utilize that

$$\|\nabla(\chi - \sigma_n)\|_{L^p(\Omega)} \to 0 \implies \|\chi - \sigma_n\|_{L^{\mathfrak{q}}(\Omega)} \to 0, \qquad n \uparrow \infty,$$

where $p < d/(d-1)$, to conclude that $\mathsf{I} \to 0$ as $n \uparrow \infty$.

Finally, we invoke (4.13) and (4.14) to immediately arrive at $(\bar{\mathsf{y}} - \mathsf{y}_d, \chi)_{L^2(\Omega)} = \sum_{\mathsf{z}\in\mathcal{D}} \bar{\mathsf{p}}(\mathsf{z})(\mathsf{u_z} - \bar{\mathsf{u}}_\mathsf{z})$. This concludes the proof. ∎

We end this section with the following projection formula: If $\bar{\mathbf{u}} = \{\bar{\mathsf{u}}_\mathsf{z}\}_{\mathsf{z}\in\mathcal{D}} \in \mathbb{U}_{ad}$ denotes a local minimizer of problem (4.2)–(4.3), then, for every $\mathsf{z} \in \mathcal{D}$, we have

$$(4.15)\qquad \bar{\mathsf{u}}_\mathsf{z} := \Pi_{[\mathsf{a_z},\mathsf{b_z}]}\left(-\alpha^{-1}\bar{\mathsf{p}}(\mathsf{z})\right),$$

where, for $t \in \mathbb{R}$, $\Pi_{[\mathsf{a_z},\mathsf{b_z}]}(t) := \max\{\mathsf{a_z}, \min\{\mathsf{b_z}, t\}\}$.

4.4. Second order optimality conditions. In this section, we follow [8, 10] and analyze necessary and sufficient second order optimality conditions for our optimal control problem (4.2)–(4.3).

4.4.1. Second order differentiability. Before deriving optimality conditions, we study second order differentiability properties for the control to state map $\mathcal{S}$ and the reduced cost functional j.

THEOREM 4.5 (second order differentiability of $\mathcal{S}$). *Let Ω be an open and bounded domain with Lipschitz boundary. If $a = a(x,y) : \Omega \times \mathbb{R} \to \mathbb{R}$ denotes a Carathéodory function that satisfies* (A.1), (A.2), and (A.4)*, then the control to state map $\mathcal{S}$: $\mathbb{U}_{ad} \to W_0^{1,p}(\Omega)$ is of class C^2 for $p < d/(d-1)$. In addition, if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{U}_{ad}$, then $\varphi = \mathcal{S}''(\mathbf{u})(\mathbf{v}, \mathbf{w}) \in W_0^{1,p}(\Omega)$ corresponds to the unique solution to*

$$(4.16)\qquad (\nabla\varphi, \nabla v)_{L^2(\Omega)} + \left(\tfrac{\partial a}{\partial y}(\cdot, \mathsf{y})\varphi, v\right)_{L^2(\Omega)} = -\left(\tfrac{\partial^2 a}{\partial y^2}(\cdot, \mathsf{y})\phi_{\mathbf{v}}\phi_{\mathbf{w}}, v\right)_{L^2(\Omega)}$$

for all $v \in W_0^{1,p'}(\Omega)$. Here, $\mathsf{y} = \mathcal{S}\mathbf{u}$ and $\phi_{\mathbf{v}}$ and $\phi_{\mathbf{w}}$ denote the solutions to (4.7) *with forcing terms $\sum_{\mathsf{z}} \mathsf{v}_{\mathsf{z}}\delta_{\mathsf{z}}$ and $\sum_{\mathsf{z}} \mathsf{w}_{\mathsf{z}}\delta_{\mathsf{z}}$, respectively.*

Proof. We follow the arguments elaborated in the proof of [24, Theorem 4.24] and prove the second order Fréchet differentiability of $\mathcal{S}$ by using the implicit function theorem. We begin by introducing the linear map $\mu \mapsto \mathsf{y}$ by

$$\mathfrak{R} : \mathcal{M}(\Omega) \to W_0^{1,p}(\Omega) : \quad (\nabla \mathsf{y}, \nabla v)_{L^2(\Omega)} = \int_\Omega v \mathrm{d}\mu \quad \forall v \in W_0^{1,p'}(\Omega), \quad p < d/(d-1).$$

We now define $\mathfrak{F} : W_0^{1,p}(\Omega) \times \mathbb{U}_{ad} \to W_0^{1,p}(\Omega)$ by $\mathfrak{F}(\mathsf{y}, \mathbf{u}) := \mathsf{y} - \mathfrak{R}[\sum_{\mathsf{z}} \mathsf{u}_{\mathsf{z}}\delta_{\mathsf{z}} - a(\cdot, \mathsf{y})]$. Note that, since (A.4) holds, the map $\mathsf{y} \mapsto a(\cdot, \mathsf{y})$ is of class C^2 as a map from $W_0^{1,p}(\Omega)$ into $L^1(\Omega)$ [7, Theorem 2.2]. This and the linearity of the map $\mathfrak{R}$ imply that $\mathfrak{F}$ is of class C^2 as well. We also observe that, by definition, $\mathfrak{F}(\mathcal{S}\mathbf{u}, \mathbf{u}) = 0$. Finally, we notice that $\partial\mathfrak{F}/\partial\mathsf{y}(\mathsf{y}, \mathbf{u})\mathsf{w} = \mathsf{w} + \mathfrak{R}\partial a/\partial \mathsf{y}(\cdot, \mathsf{y})\mathsf{w}$; it can thus be deduced that $\partial\mathfrak{F}/\partial\mathsf{y}(\mathsf{y}, \mathbf{u})$ is surjective as a map from $W_0^{1,p}(\Omega)$ into itself upon utilizing [23, Theorem 9.1]. The implicit function theorem thus implies that $\mathcal{S}$ is of class C^2. We finally mention that the fact that φ solves (4.16) follows from differentiating $\mathfrak{F}(\mathcal{S}\mathbf{u}, \mathbf{u}) = 0$. The well-posedness of problem (4.16) follows from [23, Theorem 9.1] upon noticing that $\partial^2 a/\partial y^2(\cdot, \mathsf{y})\phi_{\mathbf{v}}\phi_{\mathbf{w}} \in L^1(\Omega)$ and $0 \leq \partial a/\partial y(\cdot, \mathsf{y}) \in L^q(\Omega)$ for some $q > d/2$. ∎

The following result is instrumental.

PROPOSITION 4.6 (second order differentiability of the reduced cost j). *Let Ω be an open, bounded, and convex polytopal domain. If $a = a(x,y) : \Omega \times \mathbb{R} \to \mathbb{R}$ denotes a Carathéodory function that satisfies* (A.1), (A.3), and (A.4)*, then the reduced cost functional $j : \mathbb{U}_{ad} \to \mathbb{R}$ is of class C^2. In addition, for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{U}_{ad}$, we have*

$$(4.17)\qquad j''(\mathbf{u})(\mathbf{v}, \mathbf{w}) = \int_\Omega \phi_{\mathbf{v}}\phi_{\mathbf{w}} \mathrm{d}x + \sum_{\mathsf{z}\in\mathcal{D}} \alpha \mathsf{v}_{\mathsf{z}}\mathsf{w}_{\mathsf{z}} - \int_\Omega \mathsf{p}\frac{\partial^2 a}{\partial y^2}(x, \mathsf{y})\phi_{\mathbf{v}}\phi_{\mathbf{w}} \mathrm{d}x.$$

Here, $\mathsf{y} = \mathcal{S}\mathbf{u}$ and $\phi_{\mathbf{v}}$ and $\phi_{\mathbf{w}}$ denote the solutions to (4.7) *with forcing terms $\sum_{\mathsf{z}} \mathsf{v}_{\mathsf{z}}\delta_{\mathsf{z}}$ and $\sum_{\mathsf{z}} \mathsf{w}_{\mathsf{z}}\delta_{\mathsf{z}}$, respectively.*

Proof. Since Theorem 4.5 guarantees that $\mathcal{S} : \mathbb{U}_{ad} \to W_0^{1,p}(\Omega)$ is twice continuously Fréchet differentiable, it is immediate that j is of class C^2. The identity (4.17) follows from computations similar to those in [24, Section 4.10] upon noticing that

$$-\int_\Omega \mathsf{p}\frac{\partial^2 a}{\partial y^2}(x, \mathsf{y})\phi_{\mathbf{v}}\phi_{\mathbf{w}} \mathrm{d}x = \int_\Omega (\mathsf{y} - \mathsf{y}_d)\varphi \mathrm{d}x.$$

This identity follows from the density argument elaborated in the proof of Theorem 4.4 combined with the fact that a satisfies (A.3) and (A.4); (A.3) guarantees the $H^2(\Omega)$-regularity of $\bar{\mathsf{p}}$ (see Proposition 4.3). Here, φ denotes the solution to (4.16). ∎

4.4.2. Second order necessary optimality conditions. We begin this section by introducing some preliminary concepts. We define

$$\Psi := \{\psi_{\mathsf{z}}\}_{\mathsf{z}\in\mathcal{D}} \in \mathbb{R}^{\ell}, \qquad \psi_{\mathsf{z}} := \bar{\mathsf{p}}(\mathsf{z}) + \alpha\bar{\mathsf{u}}_{\mathsf{z}}. \tag{4.18}$$

We also introduce the cone of critical directions

$$\mathbf{C}_{\bar{\mathbf{u}}} := \left\{\mathbf{v} = \{\mathsf{v}_{\mathsf{z}}\}_{\mathsf{z}\in\mathcal{D}} \in \mathbb{R}^{\ell} \text{ satisfying (4.20) and } \mathsf{v}_{\mathsf{z}} = 0 \text{ if } \psi_{\mathsf{z}} \neq 0 \text{ for } \mathsf{z}\in\mathcal{D}\right\}. \tag{4.19}$$

The condition (4.20) reads as follows: For every $\mathsf{z}\in\mathcal{D}$, we have

$$\mathsf{v}_{\mathsf{z}} \geq 0 \text{ if } \bar{\mathsf{u}}_{\mathsf{z}} = \mathsf{a}_{\mathsf{z}}, \qquad \mathsf{v}_{\mathsf{z}} \leq 0 \text{ if } \bar{\mathsf{u}}_{\mathsf{z}} = \mathsf{b}_{\mathsf{z}}. \tag{4.20}$$

We are now in position to present second order necessary optimality conditions.

