FRACTIONAL SEMILINEAR OPTIMAL CONTROL: OPTIMALITY CONDITIONS, CONVERGENCE, AND ERROR ANALYSIS*

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Abstract. We analyze an optimal control problem for a fractional semilinear PDE; control constraints are also considered. We adopt the integral definition of the fractional Laplacian and establish the well-posedness of a fractional semilinear PDE; we also analyze suitable finite element discretizations. We thus derive the existence of optimal solutions and first and second order optimality conditions for our optimal control problem; regularity properties are also studied. We devise a fully discrete scheme that approximates the control variable with piecewise constant functions; the state and adjoint equations are discretized via piecewise linear finite elements. We analyze convergence properties of discretizations and obtain a priori error estimates.

Key words. optimal control problem, fractional diffusion, integral fractional Laplacian, regularity estimates, finite elements, convergence, a priori error estimates.

AMS subject classifications. 35R11, 49J20, 49M25, 65K10, 65N15, 65N30.

1. Introduction. In this work we shall be interested in the analysis and discretization of a distributed optimal control problem for a semilinear, elliptic, and fractional partial differential equation (PDE). To make matters precise, we let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain in $\mathbb{R}^n$ ($n \in \{2, 3\}$) with Lipschitz boundary $\partial\Omega$; additional regularity requirements on $\partial\Omega$ will be imposed in the course of our regularity and convergence rate analyses ahead. Let us define the cost functional

$$J(u, z) := \int_\Omega L(x, u(x))\mathrm{d}x + \frac{\alpha}{2}\int_\Omega |z(x)|^2\mathrm{d}x, \tag{1.1}$$

where $L : \Omega \times \mathbb{R} \to \mathbb{R}$ denotes a Carathéodory function of class C^2 with respect to the second variable and $\alpha > 0$ corresponds to the so-called regularization parameter. Further assumptions on L will be deferred until section 2.1. In this work, we shall be concerned with the following PDE-constrained optimization problem: Find $\min J(u, z)$ subject to the *semilinear, elliptic, and fractional PDE*

$$(-\Delta)^s u + a(\cdot, u) = z \text{ in } \Omega, \qquad u = 0 \text{ in } \Omega^c, \tag{1.2}$$

with $\Omega^c = \mathbb{R}^n \setminus \Omega$, and the *control constraints* $\mathfrak{a} \le z(x) \le \mathfrak{b}$ for a.e. $x \in \Omega$. The control bounds $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}$ are such that $\mathfrak{a} < \mathfrak{b}$. Assumptions on the nonlinear function a will be deferred until section 2.1. We will refer to the previously defined PDE-constrained optimization problem as the *fractional semilinear optimal control problem.*

For smooth functions $w : \mathbb{R}^n \to \mathbb{R}$, there are several equivalent definitions of the fractional Laplace operator $(-\Delta)^s$ in $\mathbb{R}^n$ [22]. Indeed, $(-\Delta)^s$ can be naturally defined by means of the following pointwise formula:

$$(-\Delta)^s w(x) = C(n, s)\,\mathrm{p.v}\int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|x - y|^{n+2s}}\mathrm{d}y, \qquad C(n, s) = \frac{2^{2s} s\Gamma(s + \frac{n}{2})}{\pi^{n/2}\Gamma(1 - s)}, \tag{1.3}$$

where p.v stands for the Cauchy principal value and $C(n, s)$ is a positive normalization constant that depends only on n and s. Equivalently, $(-\Delta)^s$ can be defined via

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Fourier transform: $\mathcal{F}((-\Delta)^s w)(\xi) = |\xi|^{2s}\mathcal{F}(w)(\xi)$. A proof of the equivalence of these two definitions can be found in [23, section 1.1]. In addition to these two definitions, several other *equivalent definitions* of $(-\Delta)^s$ in $\mathbb{R}^n$ are available in the literature [22]. Regarding *equivalence*, the scenario in bounded domains is *substantially different.* For functions supported in $\bar{\Omega}$, we may utilize the integral representation (1.3) to define $(-\Delta)^s$. This gives rise to the so-called *restricted* or *integral* fractional laplacian. Notice that we have materialized a zero Dirichlet condition by restricting the operator to act only on functions that are zero outside Ω. We must immediately mention that in bounded domains, and in addition to the *restricted* or *integral* fractional Laplacian there are, at least, two others *non-equivalent* definitions of nonlocal operators related to the fractional Laplacian: the *regional* fractional Laplacian and the *spectral* fractional Laplacian; see [7, Section 2] and [20, Section 6] for details. In this work, we adopt the restricted or integral definition of the fractional Laplace operator $(-\Delta)^s$, which, from now on, we shall simply refer to as the *integral fractional Laplacian.*

During the very recent past, there has been considerable progress in the design and analysis of solution techniques for problems involving fractional diffusion. We refer the interested reader to [8, 13] for a complete overview of the available results and limitations. In contrast to these advances, the numerical analysis of PDE-constrained optimization problems involving $(-\Delta)^s$ has been less explored. Restricting ourselves to problems considering the spectral definition, we mention [3, 16, 25] within the linear–quadratic scenario, [4] for optimization with respect to order, and [26] for bilinear optimal control. We also mention [6], where the authors analyze, at the continuous level, a semilinear optimal control problem for the spectral fractional Laplacian. Concerning the integral fractional Laplacian, it seems that the results are even scarcer; the linear–quadratic case has been recently analyzed in [14]. We conclude this paragraph by mentioning [2, 15] for numerical approximations of optimal control problems involving suitable nonlocal operators.

In addition to this exposition being the first one that studies numerical schemes for semilinear optimal control problems involving the *integral* fractional Laplacian, the analysis itself comes with its own set of difficulties. Overcoming them has required us to provide several results. Let us briefly detail some of them:

- (i) *Fractional PDEs:* Let $s \in (0,1)$, $n \geq 2$, $r > n/2s$, and $z \in L^r(\Omega)$. We show that (1.2) is well-posed for $a = a(x,u)$ being a Carathéodory function, monotone increasing in u, satisfying (3.2) and $a(\cdot,0) \in L^r(\Omega)$ (Theorem 3.1).
- (i) *FEM discretizations:* We prove convergence of finite element discretizations on Lipschitz polytopes and obtain error estimates on smooth domains; the latter under additional assumptions on a and the underlying forcing term that guarantee the regularity estimates of Theorem 5.1; see section 5.
- (iii) *Existence of an optimal control:* Assuming that, in addition, $L = L(x,u)$ is a Carathéodory function and a and L are locally Lipschitz in u, we show that our control problem admit at least a solution; see Theorem 4.1.
- (iv) *Optimality conditions:* Let $n \in \{2,3\}$ and $s > n/4$. Under additional assumptions on a and L, we derive second order necessary and sufficient optimality conditions with a minimal gap; see Section 4.3.
- (v) *Regularity estimates:* Let $n \geq 2$ and $s \in (0,1)$. We analyze regularity properties for the optimal variables. We prove that $\bar{u}, \bar{p} \in H^{s+1/2-\epsilon}(\Omega)$ and $\bar{z} \in H^{\gamma}(\Omega)$, where $\gamma = \min\{1, s+1/2-\epsilon\}$ and $\epsilon > 0$ is arbitrarily small.
- (vi) *Convergence of discretization and error estimates:* Let $n \geq 2$ and $s \in (0,1)$. We prove that the sequence $\{\bar{z}_h\}_{h>0}$ of global solutions of suitable discrete

control problems converge to a solution of the fractional semilinear optimal control problem. When $n \in \{2,3\}$ and $s > n/4$, we derive error estimates.

Over the last 20 years, several contributions have delineated the numerical analysis of semilinear optimal control problems. Without a doubt, these studies have paved the way for the achievement of the aforementioned results. In particular, we have followed [31], for the analysis of (1.2) and the optimal control problem, [11], for deriving second order optimality conditions, and [9, 10, 11], for analyzing convergence properties and deriving error estimates.

The rest of the paper is organized as follows. In section 3, we analyze the fractional state equation (1.2). Section 4 is devoted to the study of the fractional semilinear optimal control problem. In section 5, we study finite element discretizations for (1.2). Section 6 is advocated to the analysis of finite element discretizations for the fractional semilinear optimal control problem: convergence and error estimates.

2. Notation and preliminaries. Throughout this work Ω is an open and bounded domain of $\mathbb{R}^n$ $(n \geq 2)$ with Lipschitz boundary $\partial\Omega$; we will impose additional assumptions on n and $\partial\Omega$ when needed. We will denote by Ω^c the complement of Ω. If $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces, we write $\mathcal{X} \hookrightarrow \mathcal{Y}$ to denote that $\mathcal{X}$ is continuously embedded in $\mathcal{Y}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{X}$. We will denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong and weak convergence, respectively, of $\{x_n\}_{n=1}^{\infty}$ to x. The relation $\mathsf{a} \lesssim \mathsf{b}$ indicates that $\mathsf{a} \leq C\mathsf{b}$, with a positive constant C that depends neither on a, b nor on the discretization parameters but it might depend on s, n, and Ω. The value of C might change at each occurrence.

2.1. Assumptions. We will operate under the following assumptions on a and L. We must, however, immediately mention that some of the results obtained in this work are valid under less restrictive requirements; when possible we explicitly mention the assumptions on a and L that are needed to obtain a particular result.

(A.1) $a : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable such that $a(\cdot, 0) \in L^r(\Omega)$ for $r > n/2s$.

(A.2) $\frac{\partial a}{\partial u}(x,u) \geq 0$ for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}$.

(A.3) For all $\mathfrak{m} > 0$, there exists a positive constant $C_{\mathfrak{m}}$ such that

$$\sum_{i=1}^{2} \left| \frac{\partial^i a}{\partial u^i}(x,u) \right| \leq C_{\mathfrak{m}}, \qquad \left| \frac{\partial^2 a}{\partial u^2}(x,u) - \frac{\partial^2 a}{\partial u^2}(x,w) \right| \leq C_{\mathfrak{m}}|u-w|$$

for a.e. $x \in \Omega$ and $u, w \in [-\mathfrak{m}, \mathfrak{m}]$.

(B.1) $L : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable such that $L(\cdot, 0) \in L^1(\Omega)$.

(B.2) For all $\mathfrak{m} > 0$, there exist $\psi_{\mathfrak{m}}, \phi_{\mathfrak{m}} \in L^r(\Omega)$, with $r > n/2s$, such that

$$\left| \frac{\partial L}{\partial u}(x,u) \right| \leq \psi_{\mathfrak{m}}(x) \quad \left| \frac{\partial^2 L}{\partial u^2}(x,u) \right| \leq \phi_{\mathfrak{m}}(x),$$

for a.e. $x \in \Omega$ and $u \in [-\mathfrak{m}, \mathfrak{m}]$.

The following assumptions are particularly needed to derive regularity estimates:

(C.1) $a(\cdot, 0) \in L^2(\Omega) \cap H^{\frac{1}{2}-s-\epsilon}(\Omega)$,

(C.2) $\frac{\partial L}{\partial u}(\cdot, 0) \in L^2(\Omega) \cap H^{\frac{1}{2}-s-\epsilon}(\Omega)$.

In these assumptions ϵ denotes an arbitrarily small positive constant.

2.2. Function spaces. For any $s \geq 0$, we define $H^s(\mathbb{R}^n)$, the Sobolev space of order s over $\mathbb{R}^n$, by [30, Definition 15.7]

$$H^s(\mathbb{R}^n) := \left\{ v \in L^2(\mathbb{R}^n) : (1+|\xi|^2)^{s/2}\mathcal{F}(v) \in L^2(\mathbb{R}^n) \right\}.$$

With the space $H^s(\mathbb{R}^n)$ at hand, we define $\tilde{H}^s(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $H^s(\mathbb{R}^n)$. This space can be equivalently characterized by [24, Theorem 3.29]

$$\tilde{H}^s(\Omega) = \{ v|_\Omega : v \in H^s(\mathbb{R}^n), \ \text{supp } v \subset \overline{\Omega} \}. \tag{2.1}$$

When $\partial\Omega$ is Lipschitz $\tilde{H}^s(\Omega)$ is equivalent to $\mathbb{H}^s(\Omega) = [L^2(\Omega), H_0^1(\Omega)]_s$, the real interpolation between $L^2(\Omega)$ and $H_0^1(\Omega)$ for $s \in (0,1)$ and to $H^s(\Omega) \cap H_0^1(\Omega)$ for $s \in (1, 3/2)$ [24, Theorem 3.33]. We denote by $H^{-s}(\Omega)$ the dual space of $\tilde{H}^s(\Omega)$ and by $\langle \cdot, \cdot \rangle$ the duality pair between these two spaces. We define the bilinear form

$$\mathcal{A}(v,w) = \frac{C(n,s)}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{n+2s}} \mathrm{d}x\mathrm{d}y, \tag{2.2}$$

and denote by $\|\cdot\|_s$ the norm that $\mathcal{A}(\cdot,\cdot)$ induces, which is just a multiple of the $H^s(\mathbb{R}^n)$-seminorm: $\|v\|_s = \sqrt{\mathcal{A}(v,v)} = \mathfrak{C}(n,s)|v|_{H^s(\mathbb{R}^n)}$, where $\mathfrak{C}(n,s) = \sqrt{C(n,s)/2}$.

We will repeatedly use the following continuous embedding: $H^s(\Omega) \hookrightarrow L^{\mathfrak{q}}(\Omega)$ for $1 \leq \mathfrak{q} \leq 2n/(n-2s)$ [1, Theorem 7.34]; observe that $n > 2s$. If $\mathfrak{q} < 2n/(n-2s)$ the embedding $H^s(\Omega) \hookrightarrow L^{\mathfrak{q}}(\Omega)$ is compact [1, Theorem 6.3].

3. The state equation. Let $f \in H^{-s}(\Omega)$ be a forcing term. In this section, we analyze the following fractional semilinear PDE: Find $u \in \tilde{H}^s(\Omega)$ such that

$$\mathcal{A}(u,v) + \langle a(\cdot,u), v \rangle = \langle f, v \rangle \quad \forall v \in \tilde{H}^s(\Omega). \tag{3.1}$$

Here, $a = a(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ denotes a Carathéodory function that is monotone increasing in u. In addition, we assume that, for $\mathfrak{m} > 0$, there exits

$$\varphi_{\mathfrak{m}} \in L^{\mathfrak{t}}(\Omega) : \quad |a(x,u)| \leq |\varphi_{\mathfrak{m}}(x)| \text{ a.e. } x \in \Omega, \ u \in [-\mathfrak{m}, \mathfrak{m}], \quad \mathfrak{t} = 2n/(n+2s). \tag{3.2}$$

We present the following existence and uniqueness result.

