LECTURE 13 | 10/01/14.

General linear System.

From last class: We considered the example

$$\begin{cases} x + 3y + 2z - u = a \\ 2x + 6y + z + 4u + 3v = b \\ -x - 3y - 3z + 3u + v = c \\ 3x + 9y + 8z - 7u + 2v = d \end{cases}$$

We defined the augmented matrix

$$M = (A|b) = \left(\begin{array}{ccccc|c} 1 & 3 & 2 & -1 & 0 & a \\ 2 & 6 & 1 & 4 & 3 & b \\ -1 & -3 & -3 & 3 & 1 & c \\ 3 & 9 & 8 & -7 & 2 & d \end{array}\right) \longrightarrow \begin{array}{c} \left(\begin{array}{ccccc|c} \boxed{1} & 3 & 2 & -1 & 0 & a \\ 0 & 0 & \boxed{-3} & 6 & 3 & b-2a \\ 0 & 0 & 0 & 0 & \boxed{4} & d+\frac{2}{3}b-\frac{13}{3}a \\ 0 & 0 & 0 & 0 & 0 & c-\frac{1}{3}b+\frac{5}{3}a \end{array}\right) \\ \begin{array}{cccccc} & \uparrow & & \uparrow & & \\ & y & & u & & \end{array} \end{array}$$

$\Rightarrow$ rank $(A) = 3$.

We then can write the equivalent system:

$$\Rightarrow \begin{cases} x + 3y + 2z - u = a \\ -3z + 6u + 3v = b - 2a \\ 4v = d + 2/3 b - 13/3 a \\ 0 = c - 1/3 b + 5/3 a \end{cases}$$

To have solutions, we need a compatibility condition

$$\boxed{0 = c - \frac{1}{3}b + \frac{5}{3}a}$$

Otherwise the system has no solution!

$0 \neq c - \frac{1}{3}b + \frac{5}{3}a \Rightarrow$ no solution!

Consider $a = 0$, $b = 3$, $c = 1$ and $d = 1$.

Comp. condition $0 = 1 - 1/3 \cdot 3 + 5/3 \cdot 0 = 0$ ✓

Solve linear system:

$\rightarrow 4v = 3 \Rightarrow \boxed{v = 3/4}$

$\rightarrow -3z = 3 - 6u - 3v \Rightarrow z = -1 + 2u - v$

$= -1/4 + 2u$

$\boxed{z = -1/4 + 2u}$

$\rightarrow x = -3y - 2z + u$

$= -3y - 2(-1/4 + 2u) + u$

$\boxed{x = -3y + 1/2 - 3u}$

We call y and u free variables: variables corresponding to the columns without a pivot

We call x, z, v basic variables: variables corresponding to the columns with pivot.

Solution to linear system

$$\underset{\sim}{x} = \begin{pmatrix} 1/2 \\ 0 \\ -1/4 \\ 0 \\ 3/4 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} y + \begin{pmatrix} -3 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} u, \quad y, u \in \mathbb{R}$$

$\Rightarrow$ we have ∞ solutions! (3)

THM (Solution of linear systems) A system $\underset{\sim}{A}x = \underset{\sim}{b}$ of m linear equation in n unknowns has either

i) exactly one solution: for instance when $m = n$ and A is non singular (invertible)
ii) infinitely many solutions: compatibility condition ✓
iii) no solution (compatibily conditions are not satisfied.

§ Vector spaces and basis.

