§1.8. General linear system.

Let's consider $A \in \mathbb{R}^{m \times n}$ and the linear system

(1) $\qquad A\underline{x} = \underline{b}$

where $\underline{b} \in \mathbb{R}^m$. The system (1) has

m equations

n unknowns

In principle, $m \neq n$

Definition (Echelon form) A $m \times n$ matrix is said to be in echelon form if it has the following structure

$$m \text{ rows}\left\{ U = \underbrace{\begin{bmatrix} \circledast \cdots * & & & \\ & \circledast \cdots * & & \\ & & \circledast \cdots * & \\ & 0 & & \circledast \cdots * \\ & & & \end{bmatrix}}_{n \text{ columns}} \right. \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} r \\ \left.\begin{matrix} \\ \end{matrix}\right\} m-r \end{matrix}$$

1) The entries $\circledast$ are the pivots of U and must be $\neq 0$

2) The first r rows of U contains exactly 1 pivots $\Rightarrow$ r pivots

3) The last $m-r$ rows are identically zero, and do not contain any pivots

Proposition ($PA = LU$) Any matrix can be reduced to row echelon form by a sequence of operations of Types #1 and #2

$\Rightarrow$ $PA = LU$, where $P \in \mathbb{R}^{m \times m}$ – permutation matrix
$L \in \mathbb{R}^{m \times m}$ – lower triang. $l_{ii} = 1$
$U \in \mathbb{R}^{m \times n}$ – echelon form

Example.
$$\begin{cases} x + 3y + 2z - u = a \\ 2x + 6y + z + 4u + 3v = b \\ -x - 3y - 3z + 3u + v = c \\ 3x + 9y + 8z - 7u + 2v = d \end{cases}$$

$a, b, c, d \in \mathbb{R}$

We consider the augmented matrix

↓ pivot

$$M = (A \mid b) = \left(\begin{array}{ccccc|c} 1 & 3 & 2 & -1 & 0 & a \\ 2 & 6 & 1 & 4 & 3 & b \\ -1 & -3 & -3 & 3 & 1 & c \\ 3 & 9 & 8 & -7 & 2 & d \end{array}\right) \xrightarrow[\substack{+R_1 + R_3 \\ -3R_1 + R_4}]{-2R_1 + R_2} \left(\begin{array}{ccccc|c} 1 & 3 & 2 & -1 & 0 & a \\ 0 & \boxed{0} & -3 & 6 & 3 & b-2a \\ 0 & 0 & -1 & 2 & 1 & c+a \\ 0 & 0 & 2 & -4 & 2 & d-3a \end{array}\right)$$

no pivots! no permutations!

$$\xrightarrow[\frac{2}{3}R_2 + R_4]{-\frac{1}{3}R_2 + R_3} \left(\begin{array}{ccccc|c} 1 & 3 & 2 & -1 & 0 & a \\ 0 & 0 & -3 & 6 & 3 & b-2a \\ 0 & 0 & 0 & \boxed{0} & 0 & c - 1/3 b + 5/3 a \\ 0 & 0 & 0 & 0 & 4 & d + 2/3 a - 13/3 a \end{array}\right)$$

no pivots!

$$\rightarrow \left(\begin{array}{ccccc|c} 1 & 3 & 2 & -1 & 0 & a \\ 0 & 0 & -3 & 6 & 3 & b-2a \\ 0 & 0 & 0 & 0 & 4 & d+\frac{2}{3}b-\frac{13}{a} \\ 0 & 0 & 0 & 0 & 0 & c-\frac{1}{3}b+\frac{5}{3}a \end{array}\right)$$

Then, we have the following PLU factorization

$$\underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 1 & 3 & 2 & -1 & 0 \\ 2 & 6 & 1 & 4 & 3 \\ -1 & -3 & -3 & 3 & 1 \\ 3 & 9 & 8 & -7 & 2 \end{pmatrix}}_{A} = \overbrace{\begin{pmatrix} 1 & & & \\ 2 & 1 & & \\ 1 & -2/3 & 1 & \\ 3 & 1/3 & 0 & 1 \end{pmatrix}}^{L} \underbrace{\begin{pmatrix} 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & -3 & 6 & 3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{echelon form}}$$

Definition (rank of A matrix) The rank of a matrix $A \in \mathbb{R}^{m \times n}$ is the number of pivots

In our example: $\operatorname{rank}(A) = 3$

In general $\operatorname{ran}(A) = r \le \min\{m, n\}$

Let's come back to our system: next class!