THEOREM 3.1 (well-posedness). *Let $n \geq 2$, $s \in (0,1)$, and $r > n/2s$. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with Lipschitz boundary. If $f \in L^r(\Omega)$, a satisfies (3.2), and $a(\cdot,0) \in L^r(\Omega)$, then problem (3.1) has a unique solution $u \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ which satisfies the stability estimate*

$$|u|_{H^s(\mathbb{R}^n)} + \|u\|_{L^\infty(\Omega)} \lesssim \|f - a(\cdot,0)\|_{L^r(\Omega)}, \tag{3.3}$$

with a hidden constant that is independent of u, a, and f.

Proof. We proceed in four steps:

$\boxed{1}$ Let us assume, for the moment, that, in addition, $a = a(x,u)$ is globally bounded, i.e., $|a(x,u)| \leq \mathfrak{c}$ for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$, and $a(\cdot,0) = 0$. Define the mapping

$$\mathfrak{A} : \tilde{H}^s(\Omega) \to H^{-s}(\Omega) : \quad \langle \mathfrak{A}u, v \rangle = \mathcal{A}(u,v) + \langle a(\cdot,u), v \rangle \quad \forall v \in \tilde{H}^s(\Omega).$$

Since $\mathcal{A}$ is bilinear, continuous, and coercive on $\tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega)$ and $a = a(x,u)$ is globally bounded and monotone increasing in u, it is immediate that $\mathfrak{A}$ is *well-posed*, *monotone*, and *coercive*. In addition, since $a = a(x,u)$ is continuous in u for a.e. $x \in \Omega$, dominated convergence yields the *hemicontinuity* of $\mathfrak{A}$. Existence and uniqueness of

$u \in \tilde{H}^s(\Omega)$ follows from the main theorem on monotone operators [34, Theorem 26.A], [29, Theorem 2.18]. Set $v = u$ (3.1) to obtain $|u|_{H^s(\mathbb{R}^n)} \lesssim \|f\|_{H^{-s}(\Omega)}$.

$\boxed{2}$ Define, for $k > 0$, v_k by $v_k(x) = u(x) - k$ if $u(x) \geq k$, $v_k(x) = 0$ if $|u(x)| < k$, and $v_k(x) = u(x) + k$ if $u(x) \leq -k$. We also define the set

$$\Omega(k) := \{x \in \Omega : |u(x)| \geq k\}.$$

Since $a = a(x,u)$ is monotone increasing in u and $a(\cdot, 0) = 0$, we have $\langle a(\cdot,u), v_k\rangle = \int_\Omega a(x,u(x))v_k(x)\mathrm{d}x \geq 0$. This yields $\mathcal{A}(u, v_k) \leq \langle f, v_k\rangle$. The relations and inequalities (2.22)–(2.30) in [5] reveal that $\mathcal{A}(v_k, v_k) \leq \mathcal{A}(u, v_k)$. We can thus obtain $\|v_k\|_s^2 = \mathcal{A}(v_k, v_k) \leq \langle f, v_k\rangle$. Define $\mathfrak{q} := 2n/(n-2s)$. Thus, for $t < 2n/(n-2s)$,

$$\|v_k\|^2_{L^{\mathfrak{q}}(\Omega)} \lesssim |v_k|^2_{H^s(\mathbb{R}^n)} \lesssim \|v_k\|_{L^{\mathfrak{q}}(\Omega)} |\Omega(k)|^{\frac{1}{t}} \|f\|_{L^r(\Omega)}, \quad \mathfrak{q}^{-1} + r^{-1} + t^{-1} = 1.$$

On the other hand, $\|v_k\|^{\mathfrak{q}}_{L^{\mathfrak{q}}(\Omega)} = \int_{\Omega(k)} |v_k(x)|^{\mathfrak{q}}\mathrm{d}x = \int_{\Omega(k)} ||u(x)| - k|^{\mathfrak{q}}\mathrm{d}x$. Let $h > k$, then $\Omega(h) \subset \Omega(k)$ and $\int_{\Omega(k)} ||u(x)| - k|^{\mathfrak{q}}\mathrm{d}x \geq (h-k)^{\mathfrak{q}}|\Omega(h)|$. Consequently,

$$(h-k)|\Omega(h)|^{\frac{1}{\mathfrak{q}}} \leq \|v_k\|_{L^{\mathfrak{q}}(\Omega)} \lesssim (|\Omega(k)|^{\frac{1}{\mathfrak{q}}})^{\frac{\mathfrak{q}}{t}} \|f\|_{L^r(\Omega)}.$$

Since $r > n/2s$, $\mathfrak{q}/t > 1$. An application of [21, Lemma B.1] yields the existence of $\mathfrak{h} > 0$ such that $|\Omega(\mathfrak{h})| = 0$, which implies that $u \in L^\infty(\Omega)$ and $\|u\|_{L^\infty(\Omega)} \lesssim \|f\|_{L^r(\Omega)}$.

$\boxed{3}$ We remove the boundedness assumption on a. Define, for $k > 0$, a_k by $a_k(x,u) = a(x,k)$ if $u > k$, $a_k(x,u) = a(x,u)$ if $|u| \leq k$, and $a_k(x,u) = a(x,-k)$ if $u < -k$. Since a_k is bounded in the second variable and satisfies (3.2), there exists a unique solution u to problem (3.1) with a replaced by a_k. In addition, $|u|_{H^s(\mathbb{R}^n)} + \|u\|_{L^\infty(\Omega)} \leq c_\infty \|f\|_{L^r(\Omega)}$ with c_∞ being independent of a_k and k. Choose $k > c_\infty\|f\|_{L^r(\Omega)}$ so that $a_k(x,u(x)) = a(x,u(x))$ for a.e. $x \in \Omega$. Consequently, u solves (3.1). Uniqueness of solutions follows from the monotonicity properties of a.

$\boxed{4}$ We remove the condition $a(\cdot,0) = 0$ by replacing $a(\cdot,u)$ by $a(\cdot,u) - a(\cdot,0)$. $\square$

4. The optimal control problem.

In this section, we analyze the following weak version of the fractional semilinear optimal control problem: Find

$$\min\{J(u,z) : (u,z) \in \tilde{H}^s(\Omega) \times \mathbb{Z}_{ad}\} \tag{4.1}$$

subject to the *fractional, semilinear, and elliptic state equation*

$$\mathcal{A}(u,v) + (a(\cdot,u), v)_{L^2(\Omega)} = (z,v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega). \tag{4.2}$$

Here, $\mathbb{Z}_{ad} := \{v \in L^2(\Omega) : \mathfrak{a} \leq v(x) \leq \mathfrak{b} \text{ a.e. } x \in \Omega\}$.

Let $r > n/2s$ and $a = a(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a monotone increasing in u Carathéodory function satisfying (3.2) and $a(\cdot,0) \in L^r(\Omega)$. In view of Theorem 3.1, the existence of a unique solution u to problem (4.2) is guaranteed. We thus introduce the control to state map $\mathcal{S} : L^r(\Omega) \to \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ which, given a control z, associates to it the unique state u that solves (4.2). With $\mathcal{S}$ at hand, we also introduce the reduced cost functional $j : \mathbb{Z}_{ad} \to \mathbb{R}$ by the relation $j(z) = J(\mathcal{S}z, z)$.

4.1. Existence of optimal controls.

Since the PDE-constrained optimization problem (4.1)–(4.2) is not convex, we analyze existence results and optimality conditions in the context of local solutions. We begin our studies with an existence result.

THEOREM 4.1 (existence of an optimal pair). *Let $n \geq 2$, $s \in (0,1)$ and $r > n/2s$. Let $a = a(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that is monotone increasing in*

u with $a(\cdot,0) \in L^r(\Omega)$. Let $L = L(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with $L(\cdot,0) \in L^1(\Omega)$. Assume that a and L are locally Lipschitz with respect to the second variable. Thus, (4.1)–(4.2) *admits at least one solution $(\bar{u},\bar{z}) \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega) \times \mathbb{Z}_{ad}$.*

Proof. Let $\{(u_k,z_k)\}_{k=1}^\infty$ be a minimizing sequence, i.e., for $k \in \mathbb{N}$, $z_k \in \mathbb{Z}_{ad}$ and $u_k = \mathcal{S}z_k$ are such that $J(u_k,z_k) \to \mathfrak{j} := \inf J(u,z)$ as $k \uparrow \infty$. Since $\mathbb{Z}_{ad}$ is bounded in $L^\infty(\Omega)$, there exits a nonrelabeled subsequence $\{z_k\}_{k=1}^\infty$ such that $z_k \overset{*}{\rightharpoonup} \bar{z}$ in $L^\infty(\Omega)$ as $k \uparrow \infty$. Observe that, since $z_k \in \mathbb{Z}_{ad}$ for every $k \in \mathbb{N}$, there exists $\mathfrak{m} > 0$ such that $|u_k(x)| \leq \mathfrak{m}$ for $k \in \mathbb{N}$ and a.e. $x \in \Omega$. This implies that $\{a(\cdot,u_k)\}_{k=1}^\infty$ is bounded in $L^r(\Omega)$. We can thus conclude the existence of a nonrelabeled subsequence $\{u_k\}_{k=1}^\infty$ such that $u_k \rightharpoonup \bar{u}$ in $\tilde{H}^s(\Omega)$ and $u_k \to \bar{u}$ in $L^2(\Omega)$ as $k \uparrow \infty$.

For $k \in \mathbb{N}$, $u_k \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ solves

$$\mathcal{A}(u_k,v) + \langle a(\cdot,u_k),v\rangle = \langle z_k,v\rangle \quad \forall v \in \tilde{H}^s(\Omega). \tag{4.3}$$

Since $\mathfrak{M} := \{v \in L^\infty(\Omega) : |v(x)| \leq M \text{ a.e. } x \in \Omega\}$ is weakly sequentially closed, we conclude that $\bar{u} \in \mathfrak{M}$. We can thus invoke the local Lipschitz property of a in u to obtain $\|a(\cdot,\bar{u}) - a(\cdot,u_k)\|_{L^2(\Omega)} \leq \mathfrak{l}_M \|\bar{u} - u_k\|_{L^2(\Omega)} \to 0$ as $k \uparrow \infty$. In view of the previous convergence results, passing to the limit in (4.3) yields $\bar{u} = \mathcal{S}\bar{z}$.

On the other hand, the map $L^2(\Omega) \ni v \mapsto \|v\|_{L^2(\Omega)}^2 \in \mathbb{R}$ is continuous and convex; it is thus weakly lower continuous. Consequently,

$$\mathfrak{j} = \lim_{n \uparrow \infty} J(u_n,z_n) = \int_\Omega L(x,\bar{u}(x))\mathrm{d}x + \liminf_{n \uparrow \infty} \frac{\alpha}{2}\|z_n\|_{L^2(\Omega)}^2 \geq J(\bar{u},\bar{z}).$$

Notice that $\left|\int_\Omega [L(x,\bar{u}(x)) - L(x,u_n(x))]\mathrm{d}x\right| \leq \mathfrak{l}_{\mathfrak{m}} \|\bar{u} - u_n\|_{L^1(\Omega)} \to 0$ as $n \uparrow 0$ because of the local Lipschitz property of L in the second variable and $u_n \to \bar{u}$ in $L^2(\Omega)$. ☐

REMARK 4.1 (assumptions on a). To obtain the result of Theorem 4.1 we have assumed that a is locally Lipschitz in u and $a(\cdot,0) \in L^r(\Omega)$ with $r > n/2s$. Observe that condition (3.2) can thus be guaranteed because $n/2s > 2n/(n+2s)$.

4.2. First order necessary optimality conditions. In this section, we analyze differentiability properties for the control to state map $\mathcal{S}$ and derive first order necessary optimality conditions.

We begin our studies by precisely introducing the concept of local minimum. Let $q \in [1,\infty)$ and $\epsilon > 0$, we define the closed ball in $L^q(\Omega)$ of radius ϵ and centered at $\bar{z}$,

$$B_\epsilon(\bar{z}) := \{z \in L^q(\Omega) : \|\bar{z} - z\|_{L^q(\Omega)} \leq \epsilon\}.$$

DEFINITION 4.2 (local minimum). *Let $q \in [1,\infty)$. We say that $\bar{z} \in \mathbb{Z}_{ad}$ is a local minimum, or locally optimal, in $L^q(\Omega)$ for* (4.1)–(4.2) *if there exists $\epsilon > 0$ such that $j(\bar{z}) \leq j(z)$ for every $z \in B_\epsilon(\bar{z}) \cap \mathbb{Z}_{ad}$.*

REMARK 4.2 (local optimality in $L^q(\Omega) \iff$ local optimality in $L^2(\Omega)$). Since $\mathbb{Z}_{ad}$ is bounded in $L^\infty(\Omega)$, it can be proved that $\bar{z} \in \mathbb{Z}_{ad}$ is locally optimal in $L^q(\Omega)$ if and only if $\bar{z} \in \mathbb{Z}_{ad}$ is a local minimum in $L^2(\Omega)$; see [11, Section 5] for details.

In what follows we will operate in $L^2(\Omega)$ regarding local optimally.

THEOREM 4.3 (differentiability properties of $\mathcal{S}$). *Let $n \geq 2$, $s \in (0,1)$, and $r > n/2s$. Assume that* (A.1), (A.2), *and* (A.3) *hold. Then, the control to state map $\mathcal{S} : L^r(\Omega) \to \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ is of class C^2. In addition, if $z, w \in L^r(\Omega)$, then $\phi = \mathcal{S}'(z)w \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ corresponds to the unique solution to the problem*

$$\mathcal{A}(\phi,v) + \left(\tfrac{\partial a}{\partial u}(\cdot,u)\phi,v\right)_{L^2(\Omega)} = (w,v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega), \tag{4.4}$$

where $u = \mathcal{S}z$. In addition, for every $w_1, w_2 \in L^r(\Omega)$, $\psi = \mathcal{S}''(z)(w_1, w_2) \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ corresponds to the unique solution to

$$(4.5)\qquad \mathcal{A}(\psi, v) + \left(\tfrac{\partial a}{\partial u}(\cdot, u)\psi, v\right)_{L^2(\Omega)} = -\left(\tfrac{\partial^2 a}{\partial u^2}(\cdot, u)\phi_{w_1}\phi_{w_2}, v\right)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega),$$

where $u = \mathcal{S}z$ and $\phi_{w_i} = \mathcal{S}'(z)w_i$, with $i = 1, 2$.

Proof. The first order Fréchet differentiability of $\mathcal{S}$ from $L^r(\Omega)$ into $\tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ follows from a slight modification of the proof of [31, Theorem 4.17] that basically entails to replace $H^1(\Omega)$ by $\tilde{H}^s(\Omega)$ and $C(\bar{\Omega})$ by $L^\infty(\Omega)$. These arguments also show that $\phi = \mathcal{S}'(z)w \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ corresponds to the unique solution to (4.4); since $w \in L^r(\Omega)$ and $\frac{\partial a}{\partial u}(x, u) \geq 0$ for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$, problem (4.4) is well-posed.