THEOREM 4.7 (second order necessary optimality conditions). *If $\bar{\mathbf{u}} = \{\bar{\mathsf{u}}_{\mathsf{z}}\}_{\mathsf{z}\in\mathcal{D}} \in \mathbb{U}_{ad}$ denotes a local minimum for problem (4.2)–(4.3), then*

$$j''(\bar{\mathbf{u}})\mathbf{v}^2 \geq 0 \quad \forall \mathbf{v} \in \mathbf{C}_{\bar{\mathbf{u}}},$$

where $\mathbf{C}_{\bar{\mathbf{u}}}$ is defined in (4.19).

Proof. Let $\mathbf{v} := \{\mathsf{v}_{\mathsf{z}}\}_{\mathsf{z}\in\mathcal{D}} \in \mathbf{C}_{\bar{\mathbf{u}}}$. Define, for $k\in\mathbb{N}$, $\mathbf{v}_k := \{\mathbf{v}_{\mathsf{z},k}\}_{\mathsf{z}\in\mathcal{D}}$, where

$$\mathsf{v}_{\mathsf{z},k} = \begin{cases} 0 & \text{if } \mathsf{a}_{\mathsf{z}} < \bar{\mathsf{u}}_{\mathsf{z}} < \mathsf{a}_{\mathsf{z}} + k^{-1}, \quad \mathsf{b}_{\mathsf{z}} - k^{-1} < \bar{\mathsf{u}}_{\mathsf{z}} < \mathsf{b}_{\mathsf{z}}, \\ \Pi_{[-k,+k]}(\mathsf{v}_{\mathsf{z}}) & \text{otherwise.} \end{cases}$$

Since $\mathbf{v} \in \mathbf{C}_{\bar{\mathbf{u}}}$, it is immediate that, for every $k\in\mathbb{N}$, $\mathbf{v}_k \in \mathbf{C}_{\bar{\mathbf{u}}}$. In addition, we have that $\mathbf{v}_k \to \mathbf{v}$ in $\mathbb{R}^{\ell}$ as $k\uparrow\infty$. We now observe that, for every $0<\rho\leq k^{-2}$, $\bar{\mathbf{u}} + \rho\mathbf{v}_k \in \mathbb{U}_{ad}$. We can thus invoke the fact that $\bar{\mathbf{u}}$ is a local minimum to conclude that $j(\bar{\mathbf{u}}) \leq j(\bar{\mathbf{u}} + \rho\mathbf{v}_k)$ for ρ sufficiently small. We now apply Taylor's theorem and utilize that $j'(\bar{\mathbf{u}})\mathbf{v}_k = 0$, which follows from the fact that $\mathbf{v}_k \in \mathbf{C}_{\bar{\mathbf{u}}}$, to obtain

$$0 \leq j(\bar{\mathbf{u}} + \rho\mathbf{v}_k) - j(\bar{\mathbf{u}}) = \rho j'(\bar{\mathbf{u}})\mathbf{v}_k + \tfrac{\rho^2}{2} j''(\bar{\mathbf{u}} + \theta_k\rho\mathbf{v}_k)\mathbf{v}_k^2 = \tfrac{\rho^2}{2} j''(\bar{\mathbf{u}} + \theta_k\rho\mathbf{v}_k)\mathbf{v}_k^2,$$

where $\theta_k \in (0,1)$. Divide by ρ^2, let $\rho\downarrow 0$, and utilize (4.17) to arrive at $j''(\bar{\mathbf{u}})\mathbf{v}_k^2 \geq 0$. Let now $k\uparrow\infty$ and recall that $\mathbf{v}_k \to \mathbf{v}$ in $\mathbb{R}^{\ell}$ to conclude, in view of (4.17), that $j''(\bar{\mathbf{u}})\mathbf{v}^2 \geq 0$. This concludes the proof. ∎

4.4.3. Second order sufficient optimality conditions. We now provide a sufficient second order optimality condition with a minimal gap with respect to the necessary one obtained in Theorem 4.7.

THEOREM 4.8 (second order sufficient optimality conditions). *Let $\bar{\mathbf{u}} \in \mathbb{U}_{ad}$, $\bar{\mathsf{y}} \in W_0^{1,p}(\Omega)$, and $\bar{\mathsf{p}} \in H_0^1(\Omega)$ satisfy the first order optimality conditions (4.3), (4.10), and (4.12). Assume that $a = a(x,y) : \Omega\times\mathbb{R}\to\mathbb{R}$ denotes a Carathéodory function that satisfies (A.1), (A.3), and (A.4). If $j''(\bar{\mathbf{u}})\mathbf{v}^2 > 0$ for all $\mathbf{v} \in \mathbf{C}_{\bar{\mathbf{u}}} \setminus \{\mathbf{0}\}$, then there exists $\mu, \sigma > 0$ such that*

$$j(\mathbf{u}) \geq j(\bar{\mathbf{u}}) + \frac{\mu}{2}\|\bar{\mathbf{u}} - \mathbf{u}\|_{\mathbb{R}^{\ell}}^2 \quad \forall \mathbf{u} \in \mathbb{U}_{ad} : \|\bar{\mathbf{u}} - \mathbf{u}\|_{\mathbb{R}^{\ell}} \leq \sigma.$$

In particular, $\bar{\mathbf{u}}$ is a locally optimal control.

Proof. We proceed by contradiction and assume that, for every $k \in \mathbb{N}$, there exists $\mathbf{u}_k \in \mathbb{U}_{ad}$ such that

$$\|\bar{\mathbf{u}} - \mathbf{u}_k\|_{\mathbb{R}^\ell} < \tfrac{1}{k}, \qquad j(\mathbf{u}_k) < j(\bar{\mathbf{u}}) + \tfrac{1}{2k}\|\bar{\mathbf{u}} - \mathbf{u}_k\|^2_{\mathbb{R}^\ell}. \tag{4.21}$$

Define $\rho_k := \|\mathbf{u}_k - \bar{\mathbf{u}}\|_{\mathbb{R}^\ell}$ and $\mathbf{v}_k := \rho_k^{-1}(\mathbf{u}_k - \bar{\mathbf{u}})$. Since, for every $k \in \mathbb{N}$, $\|\mathbf{v}_k\|_{\mathbb{R}^\ell} = 1$, there exists a nonrelabeled subsequence such that $\mathbf{v}_k \to \mathbf{v}$ in $\mathbb{R}^\ell$ as $k \uparrow \infty$. We now proceed on the basis of three steps.

$\boxed{1}$ We first prove that $\mathbf{v} \in \mathbf{C}_{\bar{\mathbf{u}}}$. To accomplish this task, we notice that the subset of $\mathbb{R}^\ell$ conformed by the elements that satisfy (4.20) is closed. Since, for every $k \in \mathbb{N}$, $\mathbf{v}_k$ belongs to this set and $\mathbf{v}_k \to \mathbf{v}$ in $\mathbb{R}^\ell$, as $k \uparrow \infty$, we conclude that $\mathbf{v}$ satisfies (4.20). To conclude that $\mathbf{v} \in \mathbf{C}_{\bar{\mathbf{u}}}$, it thus suffices to prove that, for $\mathsf{z} \in \mathcal{D}$, $\psi_\mathsf{z} \neq 0$ implies $\mathsf{v}_\mathsf{z} = 0$. Observe that the first order optimality condition (4.12) yields

$$0 \leq \rho_k^{-1} \sum_{\mathsf{z} \in \mathcal{D}} (\bar{\mathsf{p}}(\mathsf{z}) + \alpha \bar{\mathsf{u}}_\mathsf{z})(\mathsf{u}_{\mathsf{z},k} - \bar{\mathsf{u}}_\mathsf{z}) = \rho_k^{-1} \sum_{\mathsf{z} \in \mathcal{D}} \psi_\mathsf{z}(\mathsf{u}_{\mathsf{z},k} - \bar{\mathsf{u}}_\mathsf{z}) = \sum_{\mathsf{z} \in \mathcal{D}} \psi_\mathsf{z} \mathsf{v}_{\mathsf{z},k}.$$

Consequently, $\sum_\mathsf{z} \psi_\mathsf{z} \mathsf{v}_\mathsf{z} \geq 0$. On the other hand, in view of (4.21), an application of the mean value theorem yields

$$j(\mathbf{u}_k) - j(\bar{\mathbf{u}}) = j'(\bar{\mathbf{u}} + \theta_k(\mathbf{u}_k - \bar{\mathbf{u}}))(\mathbf{u}_k - \bar{\mathbf{u}}) < \tfrac{1}{2k}\|\bar{\mathbf{u}} - \mathbf{u}_k\|^2_{\mathbb{R}^\ell} = \tfrac{\rho_k^2}{2k}, \quad \theta_k \in (0,1).$$

Divide by ρ_k and let $k \uparrow \infty$ to conclude that $j'(\bar{\mathbf{u}} + \theta_k(\mathbf{u}_k - \bar{\mathbf{u}}))\mathbf{v}_k < \frac{\rho_k}{2k} \to 0$. Define, for $k \in \mathbb{N}$, $\tilde{\mathbf{u}}_k := \bar{\mathbf{u}} + \theta_k(\mathbf{u}_k - \bar{\mathbf{u}})$, $\tilde{\mathsf{y}}_k := \mathcal{S}\tilde{\mathbf{u}}_k$, and $\tilde{\mathsf{p}}_k$ as the unique solution to (4.10) with y being replaced by $\tilde{\mathsf{y}}_k$. In view of $\tilde{\mathbf{u}}_k \to \bar{\mathbf{u}}$ in $\mathbb{R}^l$, we invoke [23, Theorem 9.1] combined with the fact that a satisfies (A.2) to arrive at $\tilde{\mathsf{y}}_k \to \bar{\mathsf{y}}$ in $W_0^{1,p}(\Omega)$, for $p < d/(d-1)$, as $k \uparrow \infty$. Invoke (A.2) again to conclude that

$$\exists q > d/2: \quad \tfrac{\partial a}{\partial y}(\cdot, \tilde{\mathsf{y}}_k) \to \tfrac{\partial a}{\partial y}(\cdot, \bar{\mathsf{y}}) \text{ in } L^q(\Omega), \quad k \uparrow \infty. \tag{4.22}$$