Vector space $\Rightarrow$ abstraction of the euclidean space $\mathbb{R}^n$

Def (Vector space). A vector space is a set V equipped with two operations $+$ (addition) and $\cdot$ multiplication by scalars s.t.

$$\text{given } \underset{\sim}{u}, \underset{\sim}{v} \in V \Rightarrow \underset{\sim}{u} + \underset{\sim}{v} \in V.$$
$$\text{given } \alpha \in \mathbb{R} \text{ and } \underset{\sim}{v} \in V \Rightarrow \alpha \cdot \underset{\sim}{v} \in V$$

and, for all $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V$ and $\alpha, \beta \in \mathbb{R}$ we have

(a) $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$ (Commutativity of addition)
(b) $\underset{\sim}{u} + (\underset{\sim}{v} + \underset{\sim}{w}) = (\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w}$ (Associativity of addition)
(c) $\exists\, \underset{\sim}{0} \in V: \underset{\sim}{0} + \underset{\sim}{v} = \underset{\sim}{v} + 0 = \underset{\sim}{v}$ (Additive Identity)
(d) For each $\underset{\sim}{v} \in V$, $\exists -\underset{\sim}{v} \in V$ and
$$\underset{\sim}{v} + (-\underset{\sim}{v}) = -\underset{\sim}{v} + \underset{\sim}{v} = 0 \quad \text{(Additive Inverse)}$$

(e) $(\alpha+\beta)\cdot \underset{\sim}{v} = \alpha \underset{\sim}{v} + \beta \underset{\sim}{v}$ and

$\alpha(\underset{\sim}{v} + \underset{\sim}{w}) = \alpha \underset{\sim}{v} + \alpha \cdot \underset{\sim}{w}$ (Distributivity)

(f) $\alpha(\beta \underset{\sim}{v}) = \alpha\beta \cdot \underset{\sim}{v}$ (Associativity)

(g) $\exists\, 1 \in \mathbb{R}$ s.t. $1 \cdot \underset{\sim}{v} = \underset{\sim}{v}$. (Identity element)

Examples.

1). $\mathbb{R}^n$ is a vector space

$+ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$(\underset{\sim}{u}, \underset{\sim}{v}) \longmapsto \underset{\sim}{u} + \underset{\sim}{v} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix} \in \mathbb{R}^n$$

$\cdot : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$(\alpha, \underset{\sim}{u}) \longmapsto \alpha \underset{\sim}{u} = \begin{pmatrix} \alpha u_1 \\ \vdots \\ \alpha u_n \end{pmatrix} \in \mathbb{R}^n$$

$$\underset{\sim}{0} = (0, \ldots, 0) \in \mathbb{R}^n$$

2) $\mathbb{R}^{m \times n}$: Space of matrices with m rows and n colums is a vector space.

3) $\mathbb{P}_n = \{\, p(x) : p(x)$ is a polynomial and $\deg(p(x)) \leq n \,\}$

$$p(x) = a_0 + a_1 x^1 + \cdots + a_n x^n$$

$$+ : \mathbb{P}_m \times \mathbb{P}_m \longrightarrow \mathbb{P}_m$$
$$(p, q) \longmapsto p(x) + q(x)$$

$$\deg(p+q) \leq m$$

$$\cdot : \mathbb{R} \times \mathbb{P}_m \longrightarrow \mathbb{P}_m$$
$$(\alpha, p) \longmapsto \alpha p$$
$$\deg(\alpha p) \leq m$$

4) $V = \{ p(x) : p(x)$ is a polynomial and $\deg(p(x)) = m \}$

Is V a vector space?

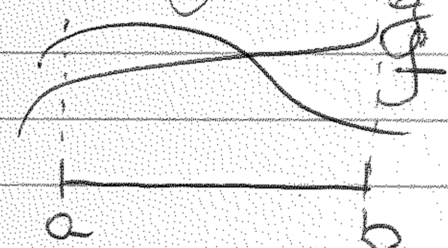
NO: $p(x) = x^2 + 1$; $q(x) = -x^2 + x + 1$

For instance take $m=2$

$\Rightarrow p(x) + q(x) = x + 1 \notin V$ because $\deg(p(x) + q(x)) = 1 \neq 2$

5) $F = \{ f(x) : f(x)$ is a continuous function on $[a, b] \}$

is a vector space



$$+ : F \times F \longrightarrow F$$
$$(f, g) \longmapsto f + g$$

$$\cdot : \mathbb{R} \times F \longrightarrow F$$
$$(\alpha, f) \longmapsto \alpha f$$