The second order Fréchet differentiability of $\mathcal{S}$ can be obtained by using the implicit function theorem. Let us introduce the linear mapping $\mathfrak{f} \mapsto \mathfrak{u}$ by

$$\mathfrak{R} : L^r(\Omega) \to \tilde{H}^s(\Omega) \cap L^\infty(\Omega) : \quad \mathcal{A}(\mathfrak{u}, v) = \langle \mathfrak{f}, v\rangle \quad \forall v \in \tilde{H}^s(\Omega).$$

Define $\mathfrak{F} : [\tilde{H}^s(\Omega) \cap L^\infty(\Omega)] \times L^r(\Omega) \to L^\infty(\Omega)$ by $\mathfrak{F}(u, z) := u - \mathfrak{R}(z - a(\cdot, u))$. We first observe that $\mathfrak{F}$ is of class C^2. Second, $\mathfrak{F}(\mathcal{S}z, z) = 0$. Third, since $\partial\mathfrak{F}/\partial u(u, z)v = v + \mathfrak{R}\partial a/\partial u(\cdot, u)v$, it can be deduced that $\partial\mathfrak{F}/\partial u(u, z)$ is surjective from $L^\infty(\Omega)$ into $\tilde{H}^s(\Omega) \cap L^\infty(\Omega)$. The implicit function theorem thus implies that $\mathcal{S}$ is of class C^2. The fact that ψ solves (4.5) follows from differentiating $\mathfrak{F}(\mathcal{S}z, z) = 0$. ◻

The following result is standard: If $\bar{z} \in \mathbb{Z}_{ad}$ denotes a locally optimal control for problem (4.1)–(4.2), then $j'(\bar{z})(z - \bar{z}) \geq 0$ for all $z \in \mathbb{Z}_{ad}$ [31, Lemma 4.18]. To explore this inequality, we define the adjoint state $p \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ as the solution to

$$(4.6)\qquad \mathcal{A}(v, p) + \left(\tfrac{\partial a}{\partial u}(\cdot, u)p, v\right)_{L^2(\Omega)} = \left(\tfrac{\partial L}{\partial u}(\cdot, u), v\right)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega).$$

Assumption (A.1) guarantees that $\partial a/\partial u(x, u) \geq 0$ for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$. Assumption (B.2) yield $\partial L/\partial u(\cdot, u) \in L^r(\Omega)$ for $r > n/2s$. The existence and uniqueness of p in $\tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ is thus immediate.

We present first order necessary optimality conditions for (4.1)–(4.2).

THEOREM 4.4 (first order necessary optimality conditions). *Let $n \geq 2$, $s \in (0, 1)$, and $r > n/2s$. Assume that* (A.1)–(A.3) *and* (B.1)–(B.2) *hold. Then every locally optimal control $\bar{z} \in \mathbb{Z}_{ad}$ satisfies the variational inequality*

$$(4.7)\qquad (\bar{p} + \alpha\bar{z}, z - \bar{z})_{L^2(\Omega)} \geq 0 \quad \forall z \in \mathbb{Z}_{ad},$$

where $\bar{p} \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ denotes the solution to (4.6) *with u replaced by $\bar{u} = \mathcal{S}\bar{z}$.*

Proof. Define $\ell : L^\infty(\Omega) \to \mathbb{R}$ by $\ell(u) = \int_\Omega L(x, u(x))\mathrm{d}x$ and observe that (B.1)–(B.2) yield the Fréchet differentiability of ℓ on bounded sets of $L^\infty(\Omega)$. Since $\mathcal{S} : L^r(\Omega) \to \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ is differentiable, we thus deduce the Fréchet differentiability of ℓ as a map from $L^\sigma(\Omega)$ to $\mathbb{R}$, where $\sigma = \max\{n/2s, 2\}$, upon noticing that $L^2(\Omega) \ni z \mapsto \|z\|^2_{L^2(\Omega)} \in \mathbb{R}$ is also differentiable. Basic computations thus reveal

$$(4.8)\qquad j'(\bar{z})h = \int_\Omega \left(\frac{\partial L}{\partial u}(x, \mathcal{S}\bar{z}(x))\mathcal{S}'(\bar{z})h(x) + \alpha\bar{z}(x)h(x)\right)\mathrm{d}x, \quad h \in L^\sigma(\Omega).$$

Set $h = z - \bar{z} \in \mathbb{Z}_{ad}$ and define $\chi = \mathcal{S}'(\bar{z})h$. Setting $v = \chi$ in problem (4.6) and $v = \bar{p}$ in the problem that χ solves allow us to obtain $(z - \bar{z}, \bar{p})_{L^2(\Omega)} = (\frac{\partial L}{\partial u}(\cdot, \bar{u}), \chi)_{L^2(\Omega)}$. Replace this identity into (4.8) to obtain (4.7). This concludes the proof. ◻

Define $\Pi_{[\mathfrak{a},\mathfrak{b}]} : L^1(\Omega) \to \mathbb{Z}_{ad}$ by $\Pi_{[\mathfrak{a},\mathfrak{b}]}(v) := \min\{\mathfrak{b}, \max\{v, \mathfrak{a}\}\}$ a.e in Ω. The local optimal control $\bar{z}$ satisfies (4.7) if and only if [31, Section 4.6]

$$\bar{z}(x) := \Pi_{[\mathfrak{a},\mathfrak{b}]}(-\alpha^{-1}\bar{p}(x)) \text{ a.e. } x \in \Omega. \tag{4.9}$$

Since $\bar{p} \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ and $s \in (0,1)$, it is immediate that $\bar{z} \in H^s(\Omega) \cap L^\infty(\Omega)$; further regularity properties for $\bar{z}$ are obtained in Theorem 4.10 below.

4.3. Second order sufficient optimality condition. In Theorem 4.4 we obtained a first order necessary condition for local optimality. Since our optimal control problem is not convex, sufficiency requires the use of second order optimality conditions. The purpose of this section is thus to derive such conditions. To accomplish this task, we begin by introducing some preliminary concepts. Let $\bar{z} \in \mathbb{Z}_{ad}$ satisfies (4.7). Define $\bar{\mathfrak{p}} := \bar{p} + \alpha\bar{z}$. Observe that (4.7) immediately yields

$$\bar{\mathfrak{p}}(x) \begin{cases} = 0 & \text{a.e. } x \in \Omega \text{ if } \mathtt{a} < \bar{z} < \mathtt{b}, \\ \geq 0 & \text{a.e. } x \in \Omega \text{ if } \bar{z} = \mathtt{a}, \\ \leq 0 & \text{a.e. } x \in \Omega \text{ if } \bar{z} = \mathtt{b}. \end{cases} \tag{4.10}$$

Define the *cone of critial directions* $C_{\bar{z}} := \{v \in L^2(\Omega) : (4.11) \text{ holds and } \bar{\mathfrak{p}}(x) \neq 0 \implies v(x) = 0\}$, where condition (4.11) reads as follows:

$$v(x) \geq 0 \text{ a.e. } x \in \Omega \text{ if } \bar{z}(x) = \mathfrak{a}, \qquad v(x) \leq 0 \text{ a.e. } x \in \Omega \text{ if } \bar{z}(x) = \mathfrak{b}. \tag{4.11}$$

The following result is instrumental.

PROPOSITION 4.5 (j is of class C^2). *Let $n \geq 2$, $s \in (0,1)$, $r > n/2s$, and $\sigma = \max\{2, n/2s\}$. Assume that* (A.1)–(A.3) *and* (B.1)–(B.2) *hold. Then the reduced cost $j : L^\sigma(\Omega) \to \mathbb{R}$ is of class C^2. In addition, for $z, w_1, w_2 \in L^\sigma(\Omega)$, we have*

$$j''(z)(w_1, w_2) = \int_\Omega \left(\frac{\partial^2 L}{\partial u^2}(x,u)\phi_{w_1}\phi_{w_2} + \alpha w_1 w_2 - p\frac{\partial^2 a}{\partial u^2}(x,u)\phi_{w_1}\phi_{w_2} \right) \mathrm{d}x, \tag{4.12}$$

where $u = \mathcal{S}z$, p solves (4.6) *and $\phi_{w_i} = \mathcal{S}'(z)w_i$, with $i \in \{1,2\}$.*

Proof. The first order Fréchet differentiability of j has been obtained in Theorem 4.4. Theorem 4.3 guarantees that $\mathcal{S}$ is second order Fréchet differentiable as a map from $L^r(\Omega)$ into $\tilde{H}^s(\Omega) \cap L^\infty(\Omega)$. In view of (B.1)–(B.2), the map $u \mapsto \ell(u) := \int_\Omega L(x, u(x))\mathrm{d}x$ is second order Fréchet differentiable as well as a map from the set $\{u \in L^\infty(\Omega) : |u(x)| \leq \mathfrak{m}\}$ to $\mathbb{R}$. The chain rule allows us to conclude that $j \in C^2$. The identity (4.12) follows from the arguments elaborated in [31, Section 4.10]. ◻

We now formulate second order necessary optimality conditions.

THEOREM 4.6 (second order necessary optimality conditions). *Let $n \in \{2,3\}$ and $s > n/4$. Let $\bar{z} \in \mathbb{Z}_{ad}$ be a locally optimal control for problem* (4.1)–(4.2)*. Thus,*

$$j''(\bar{z})v \geq 0 \quad \forall v \in C_{\bar{z}}, \tag{4.13}$$

where $C_{\bar{z}} := \{v \in L^2(\Omega) : (4.11) \text{ holds and } \bar{\mathfrak{p}}(x) \neq 0 \implies v(x) = 0\}$.

Proof. Let $v \in C_{\bar{z}}$. Define, for every $k \in \mathbb{N}$ and for a.e. $x \in \Omega$, the function

$$v_k(x) := \begin{cases} 0 & \text{if} \quad x : \mathfrak{a} < \bar{z}(x) < \mathfrak{a} + \frac{1}{k}, \quad \mathfrak{b} - \frac{1}{k} < \bar{z}(x) < \mathfrak{b}, \\ \Pi_{[-k,k]}(v(x)) & \text{otherwise.} \end{cases}$$

Since $v \in C_{\bar{z}}$, we have that $v_k \in C_{\bar{z}}$. In addition, $|v_k(x)| \leq |v(x)|$ and $v_k(x) \to v(x)$ as $k \uparrow \infty$ for a.e. $x \in \Omega$; therefore $v_k \to v$ in $L^2(\Omega)$. Now, since $\bar{z} + \rho v_k \in \mathbb{Z}_{ad}$, for $\rho \in (0, k^{-2}]$, and $\bar{z}$ is locally optimal for j we deduce, for ρ sufficiently small,

$$0 \leq \tfrac{1}{\rho}[j(\bar{z} + \rho v_k) - j(\bar{z})] = j'(\bar{z})v_k + \tfrac{\rho}{2} j''(\bar{z} + \theta_k \rho v_k)v_k^2, \quad \theta_k \in (0,1). \tag{4.14}$$

Observe that (4.8) and $v_k \in C_{\bar{z}}$ yield $j'(\bar{z})v_k = \int_\Omega \mathfrak{p}(x)v_k(x)\mathrm{d}x = 0$. Thus, diving by ρ in (4.14), utilizing the characterization (4.12), and letting $\rho \downarrow 0$ yield $j''(\bar{z})v_k^2 \geq 0$. Let $k \uparrow \infty$ and invoke (4.12), again, and $\|v_k - v\|_{L^2(\Omega)} \to 0$ to conclude. ☐

We now provide a sufficient second order optimality condition with a minimal gap with respect to the the necessary one proved in Theorem 4.6.

THEOREM 4.7 (second order sufficient optimality conditions). *Let $n \in \{2,3\}$ and $s > n/4$. Let $\bar{z} \in \mathbb{Z}_{ad}$ be a locally optimal control for problem* (4.1)–(4.2) *satisfying*

$$j''(\bar{z})v > 0 \quad \forall v \in C_{\bar{z}} \setminus \{0\}. \tag{4.15}$$

Then, there exists $\kappa > 0$ and $\mu > 0$ such that

$$j(z) \geq j(\bar{z}) + \tfrac{\kappa}{2}\|z - \bar{z}\|^2_{L^2(\Omega)} \tag{4.16}$$

for all $z \in \mathbb{Z}_{ad}$ such that $\|\bar{z} - z\|_{L^2(\Omega)} \leq \mu$.

Proof. We proceed by contradiction. Assume that for every $k \in \mathbb{N}$ there is an element $z_k \in \mathbb{Z}_{ad}$ such that

$$\|\bar{z} - z_k\|_{L^2(\Omega)} < \tfrac{1}{k}, \quad j(z_k) < j(\bar{z}) + \tfrac{1}{2k}\|\bar{z} - z_k\|^2_{L^2(\Omega)}. \tag{4.17}$$

Define $\rho_k := \|z_k - \bar{z}\|_{L^2(\Omega)}$ and $v_k := \rho_k^{-1}(z_k - \bar{z})$. Notice that there exists a nonrelabeled subsequence $\{v_k\}_{k=1}^{\infty} \subset L^2(\Omega)$ such that $v_k \rightharpoonup v$ in $L^2(\Omega)$ as $k \uparrow \infty$.

We now proceed in three steps:

[1] We prove that $v \in C_{\bar{z}}$. Since the set of elements satisfying (4.11) is closed and convex in $L^2(\Omega)$ and, for every $k \in \mathbb{N}$, v_k belongs to this set, we deduce that v satisfies (4.11). It suffices to prove that $\mathfrak{p}(x) \neq 0$ implies $v(x) = 0$. In view of (4.7), we deduce that $\int_\Omega \mathfrak{p}(x)v(x)\mathrm{d}x \geq 0$ because $\int_\Omega \mathfrak{p}(x)v_k(x)\mathrm{d}x = \rho_k^{-1}\int_\Omega \mathfrak{p}(x)(z_k(x) - \bar{z}(x))\mathrm{d}x \geq 0$. On the other hand, observe that (4.17) and the mean value theorem reveal

$$j(z_k) - j(\bar{z}) = j'(\bar{z} + \theta_k(z_k - \bar{z}))(z_k - \bar{z}) < \tfrac{1}{2k}\|\bar{z} - z_k\|^2_{L^2(\Omega)} = \tfrac{\rho_k^2}{2k}, \quad \theta_k \in (0,1).$$

Divide by ρ_k and let $k \uparrow \infty$ to arrive at $j'(\bar{z} + \theta_k(z_k - \bar{z}))v_k < (2k)^{-1}\rho_k \to 0$ as $k \uparrow \infty$. Define $\hat{z}_k := \bar{z} + \theta_k(z_k - \bar{z})$. Since $s > n/4$ and $\hat{z}_k \to \bar{z}$ in $L^2(\Omega)$, as $k \uparrow \infty$, we have

$$\hat{u}_k := \mathcal{S}(\hat{z}_k) \to \mathcal{S}(\bar{z}) = \bar{u} \text{ in } \tilde{H}^s(\Omega) \cap L^\infty(\Omega), \quad \tfrac{\partial L}{\partial u}(\cdot, \hat{u}_k) \to \tfrac{\partial L}{\partial u}(\cdot, \bar{u}) \text{ in } L^r(\Omega),$$

upon invoking (B.2). Consequently, $\hat{p}_k \to \bar{p}$ in $H^s(\Omega) \cap L^\infty(\Omega)$ as $k \uparrow \infty$. Here, $\hat{p}_k$ denotes the solution to (4.6) with z replaced by $\hat{z}_k$ and u replaced by $\hat{u}_k$. Thus,

$$\int_\Omega \bar{\mathfrak{p}}(x)v(x)\mathrm{d}x = \lim_{k\uparrow\infty}\int_\Omega [\hat{p}_k(x) + \alpha\hat{z_k}(x)]\, v_k(x)\mathrm{d}x = \lim_{k\uparrow\infty} j'(\bar{z} + \theta_k(z_k - \bar{z}))v_k \leq 0.$$

We have thus deduced that $\int_\Omega \mathfrak{p}(x)v(x)\mathrm{d}x = \int_\Omega |\mathfrak{p}(x)v(x)|\mathrm{d}x = 0$. Consequently, $\mathfrak{p}(x) \neq 0$ implies $v(x) = 0$ for a.e. $x \in \Omega$. This proves that $v \in C_{\bar{z}}$.

