LECTURE 11 | 09/26/14.

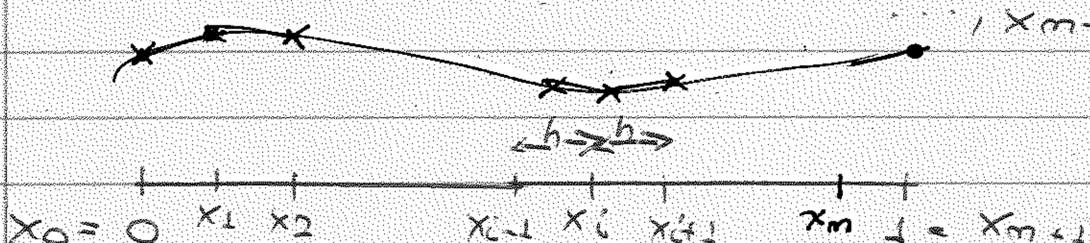
Review last class:

Consider the following problem

$$(1) \quad \begin{cases} -u''(x) + u(x) = f(x), & 0 < x < 1 \\ u(0) = \alpha; \ u(1) = \beta \end{cases}$$

§1 Finite difference method. Consider a partition of $\Omega = (0,1)$, $\{x_i\}_{i=0}^{m+1}$ where $x_i = ih$, $h = \frac{1}{m+1}$

$\Rightarrow x_0 = 0, \ x_1 = h, \ \ldots, \ x_{m+1} = 1.$



Approximation of second derivatives:

$$(2) \quad u''(x) \approx \frac{u(x-h) - 2u(x) + u(x+h)}{h^2}$$

Evaluate (1) at x_i

$$-u''(x_i) + u(x_i) = f(x_i).$$

Recall that U is the approximation of u

Then, approximate $u''(x_i)$ by formula (2)

$$-\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + U_i = \underbrace{f(x_i)}_{f_i}$$

$$\Rightarrow (2+h^2)U_i - U_{i-1} - U_{i+1} = h^2 f_i$$

$$\Rightarrow -U_{i-1} + (2+h^2)U_i - U_{i+1} = h^2 f_i$$

$i = 1, \ldots, m$

Recall that $U_0 = \alpha$ and $U_{m+1} = \beta$.

1st eq: $(2+h^2)U_1 - U_2 = h^2 f_1 + \alpha$

2nd eq: $-U_1 + (2+h^2)U_2 - U_3 = h^2 f_2$

3rd eq: $-U_2 + (2+h^2)U_3 - U_4 = h^2 f_3$

$$\vdots$$

Then, you have the following linear system

$$\begin{pmatrix} 2+h^2 & -1 & & & & \\ -1 & 2+h^2 & -1 & & & \\ & -1 & 2+h^2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2+h^2 & -1 \\ & & & & -1 & 2+h^2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_{m-1} \\ U_m \end{pmatrix} = \begin{pmatrix} h^2 f_1 + \alpha \\ h^2 f_2 \\ \vdots \\ \vdots \\ h^2 f_{m-1} \\ h^2 f_m + \beta \end{pmatrix}$$

$\iff \quad A\,\underset{\sim}{U} = \underset{\sim}{F}$

$\uparrow$

tridiagonal matrix

$|u(x) - U(x)| \le Ch^2$, where C is a constant.

§ Finite element method

(3) $$\begin{cases} -u''(x) + u(x) = f(x), & 0 < x < 1 \\ u(0) = 1; \; u(1) = 1. \end{cases}$$

s.t. $v(0) = v(1) = 0$.

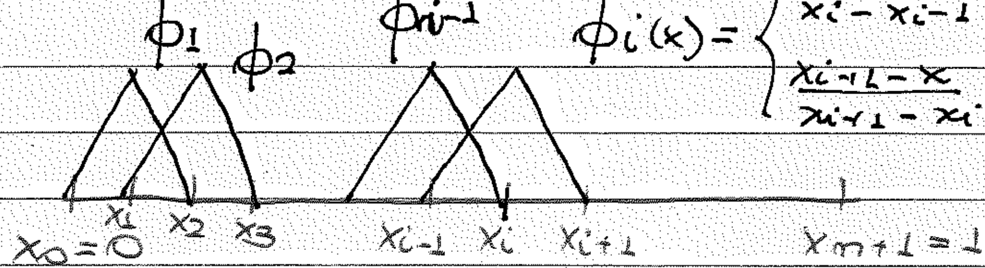
Multiply (3) by v and integrate by parts:

$$-\int_0^1 u''v + \int_0^1 uv = \int_0^1 fv$$

$$\Rightarrow \int_0^1 u'v' - u'v\Big|_{x=0}^{x=1} + \int_0^1 uv = \int_0^1 fv$$

$$\Rightarrow \int_0^1 u'v' + uv = \int_0^1 fv \qquad (4)$$

(4) is called weak formulation of (3). We have replaced u'' by u'.



$\phi_i(x_i) = 1$

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i \le x \le x_{i+1} \end{cases}$$

Approximation of u:

$$U = \sum_{i=1}^{m} \underset{\substack{\uparrow \\ \text{unknowns}}}{\alpha_i} \phi_i$$

Consider $v = \phi_j$ in (4) for $j = 1, \dots, m$, and replace u by U then

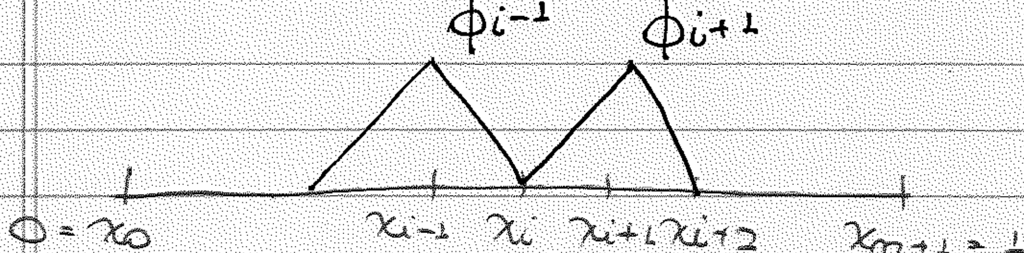
$$\int_0^1 \sum_{i=1}^{m} \alpha_i \phi_i' \phi_j' + \sum_{i=1}^{m} \alpha_i \phi_i \phi_j = \int_0^1 f \phi_j$$

$$\Rightarrow \sum_{i=1}^{m} \alpha_i \underbrace{\left(\int_0^1 \phi_i' \phi_j' + \phi_i \phi_j \right)}_{a_{ij}} = \underbrace{\int_0^1 f \phi_j}_{f_j}$$

$\forall j = 1, \dots, m.$

$$A \cdot \underset{\sim}{\alpha} = \underset{\sim}{F}$$

where $\underset{\sim}{\alpha} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}$, $\underset{\sim}{F} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$



$$a_{i-1,i+1} = \int_0^1 \phi_{i-1}'\phi_{i+1}' + \phi_{i-1}\phi_{i+1} = 0.$$

$\Rightarrow$ $a_{ij} = 0$ if $|i-j| > 1$

$\Rightarrow$ A is tridiagonal.

However
$$\left(\int_0^1 \phi_i' \phi_j'\right)_{i,j=1}^m = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

and
$$\left(\int_0^1 \phi_i \phi_j\right)_{i,j=1}^m = \frac{h}{6}\begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & & \ddots & & \\ & & 1 & 4 & 1 \\ & & & 1 & 4 \end{pmatrix}$$

$\boxtimes$