Consequently, since $\partial a/\partial y$ satisfies (A.3), we obtain $\tilde{\mathsf{p}}_k \to \bar{\mathsf{p}}$ in $H_0^1(\Omega) \cap H^2(\Omega)$ as $k \uparrow \infty$. In view of the Sobolev embedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$, we thus have that $\tilde{\mathsf{p}}_k \to \bar{\mathsf{p}}$ in $C(\bar{\Omega})$ as $k \uparrow \infty$. With all these convergence results at hand, we obtain that, for every $\mathsf{z} \in \mathcal{D}$, $\tilde{\mathsf{p}}_k(\mathsf{z}) + \alpha \tilde{\mathsf{u}}_{\mathsf{z},k} \to \bar{\mathsf{p}}(\mathsf{z}) + \alpha \bar{\mathsf{u}}_\mathsf{z}$ as $k \uparrow \infty$. Thus,

$$j'(\bar{\mathbf{u}})\mathbf{v} = \sum_{\mathsf{z} \in \mathcal{D}} (\bar{\mathsf{p}}(\mathsf{z}) + \alpha \bar{\mathsf{u}}_\mathsf{z}) \mathsf{v}_\mathsf{z} = \lim_{k \uparrow \infty} \sum_{\mathsf{z} \in \mathcal{D}} (\tilde{\mathsf{p}}_k(\mathsf{z}) + \alpha \tilde{\mathsf{u}}_{\mathsf{z},k}) \mathsf{v}_{\mathsf{z},k} = \lim_{k \uparrow \infty} j'(\tilde{\mathbf{u}}_k)\mathbf{v}_k \leq 0.$$

We have thus deduced that $j'(\bar{\mathbf{u}})\mathbf{v} = 0$. Finally, in view of the fact that $\mathbf{v}$ and $\bar{\mathbf{u}}$ satisfy (4.20) and (4.15), respectively, we arrive at $0 = j'(\bar{\mathbf{u}})\mathbf{v} = \sum_\mathsf{z} \psi_\mathsf{z} \mathsf{v}_\mathsf{z} = \sum_\mathsf{z} |\psi_\mathsf{z} \mathsf{v}_\mathsf{z}|$. This immediately implies that, if $\psi_\mathsf{z} \neq 0$ then $\mathsf{v}_\mathsf{z} = 0$ and allows us to conclude that $\mathbf{v} \in \mathbf{C}_{\bar{\mathbf{u}}}$.

$\boxed{2}$ We now prove that $\mathbf{v} = \mathbf{0}$. Apply Taylor's theorem to obtain

$$\tfrac{\rho_k^2}{2} j''(\tilde{\mathbf{u}}_k)\mathbf{v}_k^2 = j(\mathbf{u}_k) - j(\bar{\mathbf{u}}) - j'(\bar{\mathbf{u}})(\mathbf{u}_k - \bar{\mathbf{u}}) \leq j(\mathbf{u}_k) - j(\bar{\mathbf{u}}) < \tfrac{\rho_k^2}{2k}, \tag{4.23}$$

where $\tilde{\mathbf{u}}_k = \bar{\mathbf{u}} + \theta_k(\mathbf{u}_k - \bar{\mathbf{u}})$ and $\theta_k \in (0,1)$. To deduce (4.23), we have also used (4.9) and (4.21). Consequently, $j''(\tilde{\mathbf{u}}_k)\mathbf{v}_k^2 < k^{-1} \to 0$ as $k \uparrow \infty$. Assume, for the moment, that

$$j''(\bar{\mathbf{u}})\mathbf{v}^2 = \lim_{k \uparrow \infty} j''(\tilde{\mathbf{u}}_k)\mathbf{v}_k^2. \tag{4.24}$$

Since $j''(\tilde{\mathbf{u}}_k)\mathbf{v}_k^2 < k^{-1} \to 0$ as $k \uparrow \infty$, we obtain $j''(\bar{\mathbf{u}})\mathbf{v}^2 \leq 0$. We now utilize that $j''(\bar{\mathbf{u}})\mathbf{w}^2 > 0$ for all $\mathbf{w} \in \mathbf{C}_{\bar{\mathbf{u}}} \setminus \{\mathbf{0}\}$ and that $\mathbf{v} \in \mathbf{C}_{\bar{\mathbf{u}}}$ to conclude that $\mathbf{v} = \mathbf{0}$. It thus suffices to prove (4.24). To accomplish this task, we begin by noticing that

$$j''(\tilde{\mathbf{u}}_k)\mathbf{v}_k^2 = \int_\Omega \phi_{\mathbf{v}_k}^2 \mathrm{d}x + \sum_{\mathsf{z}\in\mathcal{D}} \alpha \mathsf{v}_{\mathsf{z},k}^2 - \int_\Omega \tilde{\mathsf{p}}_k \frac{\partial^2 a}{\partial y^2}(x, \tilde{\mathsf{y}}_k)\phi_{\mathbf{v}_k}^2 \mathrm{d}x =: \mathrm{I}_k + \mathrm{II}_k + \mathrm{III}_k,$$

where $\phi_{\mathbf{v}_k} := \mathcal{S}'(\tilde{\mathbf{u}}_k)\mathbf{v}_k$. Observe that $\mathbf{v}_k \to \mathbf{v}$ in $\mathbb{R}^\ell$ yields $\mathrm{II}_k \to \sum_{\mathsf{z}} \alpha \mathsf{v}_{\mathsf{z}}^2$ as $k \uparrow \infty$. In view of (4.22), we deduce that $\phi_{\mathbf{v}_k} \to \phi_{\mathbf{v}}$ in $W_0^{1,p}(\Omega)$, for $p < d/(d-1)$, as $k \uparrow \infty$. This directly implies that $\mathrm{I}_k \to \int_\Omega \phi_{\mathbf{v}}^2 \mathrm{d}x$. Let us now define

$$\begin{aligned}
(4.25)\quad \mathfrak{I}_k &:= \left| \int_\Omega \bar{\mathsf{p}} \frac{\partial^2 a}{\partial y^2}(x, \bar{\mathsf{y}})\phi_{\mathbf{v}}^2 \mathrm{d}x - \int_\Omega \tilde{\mathsf{p}}_k \frac{\partial^2 a}{\partial y^2}(x, \tilde{\mathsf{y}}_k)\phi_{\mathbf{v}_k}^2 \mathrm{d}x \right| \\
&\lesssim \left[\|\bar{\mathsf{p}} - \tilde{\mathsf{p}}_k\|_{L^r(\Omega)} \left\| \frac{\partial^2 a}{\partial y^2}(\cdot, \bar{\mathsf{y}}) \right\|_{L^s(\Omega)} + \|\tilde{\mathsf{p}}_k\|_{L^r(\Omega)} \left\| \frac{\partial^2 a}{\partial y^2}(\cdot, \bar{\mathsf{y}}) - \frac{\partial^2 a}{\partial y^2}(\cdot, \tilde{\mathsf{y}}_k) \right\|_{L^s(\Omega)} \right] \\
&\qquad \cdot \|\phi_{\mathbf{v}}^2\|_{L^t(\Omega)} + \|\tilde{\mathsf{p}}_k\|_{L^r(\Omega)} \left\| \frac{\partial^2 a}{\partial y^2}(\cdot, \tilde{\mathsf{y}}_k) \right\|_{L^s(\Omega)} \|\phi_{\mathbf{v}}^2 - \phi_{\mathbf{v}_k}^2\|_{L^t(\Omega)},
\end{aligned}$$

where $r^{-1} + s^{-1} + t^{-1} = 1$. Since a satisfies (2.4), there exists $s > 1$ if $d = 2$ and $s > 3$ if $d = 3$ such that the terms involving $\partial^2 a/\partial y^2$ are uniformly bounded in $L^s(\Omega)$. On the other hand, since $\bar{\mathsf{p}}, \tilde{\mathsf{p}}_k \in H^2(\Omega)$, we have $\bar{\mathsf{p}}, \tilde{\mathsf{p}}_k \in C(\bar{\Omega})$ so that $r = \infty$. Consequently, we must have $t < \infty$ if $d = 2$ and $t < 3/2$ if $d = 3$. Observe that $\phi_{\mathbf{v}}, \phi_{\mathbf{v}_k} \in W_0^{1,p}(\Omega)$, for $p < d/(d-1)$, so that $\phi_{\mathbf{v}}, \phi_{\mathbf{v}_k} \in L^{\mathfrak{t}}(\Omega)$, where $\mathfrak{t} < \infty$ if $d = 2$ and $\mathfrak{t} < 3$ if $d = 3$. On the other hand, since $\tilde{\mathsf{p}}_k \to \bar{\mathsf{p}}$ in $C(\bar{\Omega})$ and $\phi_{\mathbf{v}_k} \to \phi_{\mathbf{v}}$ in $W_0^{1,p}(\Omega)$ as $k \uparrow \infty$, we have

$$\|\bar{\mathsf{p}} - \tilde{\mathsf{p}}_k\|_{L^r(\Omega)} \to 0, \quad \|\phi_{\mathbf{v}}^2 - \phi_{\mathbf{v}_k}^2\|_{L^t(\Omega)} \to 0, \quad k \uparrow \infty.$$

We also have $\|\partial^2 a/\partial y^2(\cdot, \bar{\mathsf{y}}) - \partial^2 a/\partial y^2(\cdot, \tilde{\mathsf{y}}_k)\|_{L^s(\Omega)} \to 0$ as $k \uparrow \infty$, which follows from (2.4) and the Lebesgue dominated convergence theorem. All these arguments allow us to conclude that $\mathfrak{I}_k \to 0$ as $k \uparrow \infty$, which in turns implies (4.24).