[2] We prove that $v = 0$. We begin with an application of Taylor's theorem:

$$j(z_k) = j(\bar{z}) + \rho_k j'(\bar{z})v_k + \tfrac{\rho_k^2}{2} j''(\hat{z}_k)v_k^2, \quad \theta_k \in (0,1),$$

where $\hat{z}_k = \bar{z} + \theta_k(z_k - \bar{z})$ and $\rho_k v_k = z_k - \bar{z}$. Now, $j'(\bar{z})v_k \geq 0$ and (4.17) yield

$$\tfrac{\rho_k^2}{2} j''(\hat{z}_k)v_k^2 \leq j(z_k) - j(\bar{z}) < \tfrac{1}{2k}\|\bar{z} - z_k\|_{L^2(\Omega)}^2.$$

This implies that $j''(\hat{z}_k)v_k^2 < k^{-1}$. Consequently, $j''(\hat{z}_k)v_k^2 \to 0$ as $k \uparrow \infty$.

We now prove that $j''(\bar{z})v^2 \leq \liminf_k j''(\hat{z}_k)v_k^2$. We begin by noticing that

$$j''(\hat{z}_k)v_k^2 = \int_\Omega \left(\frac{\partial^2 L}{\partial u^2}(x,\hat{u}_k)\phi_{v_k}^2 - \hat{p}_k \frac{\partial^2 a}{\partial u^2}(x,\hat{u}_k)\phi_{v_k}^2 + \alpha v_k^2 \right) \mathrm{d}x.$$

As $k \uparrow \infty$, $\hat{z}_k \to \bar{z}$ and $v_k \rightharpoonup v$ in $L^2(\Omega)$. We thus have $\hat{u}_k \to \bar{u}$ and $\hat{p}_k \to \bar{p}$ in $\tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ and $\phi_{v_k} \rightharpoonup \phi_v$ in $\tilde{H}^s(\Omega)$; the latter implies that $\phi_{v_k} \to \phi_v$ in $L^{\mathfrak{q}}(\Omega)$ as $k \uparrow \infty$ for $\mathfrak{q} < 2n/(n-2s)$. Invoke (B.2) to obtain

$$\left| \int_\Omega \left(\tfrac{\partial^2 L}{\partial u^2}(x,\hat{u}_k)\phi_{v_k}^2 - \tfrac{\partial^2 L}{\partial u^2}(x,\bar{u})\phi_v^2 \right) \mathrm{d}x \right| \leq \|\phi_{v_k}\|_{L^{\mathfrak{q}}(\Omega)}^2 \left\| \tfrac{\partial^2 L}{\partial u^2}(\cdot,\hat{u}_k) - \tfrac{\partial^2 L}{\partial u^2}(\cdot,\bar{u}) \right\|_{L^r(\Omega)}$$
$$+ \|\psi_{\mathfrak{m}}\|_{L^r(\Omega)} \|\phi_v + \phi_{v_k}\|_{L^{\mathfrak{q}}(\Omega)} \|\phi_v - \phi_{v_k}\|_{L^{\mathfrak{q}}(\Omega)} \to 0, \quad k \uparrow \infty.$$

On the other hand, invoke (A.2) to derive

$$\left| \int_\Omega \left(\bar{p}\frac{\partial^2 a}{\partial u^2}(x,\bar{u})\phi_v^2 - \hat{p}_k \frac{\partial^2 a}{\partial u^2}(x,\hat{u}_k)\phi_{v_k}^2 \right) \mathrm{d}x \right| \leq C_{\mathfrak{m}} \|\phi_v\|_{L^2(\Omega)}^2 \|\bar{p} - \hat{p}_k\|_{L^\infty(\Omega)}$$
$$+ C_{\mathfrak{m}} \|\hat{p}_k\|_{L^r(\Omega)} \left(\|\bar{u} - \hat{u}_k\|_{L^\infty(\Omega)} \|\phi_v\|_{L^{\mathfrak{q}}(\Omega)}^2 + \|\phi_v + \phi_{v_k}\|_{L^{\mathfrak{q}}(\Omega)} \|\phi_v - \phi_{v_k}\|_{L^{\mathfrak{q}}(\Omega)} \right) \to 0$$

as $k \uparrow \infty$. Finally, observe that $\|v\|_{L^2(\Omega)}^2 \leq \liminf_k \|v_k\|_{L^2(\Omega)}^2$ because $\|\cdot\|_{L^2(\Omega)}^2$ is weakly lower semicontinuous. We can thus conclude that $j''(\bar{z})v^2 \leq \liminf_k j''(\hat{z}_k)v_k^2$.

Finally, since $\liminf_k j''(\hat{z}_k)v_k^2 \leq 0$ and $v \in C_{\bar{z}}$, (4.15) implies that $v = 0$.

3 Since $v = 0$, $\phi_{v_k} \to 0$ in $L^{\mathfrak{q}}(\Omega)$ as $k \uparrow \infty$ for $\mathfrak{q} < 2n/(n-2s)$. This implies

$$\alpha = \alpha\|v_k\|_{L^2(\Omega)}^2 \leq \liminf_{k\uparrow\infty} j''(\hat{z}_k)v_k^2 \leq 0,$$

which contradicts the fact that $\alpha > 0$. This concludes the proof. ∎

Define, for $\tau > 0$,

$$C_{\bar{z}}^\tau := \{v \in L^2(\Omega) : \text{(4.11) holds and } |\bar{\mathfrak{p}}(x)| > \tau \implies v(x) = 0\}. \tag{4.18}$$

THEOREM 4.8 (equivalent optimality conditions). *Let* $n \in \{2,3\}$ *and* $s > n/4$. *Let* $\bar{z} \in \mathbb{Z}_{ad}$ *be locally optimal for problem* (4.1)–(4.2). *Thus,* (4.15) *is equivalent to*

$$\exists \mu, \tau > 0: \quad j''(\bar{z})v^2 \geq \mu\|v\|_{L^2(\Omega)}^2 \quad \forall v \in C_{\bar{z}}^\tau, \tag{4.19}$$

where $C_{\bar{z}}^\tau$ *is defined in* (4.18).

Proof. Since $C_{\bar{z}} \subset C_{\bar{z}}^\tau$, we immediately conclude that (4.19) implies (4.15). To prove that (4.15) implies (4.19) we proceed by contradiction. Assume that, for $\tau > 0$,

$$\exists v_\tau \in C_{\bar{z}}^\tau: \quad j''(\bar{z})v_\tau^2 < \tau\|v_\tau\|_{L^2(\Omega)}^2.$$

Define $w_\tau := \|v_\tau\|_{L^2(\Omega)}^{-1} v_\tau$. Note that, up to a nonrelabeled subsequence if necessary,

$$w_\tau \in C_{\bar{z}}^\tau, \qquad \|w_\tau\|_{L^2(\Omega)} = 1, \qquad j''(\bar{z})w_\tau^2 < \tau, \qquad w_\tau \rightharpoonup w \text{ in } L^2(\Omega). \tag{4.20}$$

We prove that $w \in C_{\bar{z}}$. Since the set of elements satisfying (4.11) is weakly closed in $L^2(\Omega)$, we conclude that w satisfies (4.11). On the other hand,

$$\int_\Omega \mathfrak{p}(x)w(x)\mathrm{d}x = \lim_{\tau\downarrow 0}\int_\Omega \mathfrak{p}(x)w_\tau(x)\mathrm{d}x = \lim_{\tau\downarrow 0}\int_{|\mathfrak{p}(x)|\leq\tau} \mathfrak{p}(x)w_\tau(x)\mathrm{d}x \leq \lim_{\tau\downarrow 0}\tau\sqrt{|\Omega|} = 0,$$

where we have used that $\mathfrak{p} \in L^2(\Omega)$, $w_\tau \rightharpoonup w$ in $L^2(\Omega)$, $w_\tau \in C^\tau_{\bar{z}}$, and $\|w_\tau\|_{L^2(\Omega)} = 1$. As a result, $\int_\Omega |\mathfrak{p}(x)w(x)|\mathrm{d}x = \int_\Omega \mathfrak{p}(x)w(x)\mathrm{d}x = 0$. This implies that if $|\mathfrak{p}(x)| \neq 0$, then $w(x) = 0$ for a.e. $x \in \Omega$. We have thus obtained that $w \in C_{\bar{z}}$.

We prove that $w = 0$. Since $w \in C_{\bar{z}}$, (4.15) implies that $w = 0$ or $j''(\bar{z})w^2 > 0$. On the other hand, the arguments elaborated in the step 2 of the proof of Theorem 4.7 in conjunction with (4.20) yield $j''(\bar{z})w^2 \leq \liminf_{\tau\downarrow 0} j''(\bar{z})w_\tau^2 \leq \limsup_{\tau\downarrow 0} j''(\bar{z})w_\tau^2 \leq 0$. Consequently, $w = 0$ and $\lim_{\tau\downarrow 0} j''(\bar{z})w_\tau^2 = 0$.

Finally, since $w = 0$ and $w_\tau \rightharpoonup 0$ in $L^2(\Omega)$ as $\tau \downarrow 0$, we have that $\phi_{w_\tau} \to 0$ in $L^{\mathfrak{q}}(\Omega)$, as $\tau \downarrow 0$, for $\mathfrak{q} < 2n/(n-2s)$. Thus, $\alpha = \alpha\|w_\tau\|^2_{L^2(\Omega)} \leq \liminf_{\tau\downarrow 0} j''(\bar{z})w_\tau^2 = 0$, which is a contradiction. This concludes the proof. ☐

4.4. Regularity estimates. In this section, we derive regularity estimates for the optimal control variables. In doing so, the following regularity result for the linear case $a \equiv 0$ will be of importance.

PROPOSITION 4.9 (Sobolev regularity of u on smooth domains). *Let $n \geq 1$, $s \in (0,1)$, and Ω be a domain such that $\partial\Omega \in C^\infty$. Let u be the solution to $(-\Delta)^s\mathsf{u} = \mathsf{f}$ in Ω and $\mathsf{u} = 0$ in Ω^c. If $\mathsf{f} \in H^t(\Omega)$, for some $t \geq -s$, then $\mathsf{u} \in H^{s+\vartheta}(\Omega)$, where $\vartheta = \min\{s+t, 1/2-\epsilon\}$ and $\epsilon > 0$ is arbitrarily small. In addition, we have*

$$(4.21) \qquad \|\mathsf{u}\|_{H^{s+\vartheta}(\Omega)} \lesssim \|\mathsf{f}\|_{H^t(\Omega)},$$

where the hidden constant depends on Ω, n, s, and ϑ.

Proof. See [19, 32]. ☐

Observe that smoothness of f does not ensure that the solution to $(-\Delta)^s\mathsf{u} = \mathsf{f}$ in Ω and $\mathsf{u} = 0$ in Ω^c is any smoother than $\cap\{H^{s+1/2-\epsilon}(\Omega) : \epsilon > 0\}$.

To present regularity estimates, we will assume that, in addition to (A.1)–(A.3) and (B.1)–(B.2), the nonlinear functions a and L satisfy (C.1)–(C.2).

THEOREM 4.10 (regularity estimates: $s \in (0,1)$). *Let $n \geq 2$ and $s \in (0,1)$. If Ω is such that $\partial\Omega \in C^\infty$, then $\bar{u}, \bar{p} \in H^{s+1/2-\epsilon}(\Omega)$ and $\bar{z} \in H^\gamma(\Omega)$, where $\gamma = \min\{1, s+1/2-\epsilon\}$ and ϵ denotes an arbitrarily small positive constant.*

Proof. Since $\bar{z} \in \mathbb{Z}_{ad}$ and $a(\cdot, 0) \in L^2(\Omega)$, we apply Proposition 4.9 with $t = 0$ to obtain $\bar{u} \in H^{s+\nu}(\Omega)$, where $\nu = \min\{s, 1/2-\epsilon\}$ and $\epsilon > 0$ is arbitrarily small, and

$$(4.22) \qquad \|\bar{u}\|_{H^{s+\nu}(\Omega)} \lesssim \|\bar{z} - a(\cdot,\bar{u})\|_{L^2(\Omega)} \lesssim \|\bar{z}\|_{L^2(\Omega)} + \|a(\cdot,0)\|_{L^2(\Omega)},$$

upon using that a is locally Lipschitz in the second variable and $\|\bar{u}\|_s \lesssim \|\bar{z}\|_{H^{-s}(\Omega)}$. We now obtain a first regularity estimate for $\bar{p}$. To do this, we first observe that, since $\frac{\partial a}{\partial u}(\cdot, u)$ is locally Lipschitz in u and $\bar{u}, \bar{p} \in H^s(\Omega)$, we have $\frac{\partial a}{\partial u}(\cdot,\bar{u})\bar{p} \in H^s(\Omega)$. In fact, notice that, for $x, y \in \Omega$ and $u \in \mathbb{R}$, we have

$$\left|\tfrac{\partial a}{\partial u}(x,u)\bar{p}(x) - \tfrac{\partial a}{\partial u}(y,u)\bar{p}(y)\right| \leq \left|\tfrac{\partial a}{\partial u}(x,u)\right| |\bar{p}(x) - \bar{p}(y)| + |\bar{p}(y)| \left|\tfrac{\partial a}{\partial u}(x,u) - \tfrac{\partial a}{\partial u}(y,u)\right|.$$

The definition of $|\cdot|_{H^s(\Omega)}$ thus imply $|\frac{\partial a}{\partial u}(\cdot,\bar{u})\bar{p}|_{H^s(\Omega)} \lesssim |\bar{p}|_{H^s(\Omega)} + \|\bar{p}\|_{L^\infty(\Omega)}|\bar{u}|_{H^s(\Omega)}$, upon using (A.3) Invoke (C.2) and Proposition 4.9 to obtain $\bar{p} \in H^{s+\iota}(\Omega)$, where

$\iota = \min\{s+\lambda, \frac{1}{2}-\epsilon\}$, $\lambda = \min\{s, \frac{1}{2}-s-\epsilon\}$, and $\epsilon > 0$ is arbitrarily small. In addition, we have the following estimate:

$$\begin{aligned}(4.23)\quad \|\bar{p}\|_{H^{s+\iota}(\Omega)} &\lesssim \|\tfrac{\partial a}{\partial u}(\cdot,\bar{u})\bar{p}\|_{H^s(\Omega)} + \|\tfrac{\partial L}{\partial u}(\cdot,\bar{u})\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)} \\ &\lesssim |\bar{p}|_{H^s(\Omega)} + \|\bar{p}\|_{L^\infty(\Omega)}|\bar{u}|_{H^s(\Omega)} + \|\tfrac{\partial L}{\partial u}(\cdot,\bar{u})\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)}.\end{aligned}$$

A nonlinear interpolation result based on [21, Theorem A.1] and [30, Lemma 28.1] allows to obtain that $\bar{z} \in H^{\upsilon}(\Omega)$, for $\upsilon = \min\{1, s+\iota\}$, with a similar estimate.

