10/03/14. LECTURE 14

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Last class: Vector spaces

Examples

1) $V = \mathbb{R}^n$

2) $V = \mathbb{R}^{m\times n}$

+: addition of matrices

$\cdot$: multiplication by scalars.

$A, B \in \mathbb{R}^{m\times n} \Rightarrow A+B \in \mathbb{R}^{m\times n}$

$\alpha \in \mathbb{R}, A \in \mathbb{R}^{m\times n} \Rightarrow \alpha A \in \mathbb{R}^{m\times n}$

Now, given $A, B, C \in \mathbb{R}^{m\times n}$, and $\alpha, \beta \in \mathbb{R}$

(a) $A+B = B+A$.

(b) $A+(B+C) = (A+B)+C$.

(c) $\exists\, 0 \in \mathbb{R}^{m\times n}$ (0 matrix) s.t $A+0 = 0+A = 0$.

(d) For $A \in \mathbb{R}^{m\times n}$, $\exists -A \in \mathbb{R}^{m\times n}$ s.t $A+(-A) = (-A)+A = 0$

(e) $(\alpha+\beta)A = \alpha A + \beta A$.

(f) $\alpha(\beta A) = (\alpha\beta)A$.

(g) $1\cdot A = A$. ▨

3) $\mathbb{P}_n = \{p(x) : p(x) \text{ is a polynomial of degree} \le n\}$

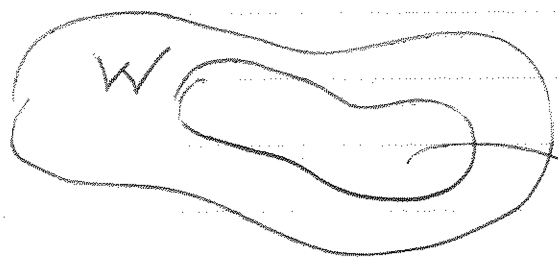
4) $C[a,b] = \{f(x) : f(x) \text{ is a continuous function on } [a,b]\}$

$+ : C[a,b]^2 \longrightarrow (f+g)(x) = f(x) + g(x)$

$\cdot : (\alpha, f) \longmapsto (\alpha f)(x) = \alpha \cdot f(x)$

§2.2 Subspaces

Def (Subspace) A subspace of a vector space V is a subset $W \subset V$ which is a space in its own right under the same operations $+$ and $\cdot$



V - vector space

$\rightarrow$ W is a vector space $\Rightarrow$ W is a subspace of V.

Proposition (Characterization of Subspaces)
A nonempty subset $W \subset V$ of a vector space is a subspace if and only if
(a) $\underline{0} \in V$
(b) $\alpha \underset{\sim}{v} + \underset{\sim}{w} \in W$ for all $\underset{\sim}{v}, \underset{\sim}{w} \in W$.

<u>Examples</u>
(a) The trivial subspace $W = \{0\}$
(b) The entire space $W = V$
(c) $W = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ is a subspace of $\mathbb{R}^3$
In fact,
(a) $\underline{0} \in W \Leftrightarrow (0, 0, 0)$ $\rightarrow$ 3rd component $= 0$.
(b) If $\underset{\sim}{v}$ and $\underset{\sim}{w} \in W$ and $\alpha \in \mathbb{R}$

$$\alpha \underset{\sim}{v} + \underset{\sim}{w} = \alpha(v_1, v_2, v_3) + (w_1, w_2, w_3)$$
$$= (\alpha v_1 + w_1, v_2 + \alpha v_2, v_3 + \alpha w_3)$$

③

Since $\underset{\sim}{v} \in W \Rightarrow v_3 = 0$

$\underset{\sim}{w} \in W \Rightarrow w_3 = 0$

Then $\alpha v_3 + w_3 = 0$

$\Rightarrow \alpha \underset{\sim}{v} + \underset{\sim}{w} = 0$.

(d) $W = \{(x, y, z) \in \mathbb{R}^3 \; 3x + 2y - z = 0\}$

(a) $0 \in W$ because $3 \cdot 0 + 2 \cdot 0 - 0 = 0$.

(b) If $\underset{\sim}{u}, \underset{\sim}{v} \in W$ and $\alpha \in \mathbb{R}$

$\alpha \underset{\sim}{u} + \underset{\sim}{v} = (\alpha u_1 + v_1, \alpha u_2 + v_3, \alpha u_3 + v_3)$

$\alpha \underset{\sim}{u} + \underset{\sim}{v} \in W$ if $3(\alpha u_1 + v_1) + 2(\alpha u_2 + v_2)$

$- (\alpha u_3 + v_3) = 0$.

But $\alpha \underbrace{(3u_1 + 2u_2 - u_3)}_{=0} + \underbrace{(3v_1 + 2v_2 + v_3)}_{=0}$

because $\underset{\sim}{u} \in W$ because $\underset{\sim}{v} \in W$

(e) $W = \{u : u'' + 2u' - 3u = 0\}$

(a) $0 \in W$ because $0'' + 20' - 3 \cdot 0 = 0$.

(b) If $u, v \in W$ and $\alpha \in \mathbb{R}$

$\alpha u + v \in W$ if $(\alpha u + v)'' + 2(\alpha u + v)' - 3(\alpha u + v) = 0$

But $\alpha \underbrace{(u'' + 2u' - 3u)}_{=0} + \underbrace{(v'' + 2v' - 3v)}_{=0} = 0$.

because $u \in W$ because $v \in W$

(f) $W = \{ f \in C[a,b] : \int_a^b f = 0 \}$

(a) $0 \in W$ because $\int_a^b 0 = 0$

(b) If $f, g \in W$ and $\alpha \in \mathbb{R}$

$$\int_a^b (\alpha f + g) = \alpha \int_a^b f + \int_a^b g = \alpha \cdot 0 + 0 = 0$$

$$\Rightarrow \alpha f + g \in W.$$

Is $W = \{(x, y, z) \in \mathbb{R}^3 : z = 1\}$ a subspace of $\mathbb{R}^3$?

NO: $0 \notin W$.

Is $W = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ a subspace of $\mathbb{R}^2$?

$0 \in W$ ✓

But $\underset{\sim}{x} = (1, 1) \Rightarrow \underset{\alpha}{\underbrace{-1}} (1, 1) = (-1, 1) \notin W$

§2.3 Span and linear independence. Next class ▨