[3] Since $\mathbf{v} = \mathbf{0}$, we have $\phi_{\mathbf{v}_k} \to 0$ in $W_0^{1,p}(\Omega)$, for $p < d/(d-1)$, as $k \uparrow \infty$. Consequently,

$$\alpha = \alpha \|\mathbf{v}_k\|_{\mathbb{R}^\ell}^2 = j''(\tilde{\mathbf{u}}_k)\mathbf{v}_k^2 - \int_\Omega \phi_{\mathbf{v}_k}^2 \mathrm{d}x + \int_\Omega \tilde{\mathsf{p}}_k \frac{\partial^2 a}{\partial y^2}(x, \tilde{\mathsf{y}}_k)\phi_{\mathbf{v}_k}^2 \mathrm{d}x.$$

Since $j''(\tilde{\mathbf{u}}_k)\mathbf{v}_k^2 < k^{-1} \to 0$ as $k \uparrow \infty$, we let $k \uparrow \infty$ in the previous expression to arrive at $\alpha \leq 0$, which is a contradiction. This concludes the proof. ∎

To present the following result, we define, for $\tau > 0$,

$$(4.26)\quad \mathbf{C}_{\bar{\mathbf{u}}}^\tau := \left\{ \mathbf{v} = \{\mathsf{v}_{\mathsf{z}}\}_{\mathsf{z}\in\mathcal{D}} \in \mathbb{R}^\ell \text{ satisfying (4.20) and } \mathsf{v}_{\mathsf{z}} = 0 \text{ if } |\psi_{\mathsf{z}}| > \tau,\ \mathsf{z} \in \mathcal{D} \right\}.$$

THEOREM 4.9 (equivalence). *Let $\bar{\mathbf{u}} \in \mathbb{U}_{ad}$, $\bar{\mathsf{y}} \in W_0^{1,p}(\Omega)$, and $\bar{\mathsf{p}} \in H_0^1(\Omega)$ satisfy the first order optimality conditions* (4.3), (4.10)*, and* (4.12)*. Assume that $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ denotes a Carathéodory function that satisfies* (A.1), (A.3), and (A.4)*. Thus, the following statements are equivalent:*

$$(4.27)\qquad j''(\bar{\mathbf{u}})\mathbf{v}^2 > 0 \quad \forall \mathbf{v} \in \mathbf{C}_{\bar{\mathbf{u}}} \setminus \{\mathbf{0}\}$$

and

$$\exists \kappa, \tau > 0 : \quad j''(\bar{\mathbf{u}})\mathbf{v}^2 \geq \kappa \|\mathbf{v}\|_{\mathbb{R}^\ell}^2 \quad \forall \mathbf{v} \in \mathbf{C}_{\bar{\mathbf{u}}}^{\tau}. \tag{4.28}$$

Proof. The proof follows similar arguments to the ones developed in the proofs of Theorems 4.7 and 4.8. For brevity, we skip details. ∎

5. Finite element approximation of the optimal control problem. In this section, we propose and analyze a finite element discretization scheme for the control problem (4.2)–(4.3). We analyze convergence properties and derive error estimates. We begin our studies by providing convergence results related to the finite element discretization of the state equation (4.3).

5.1. The discrete state equation: convergence properties. We introduce the following finite element approximation of problem (4.3): Find $y_h \in \mathbb{V}_h$ such that

$$\int_\Omega \nabla y_h \cdot \nabla v_h \mathrm{d}x + \int_\Omega a(x, y_h) v_h \mathrm{d}x = \sum_{\mathsf{z} \in \mathcal{D}} \mathsf{u}_{\mathsf{z}} v_h(\mathsf{z}) \quad \forall v_h \in \mathbb{V}_h. \tag{5.1}$$

We recall that the discrete space $\mathbb{V}_h$ is defined in (3.1).

We present the following convergence result.

THEOREM 5.1 (convergence properties). *Let Ω be an open, bounded, and convex polytopal domain. Let $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies* (A.1) and (A.2). *Assume, in addition, that* (3.3) *holds. Let $\mathsf{y} \in W_0^{1,p}(\Omega)$, for $p < d/(d-1)$, be the solution to* (4.3) *and let $\mathsf{y}_h \in \mathbb{V}_h$ be the solution to*

$$\int_\Omega \nabla \mathsf{y}_h \cdot \nabla v_h \mathrm{d}x + \int_\Omega a(x, \mathsf{y}_h) v_h \mathrm{d}x = \sum_{\mathsf{z} \in \mathcal{D}} \mathsf{u}_{\mathsf{z},h} v_h(\mathsf{z}) \quad \forall v_h \in \mathbb{V}_h, \tag{5.2}$$

where $\mathbf{u}_h = \{\mathsf{u}_{\mathsf{z},h}\}_{\mathsf{z} \in \mathcal{D}}$. If $\mathbf{u}_h \to \mathbf{u}$ in $\mathbb{R}^\ell$, then $\mathsf{y}_h \to \mathsf{y}$ in $L^2(\Omega)$ as $h \downarrow 0$.

Proof. We begin with a simple application of the triangle inequality to obtain

$$\|\mathsf{y} - \mathsf{y}_h\|_{L^2(\Omega)} \leq \|\mathsf{y} - y_h\|_{L^2(\Omega)} + \|y_h - \mathsf{y}_h\|_{L^2(\Omega)}.$$

Since y_h corresponds to the finite element approximation of y within $\mathbb{V}_h$, the arguments elaborated in the proof of Theorem 3.1 allow us to conclude that $\|\mathsf{y} - y_h\|_{L^2(\Omega)} \to 0$ as $h \downarrow 0$. To control $\|y_h - \mathsf{y}_h\|_{L^2(\Omega)}$, we first observe that $y_h - \mathsf{y}_h \in \mathbb{V}_h$ is such that

$$\int_\Omega \nabla (y_h - \mathsf{y}_h) \cdot \nabla v_h \mathrm{d}x + \int_\Omega [a(x, y_h) - a(x, \mathsf{y}_h)] v_h \mathrm{d}x = \sum_{\mathsf{z} \in \mathcal{D}} [\mathsf{u}_{\mathsf{z}} - \mathsf{u}_{\mathsf{z},h}] v_h(\mathsf{z})$$

for all $v_h \in \mathbb{V}_h$. We write $a(x, y_h) - a(x, \mathsf{y}_h) = \partial a / \partial y (x, \zeta_h)(y_h - \mathsf{y}_h)$, where $\zeta_h := \mathsf{y}_h + \theta_h (y_h - \mathsf{y}_h)$ and $\theta_h \in (0, 1)$. Define $\mathfrak{y}$ as the solution to the problem: Find $\mathfrak{y} \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \nabla \mathfrak{y} \cdot \nabla v \mathrm{d}x + \int_\Omega \tfrac{\partial a}{\partial y}(x, \zeta_h) \mathfrak{y} v \mathrm{d}x = \sum_{\mathsf{z} \in \mathcal{D}} [\mathsf{u}_{\mathsf{z}} - \mathsf{u}_{\mathsf{z},\mathsf{h}}] v(\mathsf{z}) \quad \forall v \in W_0^{1,p'}(\Omega),$$

where $p < d/(d-1)$. We estimate $\|y_h - \mathsf{y}_h\|_{L^2(\Omega)}$ as follows: $\|y_h - \mathsf{y}_h\|_{L^2(\Omega)} \leq \|\mathfrak{y} - (y_h - \mathsf{y}_h)\|_{L^2(\Omega)} + \|\mathfrak{y}\|_{L^2(\Omega)}$. The stability estimate in [23, Theorem 9.1] immediately yields the bound

$$\|\nabla \mathfrak{y}\|_{L^p(\Omega)} \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{R}^\ell} \cdot \sum_{\mathsf{z} \in \mathcal{D}} \|\delta_{\mathsf{z}}\|_{W^{-1,p}(\Omega)} \to 0, \qquad h \downarrow 0.$$

Observe that, by assumption, there exists $q > d/2$ such that $\partial a/\partial y(\cdot, \zeta_h) \in L^q(\Omega)$. Notice that, a discrete inf-sup condition, which follows from [4, Theorem 8.5.3] and requires the convexity of Ω, guarantees that $\|\nabla \zeta_h\|_{L^p(\Omega)} < \infty$ for $p < d/(d-1)$. On the other hand, since $y_h - \mathsf{y}_h$ corresponds to the finite element approximation of $\mathfrak{y}$ within $\mathbb{V}_h$, basic arguments reveal that $\|\mathfrak{y} - (y_h - \mathsf{y}_h)\|_{L^2(\Omega)} \to 0$ as $h \downarrow 0$. All of our previous findings allow us to conclude that $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{R}^\ell} \to 0$ implies that $\|\mathsf{y} - \mathsf{y}_h\|_{L^2(\Omega)} \to 0$ as $h \downarrow 0$. This concludes the proof. ∎

5.2. The discrete adjoint equation: an error estimate. We introduce the following finite element approximation of problem (4.10): Find $p_h \in \mathbb{V}_h$ such that

$$(5.3) \qquad \int_\Omega \nabla p_h \cdot \nabla v_h \mathrm{d}x + \int_\Omega \frac{\partial a}{\partial y}(x, \mathsf{y}) p_h v_h \mathrm{d}x = \int_\Omega (\mathsf{y} - \mathsf{y}_d) v_h \mathrm{d}x \quad \forall v_h \in \mathbb{V}_h.$$

THEOREM 5.2 (error estimate). *Let Ω be an open, bounded, and convex polytopal domain. Let $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies (A.1) and (A.3). Let us assume that, in addition, there exists $\phi_1 \in L^q(\Omega)$, with $q > 2$ if $d = 2$ and $q > 6$ if $d = 3$, such that*

$$(5.4) \qquad 0 \leq \frac{\partial a}{\partial y}(x, y) \leq |\phi_1(x)| + C_a |y|^r \quad \text{a.e. } x \in \Omega, \ \forall y \in \mathbb{R}.$$

Here, $C_a > 0$ is a constant, $r < \infty$ if $d = 2$, and $r < 1/2$ if $d = 3$. Let $\Omega_1 \Subset \Omega_0 \Subset \Omega$ with Ω_0 smooth. If, in addition, $\mathsf{y}_d \in L^t(\Omega_0)$ and $\partial a/\partial y(\cdot, y) \in L^t(\Omega_0)$, for every $y \in \mathbb{R}$, where $t < \infty$ if $d = 2$ and $t < 3$ if $d = 3$, then we have the following local error estimates in maximum–norm:

$$(5.5) \qquad \|\mathsf{p} - p_h\|_{L^\infty(\Omega_1)} \lesssim h^2 |\log h|^2, \qquad \|\mathsf{p} - p_h\|_{L^\infty(\Omega_1)} \lesssim h^{1-\epsilon} |\log h|,$$

for $d = 2$ and $d = 3$, respectively. Here, $\epsilon > 0$ is arbitrarily small. In both inequalities the hidden constant is independent of h.