We now consider four cases.

$\boxed{1}$ $s \in (\frac{1}{2}, 1)$: Observe that $\nu = \iota = \frac{1}{2}-\epsilon$. Thus, $\bar{u}, \bar{p} \in H^{s+\frac{1}{2}-\epsilon}(\Omega)$, for $\epsilon > 0$ being arbitrarily small, and $\bar{z} \in H^1(\Omega)$. In addition, (4.22) and (4.23) yield

$$\begin{aligned}\|\bar{u}\|_{H^{s+\frac{1}{2}-\epsilon}(\Omega)} + \|\bar{p}\|_{H^{s+\frac{1}{2}-\epsilon}(\Omega)} + \|\bar{z}\|_{H^1(\Omega)} &\lesssim |\bar{p}|_{H^s(\Omega)} + \|\bar{p}\|_{L^\infty(\Omega)}|\bar{u}|_{H^s(\Omega)} \\ &+ \|\bar{z}\|_{L^2(\Omega)} + \|\tfrac{\partial L}{\partial u}(\cdot,\bar{u})\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)} + \|a(\cdot,0)\|_{L^2(\Omega)} =: \mathfrak{B}.\end{aligned}$$

$\boxed{2}$ $s = \frac{1}{2}$. The proof follows similar arguments. For brevity, we skip the details.

$\boxed{3}$ $s \in [\frac{1}{4}, \frac{1}{2})$. Here, $\nu = s$ and $\iota = \frac{1}{2}-\epsilon$. Thus, $\bar{u} \in H^{2s}(\Omega)$ and $\bar{p} \in H^{s+\frac{1}{2}-\epsilon}(\Omega)$, where $\epsilon > 0$ is arbitrarily small. In view of (4.9), a nonlinear interpolation argument yields $\bar{z} \in H^{s+\frac{1}{2}-\epsilon}(\Omega)$. In addition, we have the estimate

$$\|\bar{u}\|_{H^{2s}(\Omega)} + \|\bar{p}\|_{H^{s+\frac{1}{2}-\epsilon}(\Omega)} + \|\bar{z}\|_{H^{s+\frac{1}{2}-\epsilon}(\Omega)} \lesssim \mathfrak{B}.$$

Invoke Proposition 4.9 with $t = \frac{1}{2}-s-\epsilon$ and (C.1) to obtain that $\bar{u} \in H^{s+\frac{1}{2}-\epsilon}(\Omega)$. Observe that $s + \frac{1}{2} - \epsilon > \frac{1}{2} - s - \epsilon$ and $2s > \frac{1}{2} - s - \epsilon$ for $\epsilon > 0$ arbitrarily small.

$\boxed{4}$ $s \in (0, \frac{1}{4})$. We proceed on the basis of a bootstrap argument as in [14]. Since $s < \frac{1}{4}$, $\nu = s$ and $\iota = 2s$. Thus $\bar{u} \in H^{2s}(\Omega)$, $\bar{p} \in H^{3s}(\Omega)$, $\bar{z} \in H^{3s}(\Omega)$, and

$$\|\bar{u}\|_{H^{2s}(\Omega)} + \|\bar{p}\|_{H^{3s}(\Omega)} + \|\bar{z}\|_{H^{3s}(\Omega)} \lesssim \mathfrak{B}.$$

$\boxed{4.1}$ $s \in [\frac{1}{6}, \frac{1}{4})$. Observe that $2s > \frac{1}{2} - s - \epsilon$ for $\epsilon > 0$ being arbitrarily small. Invoke Proposition 4.9 with $t = 1/2 - s - \epsilon$ to obtain

$$\begin{aligned}\|\bar{u}\|_{H^{s+\frac{1}{2}-\epsilon}(\Omega)} &\lesssim \|z\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)} + \|a(\cdot,\bar{u}) - a(\cdot,0)\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)} + \|a(\cdot,0)\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)} \\ &\lesssim \|\bar{z}\|_{H^{2s}(\Omega)} + \|\bar{u}\|_{H^{2s}(\Omega)} + \|a(\cdot,0)\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)} \lesssim \mathfrak{B} + \|a(\cdot,0)\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)}.\end{aligned}$$

By assumption $\frac{\partial L}{\partial u}(\cdot,0) \in L^2(\Omega) \cap H^{\frac{1}{2}-s-\epsilon}(\Omega)$. On the other hand, $\frac{\partial a}{\partial u}(\cdot,\bar{u})\bar{p} \in H^{2s}(\Omega)$, because $\bar{p} \in H^{3s}(\Omega)$ and $\bar{u} \in H^{2s}(\Omega)$. Thus, Proposition 4.9 with $t = \frac{1}{2} - s - \epsilon$ yields

$$\|\bar{p}\|_{H^{s+\frac{1}{2}-\epsilon}(\Omega)} \lesssim |\bar{p}|_{H^{2s}(\Omega)} + \|\bar{p}\|_{L^\infty(\Omega)}|\bar{u}|_{H^{2s}(\Omega)} + \|\tfrac{\partial L}{\partial u}(\cdot,\bar{u})\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)}.$$

A nonlinear interpolation argument yields $\bar{z} \in H^{s+\frac{1}{2}-\epsilon}(\Omega)$ with a similar estimate.

$\boxed{4.2}$ $s \in (0, \frac{1}{6})$. Invoke Proposition 4.9 with $t = 2s$ to obtain $\bar{u} \in H^{3s}(\Omega)$ and

$$\|\bar{u}\|_{H^{3s}(\Omega)} \lesssim \|\bar{z}\|_{H^{2s}(\Omega)} + \|\bar{u}\|_{H^{2s}(\Omega)} + \|a(\cdot,0)\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)} \lesssim \mathfrak{B} + \|a(\cdot,0)\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)}.$$

$\boxed{4.2.1}$ $s \in [\frac{1}{8}, \frac{1}{6})$. Observe that $3s > \frac{1}{2} - s - \epsilon$ for $\epsilon > 0$ being arbitrarily small. Invoke Proposition 4.9 with with $t = \frac{1}{2} - s - \epsilon$ to obtain $\bar{u} \in H^{s+\frac{1}{2}-\epsilon}(\Omega)$ with

$$\|\bar{u}\|_{H^{s+\frac{1}{2}-\epsilon}(\Omega)} \lesssim \|\bar{z}\|_{H^{3s}(\Omega)} + \|\bar{u}\|_{H^{3s}(\Omega)} + \|a(\cdot,0)\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)} \lesssim \mathfrak{B} + \|a(\cdot,0)\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)}.$$

Invoke Proposition 4.9 again to deduce that $\bar{p}, \bar{z} \in H^{s+\frac{1}{2}-\epsilon}(\Omega)$.

$\boxed{4.2.2}$ $s \in (0, \frac{1}{8})$. Since $\bar{u}, \bar{p} \in H^{3s}(\Omega)$, we have that $\frac{\partial a}{\partial u}(\cdot, \bar{u})\bar{p} \in H^{3s}(\Omega)$. Apply Proposition 4.9 with $t = 3s$ to obtain

$$\|\bar{p}\|_{H^{4s}(\Omega)} \lesssim |\bar{p}|_{H^{3s}(\Omega)} + \|\bar{p}\|_{L^\infty(\Omega)}|\bar{u}|_{H^{3s}(\Omega)} + \|\tfrac{\partial L}{\partial u}(\cdot, \bar{u})\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)}.$$

Observe that $3s \leq \frac{1}{2} - s - \epsilon$ for $\epsilon > 0$ being arbitrarily small. A nonlinear interpolation argument yields $\bar{z} \in H^{4s}(\Omega)$ with a similar estimate. Similarly, we obtain $\bar{u} \in H^{4s}(\Omega)$.

$\boxed{4.2.2.1}$ $s \in [\frac{1}{10}, \frac{1}{8})$. Observe that $4s > \frac{1}{2} - s - \epsilon$ for $\epsilon > 0$ being arbitrarily small. We thus immediately obtain that $\bar{u}, \bar{p}, \bar{z} \in H^{s+\frac{1}{2}-\epsilon}(\Omega)$ with a suitable estimate.

$\boxed{4.2.2.2}$ $s \in (0, \frac{1}{10})$. Since $\bar{u}, \bar{p} \in H^{4s}(\Omega)$, we have that $\frac{\partial a}{\partial u}(\cdot, \bar{u})\bar{p} \in H^{4s}(\Omega)$. Apply Proposition 4.9 with $t = 4s$ to obtain $\bar{p} \in H^{5s}(\Omega)$. A nonlinear interpolation argument yields $\bar{z} \in H^{5s}(\Omega)$ with a suitable estimate.

From this procedure we note that, at every step, there is a regularity gain. Consequently, after a finite number of steps, which is proportional to s^{-1}, we can conclude that the desired regularity results hold. This concludes the proof. ∎

5. Finite element approximation of fractional semilinear PDEs. In this section, we analyze the convergence properties of suitable discretizations and derive a priori error estimates. For analyzing convergence properties, it will be sufficient to assume that Ω is an open and bounded Lipschitz polytope. However, additional assumptions on Ω will be imposed for deriving error estimates: Ω is smooth and convex; convexity being assumed for simplicity. Since in this case Ω cannot be meshed exactly, we consider curved simplices to discretize $\Omega \setminus \Omega_h$. Here, Ω_h denotes a suitable polytopal domain that *approximates* Ω.

For the sake of brevity, we restrict the presentation to open and bounded domains $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ such that $\partial\Omega \in C^2$; for Lipschitz polytopes the presentation is simpler (see Remark 5.1). We follow [27, Section 5.2] and consider a family of open, bounded, and convex polytopal domains $\{\Omega_h\}_{h>0}$, based on a family of quasi-uniform partitions made of closed simplices $\{\mathscr{T}_h\}_{h>0}$, that approximate Ω in the following sense:

$$\mathcal{N}_h \subset \bar{\Omega}_h, \quad \mathcal{N}_h \cap \partial\Omega_h \subset \partial\Omega, \quad |\Omega \setminus \Omega_h| \lesssim h^2. \tag{5.1}$$

Here, $h = \max_{T \in \mathscr{T}_h} h_T$ denotes the mesh-size of the quasi-uniform partition $\mathscr{T}_h = \{T\}$, where $h_T = \mathrm{diam}(T)$, and $\mathcal{N}_h$ correspond to the set of all nodes of the mesh $\mathscr{T}_h$. We shall also assume that Ω is convex so that $\Omega_h \subset \Omega$ for every $h > 0$.

Given a mesh $\mathscr{T}_h$, we define the finite element space of continuous piecewise polynomials of degree one as

$$\mathbb{V}_h = \left\{ v_h \in C^0(\overline{\Omega}) : v_{h|T} \in \mathbb{P}_1(T) \ \forall T \in \mathscr{T}_h, \ v_h = 0 \text{ on } \overline{\Omega} \setminus \Omega_h \right\}. \tag{5.2}$$

Note that discrete functions are trivially extended by zero to Ω^c and that we enforce a classical homogeneous Dirichlet boundary condition at the degrees of freedom that are located at the boundary of Ω_h.

REMARK 5.1 (polytopes). If Ω is a Lipschitz polytope the previous construction is not necessary: $\Omega = \Omega_h$ and $\mathbb{V}_h = \{v_h \in C^0(\overline{\Omega}) : v_{h|T} \in \mathbb{P}_1(T) \ \forall T \in \mathscr{T}_h\}$.

5.1. The discrete problem. We introduce the following finite element approximation of problem (3.1): Find $u_h \in \mathbb{V}_h$ such that

$$\mathcal{A}(u_h, v_h) + \int_{\Omega_h} a(x, u_h(x)) v_h(x) \mathrm{d}x = \int_{\Omega_h} f(x) v_h(x) \mathrm{d}x \quad \forall v_h \in \mathbb{V}_h. \tag{5.3}$$

Let $r > n/2s$ and $f \in L^r(\Omega)$. Let $a = a(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that is monotone increasing in u. Assume, in addition, that a satisfies (3.2) and $a(\cdot,0) \in L^r(\Omega)$. Withing this setting, Theorem 3.1 guarantees that the continuous problem (3.1) admits a unique solution $u \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ satisfying (3.3). Since $\mathcal{A}$ is coercive and a is monotone increasing in u, an appllication of Brouwer's fixed point theorem [33, Proposition 2.6] yields the existence of a unique solution for (5.3); see also the proof of [34, Theorem 26.A]. In addition, $\|u_h\|_s \lesssim \|f\|_{H^{-s}(\Omega)}$ for every $h > 0$.

5.2. Regularity estimates. Before deriving error estimates, it is of fundamental importance the understanding of regularity estimates for the solution of (3.1).

THEOREM 5.1 (regularity estimates: $s \in (0,1)$). *Let $n \geq 2$, $s \in (0,1)$, and Ω be a domain such that $\partial\Omega \in C^\infty$. Assume, in addition, that a is locally Lipschitz with respect to the second variable. If both $a(\cdot,0)$ and f belong to $H^{1/2-s-\epsilon}(\Omega)$, with ϵ arbitrarily small, then $u \in H^{s+1/2-\epsilon}(\Omega)$.*

Proof. Define $\lambda := \min\{s, \frac{1}{2} - s - \epsilon\}$, where $\epsilon > 0$ is arbitrarily small. Apply Proposition 4.9 with $t = \lambda$ to obtain $u \in H^{s+\nu}(\Omega)$, where $\nu = \min\{s + \lambda, \frac{1}{2} - \epsilon\}$, and

$$\|u\|_{H^{s+\nu}(\Omega)} \lesssim \|f\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)} + \|a(\cdot,u) - a(\cdot,0)\|_{H^\lambda(\Omega)} + \|a(\cdot,0)\|_{H^{1/2-s-\epsilon}(\Omega)}. \tag{5.4}$$

We now proceed in several steps on the basis of a bootstrap argument.

