LECTURE 10 | 09/23/14

Application # 1. Two point boundary value problems.

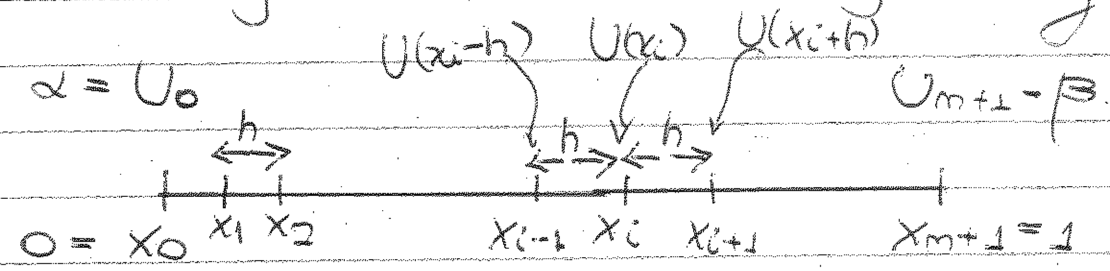
Consider the following problem

$$(1)\begin{cases} -u''(x) + u(x) = f(x), & 0 < x < 1 \\ u(0) = \alpha \ ; \ u(1) = \beta \end{cases}$$

Given a function f and $\alpha, \beta \in \mathbb{R}$, we want to find $u = u(x)$, $x \in (0,1)$ solution of (1).

§1. Finite difference method

Notation: $\Omega = (0,1)$. We first need a partition of Ω : $\{x_i\}_{i=0}^{m+1}$ defined by



$x_{i+1} - x_i = h \quad \forall i \in \{0, \dots, m\}$

This is a partition of Ω in m subintervals of the same size!

We define $U = \{U_i\}_{i=0}^{m+1}$ to be the approximation of u. Then, immediately we have from boundary conditions.

$$u(0) = \alpha \quad \text{and} \quad u(1) = \beta$$
$$\Rightarrow U_0 = \alpha \quad \text{and} \quad U_{m+1} = \beta$$

How do we find $U_1, \ldots, U_m$?

Fundamental tool: Taylor polynomials

Let $u : \Omega \to \mathbb{R}$, and $x_0 \in \Omega$ and assume that u has $k+1$ derivatives. Then for each $x \in \Omega = (0,1)$, there exists $\xi \in (x, x_0)$ s.t.

$$u(x) = u(x_0) + u'(x_0)(x-x_0) + u''(x_0)\frac{(x-x_0)^2}{2} + \cdots + u^{(k)}(x_0)\frac{(x-x_0)^k}{k} + \frac{u^{(k+1)}(\xi)}{(k+1)!}(x-x_0)^{k+1}$$

$$= \underbrace{p_k(x)}_{\text{Taylor polynomial of degree } k} + \underbrace{r_k(x)}_{\text{residual}}$$

Notice that

$$|u(x) - p_k(x)| = \left| \frac{u^{(k+1)}}{(k+1)!}(\xi)(x-x_0)^{k+1} \right|$$
$$< \frac{|u^{(k+1)}(\xi)|}{(k+1)!} \xrightarrow[k \to \infty]{} 0$$

p_K is a good approx. of u when K is sufficiently large.

However,
$$\begin{cases} p_K(x_0) = u(x_0) \\ p_K'(x_0) = u'(x_0) \\ p_K''(x_0) = u''(x_0) \\ \vdots \\ p_K^{(K)}(x_0) = u^{(K)}(x_0) \end{cases}$$

How do we use this tool in our problem?

Suppose u has two derivatives

$$u(x+h) = u(x) + u'(x)h + \frac{u''(\xi)}{2!}h^2$$

$$\left.\begin{aligned} x &\to x+h \\ x_0 &\to x \end{aligned}\right\} \Rightarrow x - x_0 = (x+h) - x = h$$

$$\Rightarrow u(x+h) - u(x) = u'(x)h + \frac{u''(\xi)}{2}h^2$$

$$\Rightarrow u'(x) = \frac{u(x+h) - u(x)}{h} - \frac{u''(\xi)}{2}h$$

Then, we can approximate

$$u'(x) \approx \frac{u(x+h) - u(x)}{h} \quad \text{for } h \text{ small.}$$

Suppose now, we have 4 derivatives in u. Then

$$u(x+h) = u(x) + u'(x)h + u''(x)\frac{h^2}{2} + u'''(x)\frac{h^3}{3!} + \frac{u^{(4)}}{4!}(\xi)h^4$$

and

$$u(x-h) = u(x) - u'(x)h + u''(x)\frac{h^2}{2} - u'''(x)\frac{h^3}{3!}$$

$$\left.\begin{array}{l} x \to x-h \\ x_0 \to x \end{array}\right\} \Rightarrow \cancel{x - x_0 = x - h - x = -h}$$

$$+ \frac{u^{(4)}}{4!}(\xi)h^4$$

$$\Rightarrow u(x+h) + u(x-h) = 2u(x) + u''(x)h^2 + \frac{u^{(4)}}{4!}(\xi)h^4$$

$$\Rightarrow u''(x)h^2 = u(x+h) - 2u(x) + u(x+h) - \frac{u^{(4)}}{4!}(\xi)h^4$$

$$\Rightarrow u''(x) = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} - \frac{u^{(4)}}{4!}(\xi)h^2$$

Then, we can approximate

$$u''(x) \simeq \frac{u(x-h) - 2u(x) + u(x+h)}{h^2}$$

for h small.

In our problem

$$-\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + U_i = \underbrace{f(x_i)}_{f_i}$$

$i = 1, \dots, m$

$$\Rightarrow -U_{i-1} + 2U_i - U_{i+1} + h^2 U_i = h^2 f_i$$

First eqn $i = 1$: $(2 + h^2)U_1 - U_2 = h^2 f_1 + U_0$
$= h^2 f_1 + \alpha$

Second eqn $i = 2$: $-U_1 + (2+h^2)U_2 + U_3 = h^2 f_2$

Third eqn $i = 3$: $-U_2 + (2+h^2)U_3 + U_4 = h^2 f_3$

In matricial form

$$\underbrace{\begin{pmatrix} 2+h^2 & -1 & & & \\ -1 & 2+h^2 & -1 & & \\ & -1 & 2+h^2 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & -1 & 2+h^2 & -1 \\ & & & -1 & 2+h^2 \end{pmatrix}}_{\text{tri diagonal matrix!}} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_{m-1} \\ U_m \end{pmatrix} = \begin{pmatrix} h^2 f_1 + \alpha \\ \\ \\ \vdots \\ \\ h^2 f_m + \beta \end{pmatrix}$$

$\Rightarrow$ LU decomposition.

Is this method good? (6)

$$|u(x) - U(x)| \leq Ch^2 \xrightarrow[h \to 0]{} 0$$

When u has 4 derivatives ☒