Proof. We begin with a simple application of the triangle inequality and write

$$\|\mathsf{p} - p_h\|_{L^\infty(\Omega_1)} \leq \|\mathsf{p} - q_h\|_{L^\infty(\Omega_1)} + \|q_h - p_h\|_{L^\infty(\Omega_1)},$$

where $q_h \in \mathbb{V}_h$ solves (5.3) with $\partial a/\partial y(\cdot, \mathsf{y}) p_h$ replaced by $\partial a/\partial y(\cdot, \mathsf{y})\mathsf{p}$. A key ingredient in favor of the definition of q_h is that $(\nabla(\mathsf{p} - q_h), \nabla v_h)_{L^2(\Omega)} = 0$ for every $v_h \in \mathbb{V}_h$. Let Λ_1 be a smooth domain such that $\Omega_1 \Subset \Lambda_1 \Subset \Omega_0$. Invoke [22, Corollary 5.1], [2, Proposition 4.3] to conclude the existence of $h_0 \in (0, 1)$ such that, for $v_h \in \mathbb{V}_h$,

$$\|\mathsf{p} - q_h\|_{L^\infty(\Omega_1)} \lesssim |\log h| \|\mathsf{p} - v_h\|_{L^\infty(\Lambda_1)} + l^{-\frac{d}{2}} \|\mathsf{p} - q_h\|_{L^2(\Lambda_1)}, \quad h \leq h_0.$$

Here, l is such that $\mathrm{dist}(\Omega_1, \partial\Lambda_1) \geq l$, $\mathrm{dist}(\Lambda_1, \partial\Omega) \geq l$, and $Ch \leq l$, where $C > 0$. The regularity results of Proposition 4.3 guarantee that $\mathsf{p} \in H^2(\Omega) \cap W^{2,t}(\Lambda_1)$ for t as in the statement of that theorem. We can thus obtain the error estimate

$$(5.6) \qquad \begin{aligned} \|\mathsf{p} - q_h\|_{L^\infty(\Omega_1)} &\leq C_1 |\log h| h^{2-\frac{d}{t}} |\nabla^2 \mathsf{p}|_{L^t(\Lambda_1)} + C_2 h^2 |\mathsf{p}|_{H^2(\Omega)} \\ &\leq C_3 |\log h| t h^{2-\frac{d}{t}} \mathfrak{C} + C_2 h^2 |\mathsf{p}|_{H^2(\Omega)}, \end{aligned}$$

where C_1, C_2 and C_3 are positive constants that are independent of h, t, and p and $\mathfrak{C} = \mathfrak{C}(\mathsf{y}, \mathsf{y}_d, a, \mathsf{p})$ collects the terms appearing in the right hand side of estimate (4.11). Set, in two dimensions, $t = |\log h|$ and obtain $\|\mathsf{p} - q_h\|_{L^\infty(\Omega_1)} \lesssim h^2 |\log h|^2$. In three dimensions, $t < 3$, so that we obtain $\|\mathsf{p} - q_h\|_{L^\infty(\Omega_1)} \lesssim h^{1-\epsilon} |\log h|$ for every $\epsilon > 0$.

We now control $\|q_h - p_h\|_{L^\infty(\Omega_1)}$. To accomplish this task, we first notice that

$$q_h - p_h \in \mathbb{V}_h : \quad \int_\Omega \nabla(q_h - p_h) \cdot \nabla v_h \mathrm{d}x = \int_\Omega \tfrac{\partial a}{\partial y}(x, \mathsf{y})(p_h - \mathsf{p}) v_h \mathrm{d}x \quad \forall v_h \in \mathbb{V}_h.$$

Let $\mathfrak{p}$ be the solution to the associated continuous problem, i.e.,

$$\mathfrak{p} \in H_0^1(\Omega) : \quad \int_\Omega \nabla \mathfrak{p} \cdot \nabla v \mathrm{d}x = \int_\Omega \tfrac{\partial a}{\partial y}(x, \mathsf{y})(p_h - \mathsf{p}) v \mathrm{d}x \quad \forall v \in H_0^1(\Omega).$$

A basic Sobolev embedding combined with a stability estimate reveal that

$$\|\mathfrak{p}\|_{L^\infty(\Omega)} \lesssim \|\nabla \mathfrak{p}\|_{L^r(\Omega)} \lesssim \|\tfrac{\partial a}{\partial y}(\cdot, \mathsf{y})\|_{L^q(\Omega)} \|p_h - \mathsf{p}\|_{L^2(\Omega)},$$

where $r > d$ and q is as in the statement of the theorem. We thus control $\|q_h - p_h\|_{L^\infty(\Omega_1)}$ as follows: $\|q_h - p_h\|_{L^\infty(\Omega_1)} \leq \|(q_h - p_h) - \mathfrak{p}\|_{L^\infty(\Omega_1)} + \|\mathfrak{p}\|_{L^\infty(\Omega_1)}$. Since $q_h - p_h$ corresponds to the finite element approximation of $\mathfrak{p}$ within $\mathbb{V}_h$, we obtain

$$\|(q_h - p_h) - \mathfrak{p}\|_{L^\infty(\Omega_1)} \lesssim h^{1-d/r} \|\nabla \mathfrak{p}\|_{L^r(\Omega)} \lesssim h^{1-d/r} \|p_h - \mathsf{p}\|_{L^2(\Omega)}.$$

Consequently, $\|q_h - p_h\|_{L^\infty(\Omega_1)} \lesssim \|p_h - \mathsf{p}\|_{L^2(\Omega)} \lesssim h^2 |\mathsf{p}|_{H^2(\Omega)}$. Combining this estimate with (5.6) yield (5.5). This concludes the proof. ∎

5.3. An error estimate in L^1. In this section, we follow the ideas developed in [20, Theorem 4.1] and [17, Lemma 4.2] and derive an error estimate in $L^1(\Omega)$ for the finite element approximation of the *semilinear state equation* (4.3). Notice that, since $D \subset \Omega$ and D is finite, $\mathrm{dist}(\mathcal{D}, \partial\Omega) > 0$ so that we can conclude the existence of smooth subdomains Ω_0 and Ω_1 such that $\mathcal{D} \subset \Omega_1 \Subset \Omega_0 \Subset \Omega$.

THEOREM 5.3 (a priori error estimate in $L^1(\Omega)$). *Let Ω be an open, bounded, and convex polytopal domain. Let $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies (A.1) and (A.3). Assume, in addition, that a satisfies $a(\cdot, 0) \in L^2(\Omega)$ and (3.3) with $\psi \in L^s(\Omega)$, $s > 2$ if $d = 2$, and $s > 6$ if $d = 3$. Let Ω_0 and Ω_1 be smooth subdomains such that $\mathcal{D} \subset \Omega_1 \Subset \Omega_0 \Subset \Omega$. If $\psi \in L^t(\Omega_0)$, for every $t < \infty$, then we have the following error estimate for h sufficiently small:*

$$\|y - y_h\|_{L^1(\Omega)} \lesssim h^2 |\log h|^2, \tag{5.7}$$

with a hidden constant that is independent of h.

Proof. Define χ, $\mathfrak{z}$, and $\mathfrak{z}_h$ as in the proof of Theorem 3.1. Set $\mathfrak{f} = \mathrm{sgn}(\mathsf{y} - y_h) \in L^\infty(\Omega)$ as a forcing term in the problem that $\mathfrak{z}$ solves. A density argument yields

$$\|y - y_h\|_{L^1(\Omega)} = (\nabla(y - y_h), \nabla(\mathfrak{z} - \mathfrak{z}_h))_{L^2(\Omega)} + (a(\cdot, y) - a(\cdot, y_h), \mathfrak{z} - \mathfrak{z}_h)_{L^2(\Omega)}.$$

The arguments elaborated in the proof of proof of Theorem 3.1 thus reveal that

$$\|y - y_h\|_{L^1(\Omega)} = \sum_{\mathsf{z}} \mathsf{u}_{\mathsf{z}}(\mathfrak{z}(\mathsf{z}) - \mathfrak{z}_h(\mathsf{z})) + (\chi(\mathfrak{z} - \mathfrak{z}_h), y_h)_{L^2(\Omega)} - (a(\cdot, y_h), \mathfrak{z} - \mathfrak{z}_h). \tag{5.8}$$

We first estimate the second and the third terms on the right hand side of the previous expression upon exploiting the $H^2(\Omega)$-regularity of $\mathfrak{z}$. In fact, we have

$$\begin{aligned} &|(\chi(\mathfrak{z} - \mathfrak{z}_h), y_h)_{L^2(\Omega)} - (a(\cdot, y_h), \mathfrak{z} - \mathfrak{z}_h)| \\ &\qquad \leq \left(\|\chi y_h\|_{L^2(\Omega)} + \|a(\cdot, y_h)\|_{L^2(\Omega)}\right) \|\mathfrak{z} - \mathfrak{z}_h\|_{L^2(\Omega)} \lesssim h^2 |\mathfrak{z}|_{H^2(\Omega)}. \end{aligned}$$