[1] If $s \geq \frac{1}{4}$, then $s > \frac{1}{2} - s - \epsilon$, with $\epsilon > 0$ arbitrarily small. Thus, $\lambda = \frac{1}{2} - s - \epsilon$ and

$$\|u\|_{H^{s+\frac{1}{2}-\epsilon}(\Omega)} \lesssim \|f\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)} + \|u\|_{H^s(\Omega)} + \|a(\cdot,0)\|_{H^{1/2-s-\epsilon}(\Omega)},$$

where we have used the local Lipschitz property of a in the second variable. Invoke the basic estimate $\|u\|_s \lesssim \|f\|_{H^{-s}(\Omega)}$ to bound $\|u\|_{H^s(\Omega)}$.

[2] If $s \in (0, \frac{1}{4})$, we have $\lambda = s$ and $\nu = 2s$. Consequently, $u \in H^{3s}(\Omega)$ with the estimate (5.4). Define $\iota := \min\{3s, \frac{1}{2} - s - \epsilon\}$, where $\epsilon > 0$ is arbitrarily small. Apply Proposition 4.9 with $t = \iota$ to obtain $u \in H^{s+\kappa}(\Omega)$, where $\kappa = \min\{s + \iota, \frac{1}{2} - \epsilon\}$, and

$$\|u\|_{H^{s+\kappa}(\Omega)} \lesssim \|f\|_{H^{\frac{1}{2}-s-\epsilon}(\Omega)} + \|a(\cdot,u) - a(\cdot,0)\|_{H^\iota(\Omega)} + \|a(\cdot,0)\|_{H^{1/2-s-\epsilon}(\Omega)}. \tag{5.5}$$

[2.1] If $s \in [\frac{1}{8}, \frac{1}{4})$, then $\iota = \frac{1}{2} - s - \epsilon$, $\kappa = \frac{1}{2} - \epsilon$, and $u \in H^{s+\frac{1}{2}-\epsilon}(\Omega)$.

[2.2] If $s \in (0, \frac{1}{8})$, we have $\iota = 3s$ and $\kappa = 4s$. Consequently, $u \in H^{5s}(\Omega)$ with the estimate (5.5). Define $\mu := \min\{5s, \frac{1}{2} - s - \epsilon\}$, where $\epsilon > 0$ is arbitrarily small. Invoke Proposition 4.9 with $t = \mu$ to obtain that $u \in H^{s+\upsilon}(\Omega)$, where $\upsilon = \min\{s + \mu, \frac{1}{2} - \epsilon\}$.

[2.2.1] If $s \in [\frac{1}{12}, \frac{1}{4})$, then $\mu = \frac{1}{2} - s - \epsilon$, $\upsilon = \frac{1}{2} - \epsilon$, and $u \in H^{s+\frac{1}{2}-\epsilon}(\Omega)$.

From this procedure we note that, at every step, there is a regularity gain of $2s$. Consequently, after a finite number of steps we can conclude that $u \in H^{s+\frac{1}{2}-\epsilon}(\Omega)$, with $\epsilon > 0$ arbitrarily small. ∎

5.3. Error estimates. We now present error estimates. In doing so, we will assume, in addition, that there exists $\phi \in L^r(\Omega)$, with $r > n/2s$, such that

$$|a(x,u) - a(x,v)| \leq |\phi(x)||u - v| \text{ a.e. } x \in \Omega, \; u, v \in \mathbb{R}. \tag{5.6}$$

THEOREM 5.2 (error estimates). *Let $n \geq 2$, $s \in (0,1)$, and $r > n/2s$. Let Ω be an open and bounded domain with Lipschitz boundary. Assume that a is as in the statement of Theorem 3.1. Assume, in addition, that a satisfies* (5.6). *Let $u \in \tilde{H}^s(\Omega)$ be the solution to* (3.1) *and let $u_h \in \mathbb{V}_h$ be its finite element approximation obtained as the solution to* (5.3). *Then, we have the quasi–best approximation result*

$$\|u - u_h\|_s \lesssim \|u - v_h\|_s \quad \forall v_h \in \mathbb{V}_h. \tag{5.7}$$

If, in addition, Ω is smooth and convex and both $a(\cdot,0)$ and f belong to $H^{1/2-s-\epsilon}(\Omega)$, with $\epsilon>0$ arbitrarily small, then

$$\|u-u_h\|_s \lesssim h^{\frac{1}{2}-\epsilon}\|u\|_{H^{s+1/2-\epsilon}(\Omega)}, \tag{5.8}$$

If, in addition, (5.6) *holds with $r \geq n/s$, then*

$$\|u-u_h\|_{L^2(\Omega)} \lesssim h^{\vartheta+\frac{1}{2}-\epsilon}\|u\|_{H^{s+1/2-\epsilon}(\Omega)}. \tag{5.9}$$

Here, $\vartheta=\min\{s,\frac{1}{2}-\epsilon\}$ with $\epsilon>0$ being arbitrarily small. In all three estimates the hidden constant is independent of u, u_h, and h.

Proof. Since a is monotone increasing in the second variable, we obtain

$$\begin{aligned}\|u-u_h\|_s^2 &= \mathcal{A}(u-u_h,u-u_h) \leq \mathcal{A}(u-u_h,u-u_h)+(a(\cdot,u)-a(\cdot,u_h),u-u_h)_{L^2(\Omega)}\\ &= \mathcal{A}(u-u_h,u-v_h)+(a(\cdot,u)-a(\cdot,u_h),u-v_h)_{L^2(\Omega)}, \quad v_h\in\mathbb{V}_h\end{aligned}$$

upon utilizing Galerkin orthogonality. Invoke (5.6) and $H^s(\Omega)\hookrightarrow L^{\mathfrak{q}}(\Omega)$ with $\mathfrak{q}\leq 2n/(n-2s)$ to obtain (5.7).

Assume now that Ω is smooth and convex so Theorem 5.1 applies; convexity being assumed for simplicity. To bound $\|u-v_h\|_s$ we first invoke [24, Theorem 3.3.3]:

$$\|u-v_h\|_s \lesssim \|u-v_h\|_{H^s(\Omega)} \quad \forall v_h\in\mathbb{V}_h, \qquad s\in(0,1)\setminus\{\tfrac{1}{2}\}.$$

The second ingredient is the localization of fractional order Sobolev seminorms [17, 18]:

$$|v|_{H^s(\Omega)}^2 \leq \sum_T\left[\int_T\int_{S_T}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\mathrm{d}y\mathrm{d}x+\frac{\mathfrak{c}}{sh_T^{2s}}\|v\|_{L^2(T)}^2\right], \quad s\in(0,1), \quad \mathfrak{c}>0,$$

for $v\in H^s(\Omega)$; S_T denotes a suitable patch associated to T. We stress that curved domains/simplices are also handled in [17, 18]. It thus suffices to note that, if $\mathfrak{T}$ denotes a boundary curved simplex, the fact that $u\in\tilde{H}^s(\Omega)\cap H^{s+1/2-\epsilon}(\Omega)$, with $\epsilon>0$ arbitrarily small, implies

$$\|u-u_h\|_{L^2(\mathfrak{T})}=\|u\|_{L^2(\Omega\setminus\Omega_h)} \lesssim h^{2\upsilon}\|u\|_{H^{\upsilon}(\Omega)}, \quad \upsilon=\min\{1,s+1/2-\epsilon\},$$

which follows from interpolating [27, estimate (5.2.18)] and $\|v\|_{L^2(\Omega\setminus\Omega_h)}\leq\|v\|_{L^2(\Omega)}$. On the other hand, if $v\in\tilde{H}^s(\Omega)\cap H^{s+1/2-\epsilon}(\Omega)$, with $\epsilon>0$ arbitrarily small, then

$$\int_{\mathfrak{T}}\int_{S_{\mathfrak{T}}}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\mathrm{d}y\mathrm{d}x \leq |v|_{H^s(S_{\mathfrak{T}})}^2 \lesssim h^{2(1/2-\epsilon)}\|v\|_{H^{s+1/2-\epsilon}(\Omega)}^2.$$

We thus utilize interpolation error estimates for the Scott–Zhang operator [8, Proposition 3.6] and Theorem 5.1 to arrive at the estimate (5.8); see [8, Section 3.2] for details and the particular treatment of the case $s=1/2$.

The error estimate in $L^2(\Omega)$ follows from duality. Define $0\leq\chi\in L^r(\Omega)$ by

$$\chi(x)=\frac{a(x,u(x))-a(x,u_h(x))}{u(x)-u_h(x)} \text{ if } u(x)\neq u_h(x), \quad \chi(x)=0 \text{ if } u(x)=u_h(x).$$

Let $\mathfrak{z}\in\tilde{H}^s(\Omega)$ be the solution to $\mathcal{A}(v,\mathfrak{z})+(\chi\mathfrak{z},v)_{L^2(\Omega)}=\langle\mathfrak{f},v\rangle$ for all $v\in\tilde{H}^s(\Omega)$; $\mathfrak{f}\in H^{-s}(\Omega)$. Let $\mathfrak{z}_h$ be the finite element approximation of $\mathfrak{z}$ within $\mathbb{V}_h$. Thus,

$$\begin{aligned}\langle\mathfrak{f},u-u_h\rangle &= \mathcal{A}(u-u_h,\mathfrak{z})+(\chi\mathfrak{z},u-u_h)_{L^2(\Omega)}=\mathcal{A}(u-u_h,\mathfrak{z}-\mathfrak{z}_h)+\mathcal{A}(u-u_h,\mathfrak{z}_h)\\ &\quad+(\chi\mathfrak{z},u-u_h)_{L^2(\Omega)}=\mathcal{A}(u-u_h,\mathfrak{z}-\mathfrak{z}_h)+(a(\cdot,u)-a(\cdot,u_h),\mathfrak{z}-\mathfrak{z}_h)_{L^2(\Omega)}\\ &\quad\leq\|u-u_h\|_s\|\mathfrak{z}-\mathfrak{z}_h\|_s+\|\phi\|_{L^r(\Omega)}\|u-u_h\|_{L^{\mathfrak{q}}(\Omega)}\|\mathfrak{z}-\mathfrak{z}_h\|_{L^{\mathfrak{q}}(\Omega)}.\end{aligned}$$

Here, q satisfies $2q^{-1} + r^{-1} = 1$, i.e., $q < 2n/(n-2s)$. Set $\mathfrak{f} = u - u_h \in L^2(\Omega)$. Notice that, since $\phi \in L^r(\Omega)$, with $r \geq n/s$, $\chi\mathfrak{z}$ belong to $L^2(\Omega)$. We can thus invoke Proposition 4.9 with $t = 0$ to obtain $\|\mathfrak{z}\|_{H^{s+\theta}(\Omega)} \lesssim \|u - u_h\|_{L^2(\Omega)}$. Consequently,

$$\|u - u_h\|^2_{L^2(\Omega)} \lesssim \|u - u_h\|_s \|\mathfrak{z} - \mathfrak{z}_h\|_s \lesssim h^{\frac{1}{2}-\epsilon} \|u\|_{H^{s+\frac{1}{2}-\epsilon}(\Omega)} h^{\vartheta} \|u - u_h\|_{L^2(\Omega)},$$

where $\vartheta = \min\{s, 1/2 - \epsilon\}$ and $\epsilon > 0$ is arbitrarily small. This concludes the proof. □

5.4. Convergence properties. Let $1 < p < \infty$ and let $\{f_h\}_{h>0}$ be a sequence such that $f_h \in L^p(\Omega_h)$. We will say that $f_h \rightharpoonup f$ in $L^p(\Omega)$ as $h \downarrow 0$ if $f \in L^p(\Omega)$ and

$$\int_{\Omega_h} f_h(x)v(x)\mathrm{d}x \to \int_{\Omega} f(x)v(x)\mathrm{d}x \quad \forall v \in L^q(\Omega), \quad h \downarrow 0, \quad p^{-1} + q^{-1} = 1. \tag{5.10}$$

If $p = \infty$, we will say that $f_h \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$ if $f \in L^\infty(\Omega)$ and (5.10) holds for every $v \in L^1(\Omega)$. Observe that, upon considering a suitable extension of f_h to $\Omega \setminus \Omega_h$, f_h can be understood as an element of $L^p(\Omega)$. Since $|\Omega \setminus \Omega_h| \to 0$ as $h \downarrow 0$, (5.10) is equivalent to $\int_\Omega \tilde{f}_h(x)v(x)\mathrm{d}x \to \int_\Omega f(x)v(x)\mathrm{d}x$ for $\{\tilde{v}_h\}_{h>0}$ being a uniformly bounded extension of $\{v_h\}_{h>0}$ to Ω.

REMARK 5.2 (polytopes). If Ω is a Lipschitz polytope, then (5.10) reduces to the standard concept of weak convergence in $L^p(\Omega)$ because $\Omega_h = \Omega$ for every $h > 0$.

PROPOSITION 5.3 (convergence). *Let $n \geq 2$, $s \in (0,1)$, and $r > n/2s$. Let Ω be an open, bounded, and convex domain such that $\partial\Omega \in C^2$. Assume that a is as in the statement of Theorem 3.1 and satisfies, in addition,* (5.6)*. Let $u \in \tilde{H}^s(\Omega)$ solves* (3.1)*. Let $\mathfrak{u}_h \in \mathbb{V}_h$ be the solution to* (5.3) *with f replaced by $f_h \in L^r(\Omega_h)$. Then,*

$$f_h \rightharpoonup f \text{ in } L^r(\Omega) \implies \mathfrak{u}_h \to u \text{ in } L^{\mathfrak{t}}(\Omega), \quad h \downarrow 0, \quad \mathfrak{t} \leq 2n/(n-2s).$$

Here, $f_h \rightharpoonup f$ in $L^r(\Omega)$ is understood in the sense of (5.10)*.*

Proof. A simple application of the triangle inequality yields

$$\|u - \mathfrak{u}_h\|_{L^{\mathfrak{t}}(\Omega)} \leq \|u - u_h\|_{L^{\mathfrak{t}}(\Omega)} + \|u_h - \mathfrak{u}_h\|_{L^{\mathfrak{t}}(\Omega)}, \quad \mathfrak{t} \leq 2n/(n-2s),$$

where u_h denotes the solution to (5.3). Since $H^s(\Omega) \hookrightarrow L^{\mathfrak{q}}(\Omega)$ for $\mathfrak{q} \leq 2n/(n-2s)$, the quasi–best approximation estimate (5.7) yields $\|u - u_h\|_{L^{\mathfrak{q}}(\Omega)} \lesssim \|u - v_h\|_s$ for an arbitrary $v_h \in \mathbb{V}_h$. A density argument as in [12, Theorem 3.2.3] reveals the convergence result $\|u - u_h\|_{L^{\mathfrak{q}}(\Omega)} \to 0$ as $h \downarrow 0$.

To control $\|u_h - \mathfrak{u}_h\|_{L^{\mathfrak{t}}(\Omega)}$ we invoke the problems that u_h and $\mathfrak{u}_h$ solve:

$$\begin{aligned} \|u_h - \mathfrak{u}_h\|_s^2 &= \mathcal{A}(u_h - \mathfrak{u}_h, u_h - \mathfrak{u}_h) = (f - f_h, u_h - \mathfrak{u}_h)_{L^2(\Omega)} \\ &\quad - (a(\cdot, u_h) - a(\cdot, \mathfrak{u}_h), u_h - \mathfrak{u}_h)_{L^2(\Omega)} \leq \|f - f_h\|_{H^{-s}(\Omega)} \|u_h - \mathfrak{u}_h\|_s. \end{aligned} \tag{5.11}$$

This immediately yields $\|u_h - \mathfrak{u}_h\|_{L^{\mathfrak{t}}(\Omega)} \lesssim \|f - f_h\|_{H^{-s}(\Omega)}$. Since $f_h \rightharpoonup f$ in $L^r(\Omega)$ we can thus obtain that $\|u_h - \mathfrak{u}_h\|_{L^{\mathfrak{t}}(\Omega)} \to 0$ as $h \downarrow 0$. This concludes the proof. □

REMARK 5.3 (convergence on polytopes). The result of Proposition 5.3 can also be obtained for Lipschitz polytopes; observe that the involved arguments do not utilize further regularity beyond what is natural for the problem: $u \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$.

6. Finite element approximation for the optimal control problem. In this section, we propose a finite element discretization for our control problem. We analyze convergence properties and derive, when possible, error estimates. To accomplish this task, we operate within the discrete setting introduced in section 5 and introduce, in addition, the finite element space of piecewise constant functions

$$\mathbb{Z}_h = \left\{ v_{\mathscr{T}} \in L^\infty(\Omega_h) : v_{\mathscr{T}|T} \in \mathbb{P}_0(T) \ \forall T \in \mathscr{T}_h \right\} \tag{6.1}$$

and the space of discrete admissible controls $\mathbb{Z}_{\text{ad},h} = \mathbb{Z}_{\text{ad}} \cap \mathbb{Z}_h$.

6.1. The discrete optimal control problem. We consider the following discrete counterpart of the continuous optimal control problem (4.1)–(4.2): Find

$$\min\{J_h(u_h, z_h) : (u_h, z_h) \in \mathbb{V}_h \times \mathbb{Z}_{ad,h}\} \tag{6.2}$$

subject to the *discrete state equation*

$$\mathcal{A}(u_h, v_h) + \int_{\Omega_h} a(x, u_h(x)) v_h(x) \mathrm{d}x = \int_{\Omega_h} z_h(x) v_h(x) \mathrm{d}x \quad \forall v \in \mathbb{V}_h. \tag{6.3}$$

Here, $J_h : \mathbb{V}_h \times \mathbb{Z}_{ad,h} \ni (u_h, z_h) \mapsto J_h(u_h, z_h) := \int_{\Omega_h} L(x, u_h(x)) \mathrm{d}x + \frac{\alpha}{2} \|z_h\|^2_{L^2(\Omega_h)} \in \mathbb{R}$.

We present the following result.

Theorem 6.1 (existence of an optimal pair and optimality system). *Let $n \geq 2$ and $s \in (0,1)$. Assume that* (A.1)–(A.3) *and* (B.1)–(B.2) *hold. Thus, the discrete optimal control problem* (6.2)–(6.3) *admits at least one solution $(\bar{u}_h, \bar{z}_h) \in \mathbb{V}_h \times \mathbb{Z}_{ad,h}$. In addition, if $(\bar{u}_h, \bar{z}_h)$ denotes an optimal solution for* (6.2)–(6.3)*, then the triple $(\bar{u}_h, \bar{p}_h, \bar{z}_h) \in \mathbb{V}_h \times \mathbb{V}_h \times \mathbb{Z}_{ad,h}$ satisfies*

$$\mathcal{A}(\bar{u}_h, v_h) + (a(\cdot, \bar{u}_h), v_h)_{L^2(\Omega_h)} = (\bar{z}_h, v_h)_{L^2(\Omega_h)} \quad \forall v_h \in \mathbb{V}_h, \tag{6.4}$$

$$\mathcal{A}(v_h, \bar{p}_h) + \left(\tfrac{\partial a}{\partial u}(\cdot, \bar{u}_h)\bar{p}_h, v_h\right)_{L^2(\Omega_h)} = \left(\tfrac{\partial L}{\partial u}(\cdot, \bar{u}_h), v_h\right)_{L^2(\Omega_h)} \quad \forall v_h \in \mathbb{V}_h, \tag{6.5}$$

and the variational inequality

$$(\bar{p}_h + \alpha \bar{z}_h, z_h - \bar{z}_h)_{L^2(\Omega_h)} \geq 0 \tag{6.6}$$

for every $z_h \in \mathbb{Z}_{ad,h}$.

Proof. The proof follows from the arguments developed, for the continuous counterpart, in Theorems 4.1 and 4.4. For brevity, we skip details. ∎

6.2. Convergence of discretizations. We begin with the following convergence result: *the sequence $\{\bar{z}_h\}_{h>0}$ of global solutions of the discrete control problems* (6.2)–(6.3) *converge, as $h \downarrow 0$, to a solution of the continuous fractional semilinear optimal control problem* (4.1)–(4.2).

Theorem 6.2 (convergence). *Let $n \geq 2$ and $s \in (0,1)$. Let Ω be a Lipschitz domain satisfying the exterior ball condition. Assume that (A.1)–(A.3) and (B.1)–(B.2) hold. Assume that $a = a(x,u)$ satisfies, in addition,* (5.6)*. Assume that $\frac{\partial L}{\partial u}(\cdot, 0) \in L^\infty(\Omega)$. Let $\bar{z}_h$, for every $h > 0$, be a global solution of the discrete optimal control problem. Then, there exist nonrelabeled subsequences $\{\bar{z}_h\}_{h>0}$ such that $\bar{z}_h \overset{*}{\rightharpoonup} \bar{z}$, in $L^\infty(\Omega)$, with $\bar{z}$ being a solution to* (4.1)–(4.2)*. In addition, we have*

$$\|\bar{z} - \bar{z}_h\|_{L^2(\Omega)} \to 0, \qquad j_h(\bar{z}_h) \to j(\bar{z}), \tag{6.7}$$

as $h \downarrow 0$.

Proof. Since $\{\bar{z}_h\}_{h>0}$ is uniformly bounded in $L^\infty(\Omega)$, we deduce the existence of a nonrelabeled subsequence $\{\bar{z}_h\}_{h>0}$ such that $\bar{z}_h \overset{*}{\rightharpoonup} \bar{z}$ in $L^\infty(\Omega)$ as $h \downarrow 0$. In what follows we prove that $\bar{z}$ is locally optimal for the continuous optimal control problem and that $j_h(\bar{z}_h) \to j(\bar{z})$ as $h \downarrow 0$.

Let $\tilde{z} \in \mathbb{Z}_{ad}$ be locally optimal for (4.1)–(4.2) and define $\tilde{p}$ as the solution to (4.6) with u replaced by $\tilde{u} := \mathcal{S}\tilde{z}$. Define $\tilde{z}_h \in \mathbb{Z}_{ad,h}$ by $\tilde{z}_h|_T := \int_T \tilde{z}(x)\mathrm{d}x/|T|$ for $T \in \mathscr{T}_h$. Now, observe that, since (A.3) holds and $\tilde{p}, \partial L/\partial u(\cdot, 0) \in L^\infty(\Omega)$, we can deduce that $\partial L/\partial u(\cdot,\tilde{u}) - \partial a/\partial u(\cdot,\tilde{u})\tilde{p} \in L^\infty(\Omega)$. In view of the fact that Ω is Lipschitz and satisfies the exterior ball condition, we can thus invoke [28, Proposition 1.1] to obtain that $\tilde{p} \in C^s(\mathbb{R}^n)$. The projection formula (4.9) thus yields $\tilde{z} \in C^s(\overline{\Omega})$. Consequently, $\|\tilde{z} - \tilde{z}_h\|_{L^\infty(\Omega_h)} \to 0$ as $h \downarrow 0$. Invoke that $\tilde{z}$ is locally optimal for (4.1)–(4.2) and that $\bar{z}_h$ corresponds to the global solution of the discrete control problem to arrive at

$$j(\tilde{z}) \le j(\bar{z}) \le \liminf_{h\downarrow 0} j_h(\bar{z}_h) \le \limsup_{h\downarrow 0} j_h(\bar{z}_h) \le \limsup_{h\downarrow 0} j_h(\tilde{z}_h) = j(\tilde{z}).$$

To obtain the last equality, we used that $\|\tilde{z} - \tilde{z}_h\|_{L^\infty(\Omega)} \to 0$ implies $j_h(\tilde{z}_h) \to j(\tilde{z})$ as $h \downarrow 0$. We have thus proved that $\bar{z}$ is locally optimal and $j_h(\bar{z}_h) \to j(\bar{z})$ as $h \downarrow 0$.

We now prove that $\|\bar{z} - \bar{z}_h\|_{L^2(\Omega)} \to 0$ as $h \downarrow 0$. In view of Proposition 5.3, we have that $\bar{u}_h \to \bar{u}$ in $L^{\mathfrak{q}}(\Omega)$, for $\mathfrak{q} \le 2n/(n-2s)$, as $h \downarrow 0$. Invoke the local Lipschitz property of L in the second variable to arrive at

$$\left| \int_{\Omega_h} [L(x,\bar{u}(x)) - L(x,\bar{u}_h(x))]\,\mathrm{d}x \right| \le \mathfrak{l}_M \|\bar{u} - \bar{u}_h\|_{L^1(\Omega)} \to 0, \quad h \downarrow 0.$$

Similar arguments yield $\|L(\cdot,\bar{u}) - L(\cdot,0)\|_{L^1(\Omega\setminus\Omega_h)} \le \mathfrak{l}_M\|\bar{u}\|_{L^1(\Omega\setminus\Omega_h)} \to 0$ as $h \downarrow 0$. Consequently, in view of the convergence result $j_h(\bar{z}_h) \to j(\bar{z})$, we obtain

$$\tfrac{\alpha}{2}\|\bar{z}_h\|^2_{L^2(\Omega)} \to \tfrac{\alpha}{2}\|\bar{z}\|^2_{L^2(\Omega)}, \quad h \downarrow 0.$$

This and the weak converge $\bar{z}_h \rightharpoonup \bar{z}$ in $L^2(\Omega)$ imply that $\bar{z}_h \to \bar{z}$ in $L^2(\Omega)$, as $h \downarrow 0$, and concludes the proof. ☐

We now prove a somehow reciprocal result: *every strict local minimum of the continuous problem* (4.1)–(4.2) *can be approximated by local minima of the discrete optimal control problems.*

THEOREM 6.3 (convergence). *Let $n \ge 2$ and $s \in (0,1)$. Let Ω be a Lipschitz domain satisfying the exterior ball condition. Assume that (A.1)–(A.3) and (B.1)–(B.2) hold. Assume that $a = a(x,u)$ satisfies, in addition,* (5.6). *Assume that $\frac{\partial L}{\partial u}(\cdot,0) \in L^\infty(\Omega)$. Let $\bar{z}$ be a strict local minimum of problem* (4.1)–(4.2). *Then, there exists a sequence $\{\bar{z}_h\}_{h>0}$ of local minima of the discrete problems such that*

$$\|\bar{z} - \bar{z}_h\|_{L^2(\Omega)} \to 0, \qquad j_h(\bar{z}_h) \to j(\bar{z}), \tag{6.8}$$

as $h \downarrow 0$.

Proof. Since $\bar{z}$ is a strict local minimum for problem (4.1)–(4.2), we deduce the existence of $\epsilon > 0$ such that the minimization problem

$$\min\{j(z) : z \in \mathbb{Z}_{ad} \text{ and } \|\bar{z} - z\|_{L^2(\Omega)} \le \epsilon\} \tag{6.9}$$

admits a unique solution $\bar{z} \in \mathbb{Z}_{ad}$.

On the other hand, let us consider, for $h > 0$, the discrete problem

$$\min\{j_h(z_h) : z_h \in \mathbb{Z}_{ad,h} \text{ and } \|\bar{z} - z_h\|_{L^2(\Omega)} \le \epsilon\}. \tag{6.10}$$

We extend discrete functions $z_h \in \mathbb{Z}_{ad,h}$, defined over Ω_h, to Ω by setting $z_h(x) = \bar{z}(x)$ for $x \in \Omega \setminus \Omega_h$. To conclude that problem (6.10) admits at least a solution, we need to verify that the set where the minimum is sought is nonempty; notice that such a set is compact. To accomplish this task, we define, as in the proof of Theorem 6.2, $\hat{z}_h \in \mathbb{Z}_{ad,h}$ by $\hat{z}_h|_T := \int_T \bar{z}(x)\mathrm{d}x/|T|$ for $T \in \mathscr{T}_h$. Observe that, on $\Omega \setminus \Omega_h$, $\hat{z}_h = \bar{z}$. Since $\bar{z} \in C^s(\Omega)$, we have that $\|\bar{z} - \hat{z}_h\|_{L^\infty(\Omega)} \to 0$ as $h \downarrow 0$. As a result, if h is sufficiently small, $\hat{z}_h \in \mathbb{Z}_{ad,h}$ is such that $\|\bar{z} - \hat{z}_h\|_{L^2(\Omega)} \leq \epsilon$. We can thus conclude the existence of $h_\star > 0$ such that problem (6.10) admits at least a solution for $h \leq h_\star$.

Let $h \leq h_\star$ and let $\bar{z}_h$ be a solution to problem (6.10). Since $\{\bar{z}_h\}_{0<h\leq h_\star}$ is bounded in $L^\infty(\Omega)$, there exist a subsequence $\{\bar{z}_{h_k}\}_{k=1}^\infty$ of $\{\bar{z}_h\}_{0<h\leq h_\star}$ such that $\bar{z}_{h_k} \overset{*}{\rightharpoonup} \tilde{z}$ in $L^\infty(\Omega)$ as $k \uparrow \infty$. Proceed as in the proof of Theorem 6.2 to obtain that $\tilde{z}$ is a solution to the continuous problem (6.9) and $\bar{z}_{h_k} \to \tilde{z}$ in $L^2(\Omega)$ as $k \uparrow \infty$. Since problem (6.9) admits a unique solution, we must have $\tilde{z} = \bar{z}$ and $\bar{z}_h \to \bar{z}$ in $L^2(\Omega)$ as $h \downarrow 0$. Observe that, for h sufficiently small, the constraint $\|\bar{z} - \bar{z}_h\|_{L^2(\Omega)} \leq \epsilon$ is not active in problem (6.10). Consequently, $\bar{z}_h$ solves the original discrete problem. We conclude by mentioning that the arguments elaborated in the proof of Theorem 6.2 yield (6.8). ◻

6.3. Error estimates. Let $\{\bar{z}_h\} \in \mathbb{Z}_{ad,h}$ be a sequence of local minima of the discrete optimal control problems such that $\|\bar{z} - \bar{z}_h\|_{L^2(\Omega_h)} \to 0$ as $h \downarrow 0$; $\bar{z}$ being a local solution of the continuous problem (4.1)–(4.2); see Theorems 6.2 and 6.3 . The main goal of this section is to provide an error estimate for $\bar{z} - \bar{z}_h$ in $L^2(\Omega)$, namely

$$\|\bar{z} - \bar{z}_h\|_{L^2(\Omega_h)} \lesssim h^\gamma, \quad \gamma = \min\left\{1, s + \tfrac{1}{2} - \epsilon\right\}, \quad \forall h \leq h_\star. \tag{6.11}$$

Here, $\epsilon > 0$ is arbitrarily small. In what follows, if necessary, we extend discrete functions $z_h \in \mathbb{Z}_{ad,h}$, defined over Ω_h, to Ω by setting $z_h(x) = \bar{z}(x)$ for $x \in \Omega \setminus \Omega_h$. We begin with the following instrumental result.