Notice that, since $y_h \in W_0^{1,p}(\Omega)$ and $\chi \in L^s(\Omega)$ with $s > 2$ if $d = 2$ and $s > 6$ if $d = 3$, we can immediately deduce that $\chi y_h \in L^2(\Omega)$. On the other hand, observe that the term $\|a(\cdot, y_h)\|_{L^2(\Omega)}$ is uniformly bounded. It thus suffices to estimate the first term on the right hand of (5.8). Observe that $\mathfrak{z}$ is such that

$$(\nabla v, \nabla \mathfrak{z})_{L^2(\Omega)} = (\mathrm{sgn}(\mathsf{y} - \mathscr{y}_h) - \chi \mathfrak{z}, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega).$$

We can thus invoke [2, Lemma 4.2] to conclude that

$$\|\mathfrak{z}\|_{W^{2,t}(\Omega_1)} \leq C_t \left(\left\| \mathrm{sgn}(\mathsf{y} - \mathscr{y}_h) - \chi \mathfrak{z} \right\|_{L^t(\Omega_0)} + \left\| \mathrm{sgn}(\mathsf{y} - \mathscr{y}_h) - \chi \mathfrak{z} \right\|_{L^2(\Omega)} \right),$$

where C_t behaves as Ct, with $C > 0$, as $t \uparrow \infty$. A local argument as the one developed in the proof of Theorem 5.2 yields the bound

$$\left| \sum_{\mathsf{z}} \mathsf{u}_{\mathsf{z}} (\mathfrak{z}(\mathsf{z}) - \mathfrak{z}_h(\mathsf{z})) \right| \lesssim \|\mathbf{u}\|_{\mathbb{R}^\ell} \|\mathfrak{z} - \mathfrak{z}_h\|_{L^\infty(\Omega_1)} \lesssim h^2 |\log h|^2 \|\mathbf{u}\|_{\mathbb{R}^\ell}.$$

This concludes the proof. ◻

5.4. The discrete optimal control problem. We propose the following finite element discretization of the optimal control problem (4.2)–(4.3): Find

$$\min\{J(\mathsf{y}_h, \mathbf{u}_h) : (\mathsf{y}_h, \mathbf{u}_h) \in \mathbb{V}_h \times \mathbb{U}_{ad}\} \tag{5.9}$$

subject to the discrete state equation: Find $\mathsf{y}_h \in \mathbb{V}_h$ such that

$$\int_\Omega \nabla \mathsf{y}_h \cdot \nabla v_h \mathrm{d}x + \int_\Omega a(x, \mathsf{y}_h) v_h \mathrm{d}x = \sum_{\mathsf{z} \in \mathcal{D}} \mathsf{u}_{\mathsf{z},h} v_h(\mathsf{z}) \qquad \forall v_h \in \mathbb{V}_h. \tag{5.10}$$

The existence of at least one solution for the previously defined optimal control problem follows from the arguments developed in the proof of Theorem 4.1. Let us introduce the discrete control to state map $\mathcal{S}_h : \mathbb{U}_{ad} \ni \mathbf{u}_h \mapsto \mathsf{y}_h \in \mathbb{V}_h$, where y_h denotes the solution to (5.10), and define the reduced cost functional $j_h : \mathbb{U}_{ad} \ni \mathbf{u}_h \mapsto J(\mathcal{S}_h \mathbf{u}_h, \mathbf{u}_h) \in \mathbb{R}$. With these ingredients at hand, we formulate the following first order optimality condition: every discrete locally optimal control $\bar{\mathbf{u}}_h \in \mathbb{U}_{ad}$ satisfies

$$j_h'(\bar{\mathbf{u}}_h)(\mathbf{u}_h - \bar{\mathbf{u}}_h) \geq 0 \quad \forall \mathbf{u}_h = \{u_{\mathsf{z},h}\}_{\mathsf{z} \in \mathcal{D}} \in \mathbb{U}_{ad}. \tag{5.11}$$

This variational inequality leads to the following projection formula: If $\bar{\mathbf{u}}_h$ denotes a local minimizer of the discrete optimal control problem, then, for every $\mathsf{z} \in \mathcal{D}$,

$$\bar{\mathsf{u}}_{\mathsf{z},h} := \Pi_{[\mathsf{a}_\mathsf{z}, \mathsf{b}_\mathsf{z}]} \left(-\alpha^{-1} \bar{\mathsf{p}}_h(\mathsf{z}) \right), \tag{5.12}$$

where, for $t \in \mathbb{R}$, $\Pi_{[\mathsf{a}_\mathsf{z}, \mathsf{b}_\mathsf{z}]}(t) := \max\{\mathsf{a}_\mathsf{z}, \min\{\mathsf{b}_\mathsf{z}, t\}\}$. Here, $\bar{\mathsf{p}}_h$ denotes the solution to the following discrete adjoint problem: Find $\bar{\mathsf{p}}_h \in \mathbb{V}_h$ such that

$$\int_\Omega \nabla \bar{\mathsf{p}}_h \cdot \nabla v_h \mathrm{d}x + \int_\Omega \tfrac{\partial a}{\partial y}(x, \bar{\mathsf{y}}_h) \bar{\mathsf{p}}_h v_h \mathrm{d}x = \int_\Omega (\bar{\mathsf{y}}_h - \mathsf{y}_d) v_h \mathrm{d}x \quad \forall v_h \in \mathbb{V}_h. \tag{5.13}$$

5.5. An auxiliary error estimate. In this section, we derive an auxiliary error estimate that will be of fundamental importance to perform an a priori error analysis for the previously introduced discretization of our optimal control problem.

THEOREM 5.4 (error estimate). *Let Ω be an open, bounded, and convex polytopal domain. Let $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies (A.1), (A.3), and (5.4). Assume, in addition, that $\partial a/\partial y$ is locally Lipschitz in y a.e. in Ω and that a satisfies $a(\cdot, 0) \in L^2(\Omega)$ and (3.3) with $\psi \in L^s(\Omega)$, $s > 2$ if $d = 2$, and $s > 6$ if $d = 3$. Let Ω_0 and Ω_1 be smooth subdomains such that $\mathcal{D} \subset \Omega_1 \Subset \Omega_0 \Subset \Omega$. If, in addition, $\mathsf{y}_d, \psi \in L^t(\Omega_0)$ and $\partial a/\partial y(\cdot, y) \in L^t(\Omega_0)$, for every $y \in \mathbb{R}$ and $t < \infty$, then we have the following error estimates:*

$$|(j'(\mathbf{u}) - j_h'(\mathbf{u}))\mathbf{v}| \lesssim h^2 |\log h|^3 \|\mathbf{v}\|_{\mathbb{R}^\ell}, \qquad d = 2, \tag{5.14}$$

and

$$|(j'(\mathbf{u}) - j_h'(\mathbf{u}))\mathbf{v}| \lesssim h^{1-\epsilon} |\log h| \|\mathbf{v}\|_{\mathbb{R}^\ell}, \qquad d = 3. \tag{5.15}$$

Here, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^\ell$. The hidden constant, in both inequalities, is independent of h.

Proof. We begin the proof by noticing that

$$j'(\mathbf{u})\mathbf{v} = \sum_{\mathsf{z} \in \mathcal{D}} (\mathsf{p}(\mathsf{z}) + \alpha \mathsf{u}_\mathsf{z})\mathsf{v}_\mathsf{z}, \qquad j_h'(\mathbf{u})\mathbf{v} = \sum_{\mathsf{z} \in \mathcal{D}} (\hat{\mathsf{p}}_h(\mathsf{z}) + \alpha \mathsf{u}_\mathsf{z})\mathsf{v}_\mathsf{z},$$

where $\mathsf{p} \in H_0^1(\Omega) \cap H^2(\Omega)$ solves (4.10) with $\mathsf{y} = \mathcal{S}\mathbf{u}$ and $\hat{\mathsf{p}}_h$ solves (5.3) with y being replaced by $\mathscr{y}_h$, i.e., the solution to (5.1). With these identities at hand, we write

$$|(j'(\mathbf{u}) - j_h'(\mathbf{u}))\mathbf{v}| \leq \sum_{\mathsf{z} \in \mathcal{D}} \left[|\mathsf{p}(\mathsf{z}) - \mathscr{p}_h(\mathsf{z})| + |\mathscr{p}_h(\mathsf{z}) - \hat{\mathsf{p}}_h(\mathsf{z})| \right] |\mathsf{v}_\mathsf{z}| =: \sum_{\mathsf{z} \in \mathcal{D}} \left[\mathrm{I}_\mathsf{z} + \mathrm{II}_\mathsf{z} \right] |\mathsf{v}_\mathsf{z}|,$$

where $\mathscr{p}_h$ denotes the solution to (5.3); notice that $\mathscr{p}_h$ corresponds to the finite element approximation of p within $\mathbb{V}_h$. Let $\mathsf{z} \in \mathcal{D}$. Invoke Theorem 5.2 to immediately arrive at the bounds $\mathrm{I}_\mathsf{z} \lesssim h^2 |\log h|^2$ and $\mathrm{I}_\mathsf{z} \lesssim h^{1-\epsilon} |\log h|$, for $d = 2$ and $d = 3$, respectively; $\epsilon > 0$ being arbitrarily small. We now control, for $\mathsf{z} \in \mathcal{D}$, the term II_z. To accomplish this task, we first observe that $\mathscr{p}_h - \hat{\mathsf{p}}_h \in \mathbb{V}_h$ is such that

$$\begin{aligned} (\nabla(\mathscr{p}_h - \hat{\mathsf{p}}_h), \nabla v_h)_{L^2(\Omega)} &+ \left(\tfrac{\partial a}{\partial y}(\cdot, \mathsf{y})(\mathscr{p}_h - \hat{\mathsf{p}}_h), v_h \right)_{L^2(\Omega)} \\ &= \left(\mathsf{y} - \mathscr{y}_h, v_h \right)_{L^2(\Omega)} + \left(\left(\tfrac{\partial a}{\partial y}(\cdot, \mathscr{y}_h) - \tfrac{\partial a}{\partial y}(\cdot, \mathsf{y}) \right) \hat{\mathsf{p}}_h, v_h \right)_{L^2(\Omega)} \quad \forall v_h \in \mathbb{V}_h. \end{aligned}$$