Theorem 6.4 (instrumental error estimate). *Let $n \in \{2, 3\}$ and $s > n/4$. Let Ω be a convex domain such that $\partial\Omega \in C^\infty$. Assume that (A.1)–(A.3), (B.1)–(B.2), and (C.1)–(C.2) hold. Let $\bar{z} \in \mathbb{Z}_{ad}$ satisfies the second order optimality condition* (4.15), *or equivalently* (4.19). *Let as assume that* (6.11) *is false. Then, there exists a positive constants $h_\star$ such that*

$$\tfrac{\mathfrak{C}}{2}\|\bar{z} - \bar{z}_h\|^2_{L^2(\Omega_h)} \leq \left[j'(\bar{z}_h) - j'(\bar{z})\right](\bar{z}_h - \bar{z}) \tag{6.12}$$

for every $h \leq h_\star$, where $\mathfrak{C} = \min\{\mu, \alpha\}$, μ is the constant appearing in (4.19), *and α denotes the regularization parameter.*

Proof. Since (6.11) is false, there exists a sequence $\{h_\ell\}_{\ell=1}^\infty$ such that $h_\ell \downarrow 0$ as $\ell \uparrow \infty$ and $\{\bar{z}_{h_\ell}\}_{\ell=1}^\infty$ satisfies $\|\bar{z} - \bar{z}_{h_\ell}\|_{L^2(\Omega_h)}/h_\ell^\gamma \to \infty$ as $h_\ell \downarrow 0$. In what follows, to simplify notation we omit the subindex ℓ. Invoke the mean value theorem to obtain

$$\left(j'(\bar{z}_h) - j'(\bar{z})\right)(\bar{z}_h - \bar{z}) = j''(\bar{z} + \theta_h(\bar{z}_h - \bar{z}))(\bar{z}_h - \bar{z})^2, \quad \theta_h \in (0, 1). \tag{6.13}$$

Define $v_h := (\bar{z}_h - \bar{z})/\|\bar{z}_h - \bar{z}\|_{L^2(\Omega)}$. Observe that, for every $h > 0$, we have $\|v_h\|_{L^2(\Omega)} = 1$. Upon considering a subsequence, if necessary, we can assume that $v_h \rightharpoonup v$ in $L^2(\Omega)$. Since the set of elements satisfying (4.11) is weakly closed in $L^2(\Omega)$ and each v_h satisfies (4.11), we conclude that v satisfies (4.11) as well. We now prove that $|\mathfrak{p}(x)| > 0$ implies $v(x) = 0$; recall that $\mathfrak{p} = \bar{p} + \alpha\bar{z}$. Define $\bar{\mathfrak{p}}_h := \bar{p}_h + \alpha\bar{z}_h$. Observe that $\|\mathfrak{p} - \mathfrak{p}_h\|_{L^2(\Omega)} \to 0$ as $h \downarrow 0$. As a result, we obtain

$$\begin{aligned}
\int_\Omega \mathfrak{p}(x)v(x)\mathrm{d}x &= \lim_{h\to 0}\int_{\Omega_h} \mathfrak{p}_h(x)v_h(x)\mathrm{d}x \\
&= \lim_{h\to 0}\frac{1}{\|\bar{z}_h - \bar{z}\|_{L^2(\Omega)}}\left[\int_{\Omega_h} \mathfrak{p}_h(x)[(\Pi_h\bar{z}(x) - \bar{z}(x)) + (\bar{z}_h(x) - \Pi_h\bar{z}(x))]\mathrm{d}x\right],
\end{aligned}$$

where $\Pi_h : L^2(\Omega) \to \mathbb{Z}_h$ denotes the orthogonal projection operator onto piecewise constant functions over $\mathscr{T}_h$. Since $0 \leq j_h'(\bar{z}_h)(\Pi_h \bar{z} - \bar{z}_h) = \int_{\Omega_h} \mathfrak{p}_h(x)(\Pi_h \bar{z}(x) - \bar{z}_h(x))\mathrm{d}x$, because $\Pi_h \bar{z} \in \mathbb{Z}_{ad,h}$, we have

$$\int_\Omega \mathfrak{p}(x)v(x)\mathrm{d}x \leq \lim_{h\to 0} \frac{1}{\|\bar{z}_h - \bar{z}\|_{L^2(\Omega)}} \left[\int_{\Omega_h} \mathfrak{p}_h(x)(\Pi_h \bar{z}(x) - \bar{z}(x))\mathrm{d}x\right].$$

Invoke the regularity results for $\bar{z}$ obtained in Theorem 4.10, namely $\bar{z} \in H^\gamma(\Omega)$, where $\gamma = \min\{s + 1/2 - \epsilon, 1\}$ and $\epsilon > 0$ is arbitrarily small, standard error estimates for Π_h, and $\lim_{h\to 0} \|\bar{z} - \bar{z}_h\|_{L^2(\Omega_h)}/h^\gamma = \infty$ to obtain $\int_\Omega \mathfrak{p}(x)v(x)\mathrm{d}x \leq 0$. In view of (4.11), we can thus conclude that $\int_\Omega |\mathfrak{p}(x)v(x)|\mathrm{d}x = 0$ and thus that $|\mathfrak{p}(x)| > 0$ implies $v(x) = 0$ for a.e $x \in \Omega$. Consequently, $v \in C_{\bar{z}}$.

Define $\hat{z}_h := \bar{z} + \theta_h(\bar{z}_h - \bar{z})$, $\hat{u}_h := \mathcal{S}\hat{z}_h$, and $\hat{p}_h$ as the solution to (4.6) with u replaced by $\hat{u}_h$. Invoke (4.12) to obtain

$$\begin{aligned}\lim_{h\to 0} j''(\hat{z}_h)v_h^2 &= \lim_{h\to 0}\int_\Omega \left(\frac{\partial^2 L}{\partial u^2}(x,\hat{u}_h)\phi_{v_h}^2 - \hat{p}_h\frac{\partial^2 a}{\partial u^2}(x,\hat{u}_h)\phi_{v_h}^2 + \alpha v_h^2\right)\mathrm{d}x \\ &= \alpha + \int_\Omega \left(\frac{\partial^2 L}{\partial u^2}(x,\bar{u})\phi_v^2 - \bar{p}\frac{\partial^2 a}{\partial u^2}(x,\bar{u})\phi_v^2\right)\mathrm{d}x,\end{aligned} \tag{6.14}$$

where we have used $\hat{u}_h \to \bar{u}$ and $\hat{p}_h \to \bar{p}$ in $\tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ and $\phi_{v_h} \rightharpoonup \phi_v$ in $\tilde{H}^s(\Omega)$; the latter implies that $\phi_{v_h} \to \phi_v$ in $L^\mathfrak{q}(\Omega)$ as $k \uparrow \infty$ for $\mathfrak{q} < 2n/(n-2s)$; see the proof of Theorem 4.7 for details. Invoke that $v \in C_{\bar{z}}^\tau$ and $\bar{z}$ satisfies (4.19) to obtain

$$\lim_{h\to 0} j''(\hat{z}_h)v_h^2 = \alpha + j''(\bar{z})v^2 - \alpha\|v\|^2_{L^2(\Omega)} \geq \alpha + (\mu - \alpha)\|v\|^2_{L^2(\Omega)}.$$

Since $\|v\|_{L^2(\Omega)} \leq 1$, we can thus conclude that $\lim_{h\to 0} j''(\hat{z}_h)v_h^2 \geq \mathfrak{C}$, where $\mathfrak{C} = \min\{\mu, \alpha\}$. As a result, there exists $h_\star > 0$ such that, for every $h \leq h_\star$, we have $j''(\hat{z}_h)v_h^2 \geq \mathfrak{C}/2$.

In view of (6.13), we can finally derive (6.12) and conclude the proof. ☐

We now provide an error estimate for the difference $\bar{z} - \bar{z}_h$ in $L^2(\Omega)$.

THEOREM 6.5 (error estimate). *Let $n \in \{2, 3\}$ and $s > n/4$. Let Ω be a convex domain such that $\partial\Omega \in C^\infty$. Assume that (A.1)–(A.3), (B.1)–(B.2), and (C.1)–(C.2) hold. Let $\bar{z} \in \mathbb{Z}_{ad}$ satisfies the second order optimality condition* (4.15)*, or equivalently* (4.19)*. Then, there exist $h_\star > 0$ such that*

$$\|\bar{z} - \bar{z}_h\|_{L^2(\Omega_h)} \lesssim h^\gamma, \quad \gamma = \min\{1, s + 1/2 - \epsilon\}, \quad \forall h \leq h_\star, \tag{6.15}$$

where $\epsilon > 0$ is arbitrarily small.

Proof. We proceed by contradiction. Let us assume that (6.15) is false so that we have at hand the instrumental estimate of Theorem 6.4.

We begin by observing that $j_h'(\bar{z}_h)(z_h - \bar{z}_h) \geq 0$ for every $z_h \in \mathbb{Z}_{ad,h}$ and $j'(\bar{z})(\bar{z}_h - \bar{z}) \geq 0$. In view of these inequalities, we invoke (6.12) to obtain

$$\tfrac{\mathfrak{C}}{2}\|\bar{z} - \bar{z}_h\|^2_{L^2(\Omega_h)} \leq [j_h'(\bar{z}_h) - j'(\bar{z}_h)](z_h - \bar{z}_h) + j'(\bar{z}_h)(z_h - \bar{z}) \tag{6.16}$$

for every $z_h \in \mathbb{Z}_{ad,h}$. Let $\Pi_h : L^2(\Omega) \to \mathbb{Z}_h$ be the orthogonal projection operator onto piecewise constant functions over $\mathscr{T}_h$. Set $z_h = \Pi_h \bar{z} \in \mathbb{Z}_{ad,h}$ in (6.16) to obtain

$$\tfrac{\mathfrak{C}}{2}\|\bar{z} - \bar{z}_h\|^2_{L^2(\Omega_h)} \leq [j_h'(\bar{z}_h) - j'(\bar{z}_h)](\Pi_h \bar{z} - \bar{z}_h) + j'(\bar{z}_h)(\Pi_h \bar{z} - \bar{z}) =: \mathrm{I} + \mathrm{II}.$$

We bound the term II as follows. First, standard properties of Π_h reveal that

$$\text{(6.17)}\quad \mathrm{II}_{\Omega_h} := (p(\bar{z}_h) + \alpha \bar{z}_h, \Pi_h \bar{z} - \bar{z})_{L^2(\Omega_h)} = (p(\bar{z}_h), \Pi_h \bar{z} - \bar{z})_{L^2(\Omega_h)}$$
$$= (p(\bar{z}_h) - \Pi_h p(\bar{z}_h), \Pi_h \bar{z} - \bar{z})_{L^2(\Omega_h)} \lesssim h^{\gamma+\vartheta+\frac{1}{2}-\epsilon} |\bar{p}(\bar{z}_h)|_{H^{s+\frac{1}{2}-\epsilon}(\Omega)} |\bar{z}|_{H^{\gamma}(\Omega)},$$

where $\vartheta = \min\{s, 1/2-\epsilon\}$, $\gamma = \min\{1, s+1/2-\epsilon\}$, $\epsilon > 0$ is arbitrarily small, and $p(\bar{z}_h)$ denotes the solution to (4.6) with u replaced by $\mathcal{S}\bar{z}_h$. Theorem 4.10 guarantees that $\bar{z} \in H^{\gamma}(\Omega)$ so that the term $|\bar{z}|_{H^{\gamma}(\Omega)}$ is uniformly bounded. On the other, Proposition 4.9 reveals that $p(\bar{z}_h) \in H^{s+1/2-\epsilon}(\Omega)$, where $\epsilon > 0$ is arbitrarily small. The remaining term $\mathrm{II}_{\Omega \setminus \Omega_h}$ vanishes:

$$|\mathrm{II}_{\Omega\setminus\Omega_h}| = |(p(\bar{z}_h) + \alpha \bar{z}_h, \Pi_h \bar{z} - \bar{z})_{L^2(\Omega\setminus\Omega_h)}| = 0.$$

We now control I. To accomplish this task, we first observe that $\mathrm{I} = (\bar{p}_h - p(\bar{z}_h)), \Pi_h \bar{z} - \bar{z}_h)_{L^2(\Omega)}$. Second, we split $\mathrm{I} = \mathrm{I}_{\Omega_h} + \mathrm{I}_{\Omega\setminus\Omega_h}$ and control I_{Ω_h} as follows:

$$\text{(6.18)}\quad \mathrm{I}_{\Omega_h} := (\bar{p}_h - p(\bar{z}_h), \Pi_h \bar{z} - \bar{z}_h)_{L^2(\Omega_h)} = (\bar{p}_h - p(\bar{z}_h), \Pi_h(\bar{z} - \bar{z}_h))_{L^2(\Omega_h)}$$
$$\lesssim \|\bar{p}_h - p(\bar{z}_h)\|_{L^2(\Omega)} \|\bar{z} - \bar{z}_h\|_{L^2(\Omega)} \leq \frac{\mathfrak{C}}{4} \|\bar{z} - \bar{z}_h\|^2_{L^2(\Omega)} + C h^{\vartheta+1/2-\epsilon} |p(\bar{z}_h)|_{H^{s+\frac{1}{2}-\epsilon}(\Omega)},$$

where $\vartheta = \min\{s, 1/2-\epsilon\}$, $\epsilon > 0$ is arbitrarily small and $C > 0$. Since discrete functions $z_h \in \mathbb{Z}_{ad,h}$ are extended to Ω upon setting $z_h(x) = \bar{z}(x)$ in $\Omega\setminus\Omega_h$, $\mathrm{I}_{\Omega\setminus\Omega_h} = 0$.

A collection of the derived estimates yields the bound $\|\bar{z} - \bar{z}_h\|_{L^2(\Omega)} \lesssim h^{\gamma}$, which is a contradiction. This concludes the proof. ☐

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