We thus first utilize an inverse inequality and then a stability estimate for the previous problem to obtain

$$\begin{aligned} \|\mathscr{p}_h - \hat{\mathsf{p}}_h\|_{L^\infty(\Omega)}^2 \lesssim \mathfrak{i}_h^2 \|\nabla(\mathscr{p}_h - \hat{\mathsf{p}}_h)\|_{L^2(\Omega)}^2 &\lesssim \mathfrak{i}_h^2 \|\mathsf{y} - \mathscr{y}_h\|_{L^1(\Omega)} \|\mathscr{p}_h - \hat{\mathsf{p}}_h\|_{L^\infty(\Omega)} \\ &+ \mathfrak{i}_h^2 \left\| \left(\tfrac{\partial a}{\partial y}(\cdot, \mathscr{y}_h) - \tfrac{\partial a}{\partial y}(\cdot, \mathsf{y}) \right) \hat{\mathsf{p}}_h \right\|_{L^1(\Omega)} \|\mathscr{p}_h - \hat{\mathsf{p}}_h\|_{L^\infty(\Omega)}, \end{aligned} \tag{5.16}$$

where $\mathfrak{i}_h = (1 + |\log h|)^{\frac{1}{2}}$ if $d = 2$ [4, Lemma 4.9.2] and $\mathfrak{i}_h = h^{-\frac{1}{2}}$ if $d = 3$ [11, Lemma 1.142]. Invoke the Lipschitz continuity of $\frac{\partial a}{\partial y}(\cdot, \mathsf{y})$ and then the error estimate (5.7) to obtain

$$\|\mathscr{p}_h - \hat{\mathsf{p}}_h\|_{L^\infty(\Omega)} \lesssim \mathfrak{i}_h^2 \left(1 + \|\hat{\mathsf{p}}_h\|_{L^\infty(\Omega)} \right) \|\mathsf{y} - \mathscr{y}_h\|_{L^1(\Omega)} \lesssim \mathfrak{i}_h^2 h^2 |\log h|^2.$$

Collect the derived estimates for I_z and II_z to obtain (5.14) and (5.15). ∎

5.6. Convergence of discretizations. We now provide a convergence result that, in essence, guarantees that a sequence of global solutions $\{\bar{\mathbf{u}}_h\}_{h>0}$ of discrete optimal control problems admits nonrelabeled subsequences that converge, as $h \downarrow 0$, to global solutions of the continuous optimal control problem; see, for instance, [9].

THEOREM 5.5 (convergence of discretizations). *Let Ω be an open, bounded, and convex polytopal domain. Let $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies* (A.1) and (A.2). *Assume, in addition, that* (3.3) *holds. Let $\bar{\mathbf{u}}_h$, for $h > 0$, be a global solution of the discrete optimal control problem* (5.9)–(5.10)*. Then, there exist nonrelabeled subsequences $\{\bar{\mathbf{u}}_h\}_{h>0}$ such that $\bar{\mathbf{u}}_h \to \bar{\mathbf{u}}$ in $\mathbb{R}^\ell$, as $h \downarrow 0$, with $\bar{\mathbf{u}}$ being a global solution to* (4.2)–(4.3)*. In addition, we have* $\lim_{h\downarrow 0} j(\bar{\mathbf{u}}_h) = j(\bar{\mathbf{u}})$.

Proof. Since $\{\bar{\mathbf{u}}_h\}_{h>0} \subset \mathbb{U}_{ad}$, the sequence $\{\bar{\mathbf{u}}_h\}_{h>0}$ is uniformly bounded in $\mathbb{R}^\ell$. Then, there exists a nonrelabeled subsequence $\{\bar{\mathbf{u}}_h\}_{h>0}$ such that $\bar{\mathbf{u}}_h \to \bar{\mathbf{u}}$ as $h \downarrow 0$. In what follows, we prove that $\bar{\mathbf{u}}$ is a global solution to (4.2)–(4.3).

Let $\tilde{\mathbf{u}} \in \mathbb{U}_{ad}$ be a global solution to (4.2)–(4.3) and let $\{\tilde{\mathbf{u}}_h\}_{h>0} \subset \mathbb{U}_{ad}$ be such that $\tilde{\mathbf{u}}_h \to \tilde{\mathbf{u}}$ in $\mathbb{R}^\ell$ as $h \downarrow 0$. Since $\tilde{\mathbf{u}}$ is optimal for (4.2)–(4.3) and $\bar{\mathbf{u}}_h$, for $h > 0$, is a global solution of (5.9)–(5.10), we obtain

$$j(\tilde{\mathbf{u}}) \leq j(\bar{\mathbf{u}}) = \lim_{h\downarrow 0} j(\bar{\mathbf{u}}_h) \leq \lim_{h\downarrow 0} j(\tilde{\mathbf{u}}_h) = j(\tilde{\mathbf{u}}). \tag{5.17}$$

Notice that, since $\bar{\mathbf{u}}_h \to \bar{\mathbf{u}}$ and $\tilde{\mathbf{u}}_h \to \tilde{\mathbf{u}}$ in $\mathbb{R}^\ell$ as $h \downarrow 0$, Theorem 5.1 yield $\mathcal{S}_h\bar{\mathbf{u}}_h \to \mathcal{S}\bar{\mathbf{u}}$ and $\mathcal{S}_h\tilde{\mathbf{u}}_h \to \mathcal{S}\tilde{\mathbf{u}}$ in $L^2(\Omega)$ as $h \downarrow 0$. Consequently, it is immediate that $j_h(\bar{\mathbf{u}}_h) \to j(\bar{\mathbf{u}})$ and that $j_h(\tilde{\mathbf{u}}_h) \to j(\tilde{\mathbf{u}})$ as $h \downarrow 0$. In view of (5.17), we thus conclude that $\bar{\mathbf{u}}$ is a global solution to (4.2)–(4.3). This concludes the proof. ∎

In the result that follows, we prove that every strict local minimum of the continuous optimal control problem (4.2)–(4.3) can be approximated by local minima of the discrete optimal control problems (5.9)–(5.10).

THEOREM 5.6 (convergence of discretizations). *Let Ω be an open, bounded, and convex polytopal domain. Let $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies* (A.1) and (A.2). *Assume, in addition, that* (3.3) *holds. If $\bar{\mathbf{u}}$ denotes a strict local minimum of problem* (4.2)–(4.3)*, then there exists a sequence $\{\bar{\mathbf{u}}_h\}_{h>0}$, of local minima of the discrete optimal control problems, such that $\bar{\mathbf{u}}_h \to \bar{\mathbf{u}}$ in $\mathbb{R}^\ell$ as $h \downarrow 0$. In addition, we have* $\lim_{h\downarrow 0} j(\bar{\mathbf{u}}_h) = j(\bar{\mathbf{u}})$.

Proof. Since $\bar{\mathbf{u}}$ denotes a strict local minimum of problem (4.2)–(4.3), there exists $\epsilon > 0$ such that $\bar{\mathbf{u}}$ is the unique solution to $\min\{j(\mathbf{u}) : \mathbf{u} \in \mathbb{U}_{ad} \cap B_\epsilon(\bar{\mathbf{u}})\}$, where $B_\epsilon(\bar{\mathbf{u}}) := \{\mathbf{u} \in \mathbb{U}_{ad} : \|\mathbf{u} - \bar{\mathbf{u}}\|_{\mathbb{R}^\ell} \leq \epsilon\}$. Let us now consider the discrete problems:

$$\min\{j_h(\mathbf{u}) : \mathbf{u} \in \mathbb{U}_{ad} \cap B_\epsilon(\bar{\mathbf{u}})\}. \tag{5.18}$$

In view of the fact that $\mathbb{U}_{ad} \cap B_\epsilon(\bar{\mathbf{u}}) \neq \emptyset$, problem (5.18) admits at least one solution. Let $\bar{\mathbf{u}}_h \in \mathbb{U}_{ad}$, for $h > 0$, be a solution to (5.18). The arguments elaborated in the proof of Theorem 5.5 reveal the existence of nonrelabeled subsequences $\{\bar{\mathbf{u}}_h\}_{h>0}$ that converge to solutions to $\min\{j(\mathbf{u}) : \mathbf{u} \in \mathbb{U}_{ad} \cap B_\epsilon(\bar{\mathbf{u}})\}$. Since the latter minimization problems admits a unique solution, we must have that $\bar{\mathbf{u}}_h \to \bar{\mathbf{u}}$, for the whole sequence, as $h \downarrow 0$. Observe that the constraint $\bar{\mathbf{u}}_h \in B_\epsilon(\bar{\mathbf{u}})$ is thus not active for h sufficiently small. Consequently, $\bar{\mathbf{u}}_h$ is a solution to the discrete optimal control problem (5.9)–(5.10). This concludes the proof. ∎

5.7. Error estimates for the discrete optimal control problem. Let $\bar{\mathbf{u}}$ be a local minimum of the continuous optimal control problem and let $\{\bar{\mathbf{u}}_h\}_{h>0}$ be a sequence of local minima of the discrete optimal control problems such that $\|\bar{\mathbf{u}} - \bar{\mathbf{u}}_h\|_{\mathbb{R}^\ell} \to 0$ as $h \downarrow 0$; see Theorems 5.5 and 5.6. In what follows, we derive an error estimate for $\bar{\mathbf{u}} - \bar{\mathbf{u}}_h$ in $\mathbb{R}^\ell$. We begin our analysis with an instrumental error estimate.

PROPOSITION 5.7 (instrumental error estimate). *Let Ω be open, bounded, and convex polytopal domain. Let $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies* (A.1), (A.3), (A.4), and (3.3). *If $\bar{\mathbf{u}} \in \mathbb{U}_{ad}$ is a local minimum of the optimal control problem* (4.2)–(4.3) *that satisfies the second order optimality condition* (4.27)*, or equivalently* (4.28)*, then there exists $h_0 > 0$ such that*

$$\frac{\kappa}{2}\|\bar{\mathbf{u}}_h - \bar{\mathbf{u}}\|^2_{\mathbb{R}^\ell} \leq \left(j'(\bar{\mathbf{u}}_h) - j'(\bar{\mathbf{u}})\right)(\bar{\mathbf{u}}_h - \bar{\mathbf{u}}) \tag{5.19}$$

for every $h < h_0$.

Proof. We proceed on the basis of two steps.

Step 1. We first prove that, for $h > 0$ sufficiently small, the term $\bar{\mathbf{u}}_h - \bar{\mathbf{u}}$ belongs to $\mathbf{C}^\tau_{\bar{\mathbf{u}}}$ for some $\tau > 0$. Since $\bar{\mathbf{u}}_h \in \mathbb{U}_{ad}$, it is immediate that $\bar{\mathbf{u}}_h - \bar{\mathbf{u}}$ satisfies the sign condition (4.20). It thus suffices to verify the remaining condition in (4.26). To accomplish this task, we introduce the discrete variable Ψ_h as follows:

$$\Psi_h := \{\psi_{\mathsf{z},h}\}_{\mathsf{z}\in\mathcal{D}} \in \mathbb{R}^\ell, \qquad \psi_{\mathsf{z},h} := \bar{\mathsf{p}}_h(\mathsf{z}) + \alpha\bar{\mathsf{u}}_{\mathsf{z},h}.$$

Since $\bar{\mathbf{u}}_h \to \bar{\mathbf{u}}$ in $\mathbb{R}^\ell$, the results of Theorem 5.1 guarantee that $\bar{\mathsf{y}}_h \to \bar{\mathsf{y}}$ in $L^2(\Omega)$, which in turns implies that $\bar{\mathsf{p}}_h \to \bar{\mathsf{p}}$ in $C(\bar{\Omega})$. We can thus conclude the existence of $h_\dagger > 0$ such that

$$\|\Psi_h - \Psi\|_{\mathbb{R}^\ell} < \tau \quad \forall h \leq h_\dagger, \tag{5.20}$$

where, we recall that, Ψ_{z} is defined in (4.18). Let $\mathsf{z} \in \mathcal{D}$ be arbitrary but fixed and assume that $\psi_{\mathsf{z}} > \tau > 0$. In view of the projection formula (4.15), we immediately conclude that $\bar{\mathsf{u}}_{\mathsf{z}} = \mathsf{a}_{\mathsf{z}}$. On the other hand, from estimate (5.20) we can obtain that $\psi_{\mathsf{z},h} > 0$ and thus that $\bar{\mathsf{u}}_{\mathsf{z},h} > -\alpha^{-1}\bar{\mathsf{p}}_h(\mathsf{z})$. This, on the basis of the projection formula (5.12), yields $\bar{\mathsf{u}}_{\mathsf{z},h} = \mathsf{a}_{\mathsf{z}}$. Consequently, $\bar{\mathsf{u}}_{\mathsf{z}} = \bar{\mathsf{u}}_{\mathsf{z},h} = \mathsf{a}_{\mathsf{z}}$. Similar arguments allow us to obtain that, if $\psi_{\mathsf{z}} < -\tau < 0$, then $\bar{\mathsf{u}}_{\mathsf{z}} = \bar{\mathsf{u}}_{\mathsf{z},h} = \mathsf{b}_{\mathsf{z}}$. Since $\mathsf{z} \in \mathcal{D}$ is arbitrary, we can finally conclude that $\bar{\mathbf{u}}_h - \bar{\mathbf{u}} \in \mathbf{C}^\tau_{\bar{\mathbf{u}}}$.

Step 2. Since $\bar{\mathbf{u}}_h - \bar{\mathbf{u}} \in \mathbf{C}^\tau_{\bar{\mathbf{u}}}$, with $\mathbf{C}^\tau_{\bar{\mathbf{u}}}$ defined in (4.26), and $\bar{\mathbf{u}}$ satisfies (4.27), we are allowed to set $\mathbf{v} = \bar{\mathbf{u}}_h - \bar{\mathbf{u}}$ in (4.28) to arrive at

$$\kappa\|\bar{\mathbf{u}}_h - \bar{\mathbf{u}}\|^2_{\mathbb{R}^\ell} \leq j''(\bar{\mathbf{u}})(\bar{\mathbf{u}}_h - \bar{\mathbf{u}})^2. \tag{5.21}$$

On the other hand, in view of the mean value theorem we obtain, for some $\theta_h \in (0, 1)$,

$$(j'(\bar{\mathbf{u}}_h) - j'(\bar{\mathbf{u}}))(\bar{\mathbf{u}}_h - \bar{\mathbf{u}}) = j''(\hat{\mathbf{u}})(\bar{\mathbf{u}}_h - \bar{\mathbf{u}})^2,$$

where $\hat{\mathbf{u}} = \bar{\mathbf{u}} + \theta_h(\bar{\mathbf{u}}_h - \bar{\mathbf{u}})$. With (5.21) at hand, we can thus arrive at

$$\kappa\|\bar{\mathbf{u}}_h - \bar{\mathbf{u}}\|^2_{\mathbb{R}^\ell} \leq (j'(\bar{\mathbf{u}}_h) - j'(\bar{\mathbf{u}}))(\bar{\mathbf{u}}_h - \bar{\mathbf{u}}) + (j''(\bar{\mathbf{u}}) - j''(\hat{\mathbf{u}}))(\bar{\mathbf{u}}_h - \bar{\mathbf{u}})^2. \tag{5.22}$$

Invoke the fact that j'' is continuous in $\mathbb{R}^\ell$, $\theta_h \in (0, 1)$, and $\bar{\mathbf{u}}_h \to \bar{\mathbf{u}}$ in $\mathbb{R}^\ell$, to deduce the existence of $h_\ddagger > 0$ such that

$$\left|(j''(\bar{\mathbf{u}}) - j''(\hat{\mathbf{u}}))(\bar{\mathbf{u}}_h - \bar{\mathbf{u}})^2\right| \leq \frac{\kappa}{2}\|\bar{\mathbf{u}}_h - \bar{\mathbf{u}}\|^2_{\mathbb{R}^\ell} \quad \forall h \leq h_\ddagger.$$

Replacing this inequality into (5.22) yields the desired inequality (5.19). This concludes the proof. ∎

We conclude by presenting the following a priori error estimate for the approximation of the optimal control variable.

THEOREM 5.8 (a priori error estimate). *Let Ω be an open, bounded, and convex polytopal domain. Let $a = a(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies* (A.1), (A.3), (A.4), and (5.4). *Assume, in addition, that $\partial a/\partial y$ is locally Lipschitz in y a.e. in Ω and that a satisfies $a(\cdot, 0) \in L^2(\Omega)$ and* (3.3) *with $\psi \in L^s(\Omega)$, $s > 2$ if $d = 2$, and $s > 6$ if $d = 3$. Let Ω_0 and Ω_1 be smooth subdomains such that $\mathcal{D} \subset \Omega_1 \Subset \Omega_0 \Subset \Omega$. Let $\mathsf{y}_d, \psi \in L^t(\Omega_0)$ and $\partial a/\partial y(\cdot, y) \in L^t(\Omega_0)$, for every $y \in \mathbb{R}$ and $t < \infty$. If $\bar{\mathbf{u}} \in \mathbb{U}_{ad}$ is a local minimum that satisfies the second order optimality condition* (4.27)*, or equivalently* (4.28)*, then there exists $h_\star > 0$ such that*

$$
(5.23) \qquad \|\bar{\mathbf{u}} - \bar{\mathbf{u}}_h\|_{\mathbb{R}^\ell} \lesssim h^2 |\log h|^3, \qquad \|\bar{\mathbf{u}} - \bar{\mathbf{u}}_h\|_{\mathbb{R}^\ell} \lesssim h^{1-\epsilon} |\log h|, \qquad \forall h < h_\star
$$

for $d = 2$ and $d = 3$, respectively. Here, $\epsilon > 0$ is arbitrarily small. The hidden constant, in both inequalities, is independent of h.

Proof. Adding and subtracting the term $j_h'(\bar{\mathbf{u}}_h)(\bar{\mathbf{u}} - \bar{\mathbf{u}}_h)$ in the right hand side of (5.19) yields the basic error estimate

$$
\tfrac{\kappa}{2} \|\bar{\mathbf{u}} - \bar{\mathbf{u}}_h\|_{\mathbb{R}^\ell}^2 \leq \left(j'(\bar{\mathbf{u}}) - j_h'(\bar{\mathbf{u}}_h) \right) (\bar{\mathbf{u}} - \bar{\mathbf{u}}_h) + \left(j_h'(\bar{\mathbf{u}}_h) - j'(\bar{\mathbf{u}}_h) \right) (\bar{\mathbf{u}} - \bar{\mathbf{u}}_h).
$$

The continuous and discrete first order optimality conditions, (4.12) and (5.11), respectively, yield $j'(\bar{\mathbf{u}})(\bar{\mathbf{u}} - \bar{\mathbf{u}}_h) \leq 0$ and $-j_h'(\bar{\mathbf{u}}_h)(\bar{\mathbf{u}} - \bar{\mathbf{u}}_h) \leq 0$. We can thus obtain the bound

$$
\tfrac{\kappa}{2} \|\bar{\mathbf{u}}_h - \bar{\mathbf{u}}\|_{\mathbb{R}^\ell}^2 \leq \left(j_h'(\bar{\mathbf{u}}_h) - j'(\bar{\mathbf{u}}_h) \right) (\bar{\mathbf{u}} - \bar{\mathbf{u}}_h).
$$

Invoke the auxiliary error estimates of Theorem 5.4 to immediately arrive at the desired error estimates in (5.23). ∎

Remark 5.9 (optimality). In two dimensions, the error estimate of Theorem 5.8 is nearly–optimal in terms of approximation (nearly because of the presence of the log–term). In three dimensions, the error estimate is suboptimal.

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