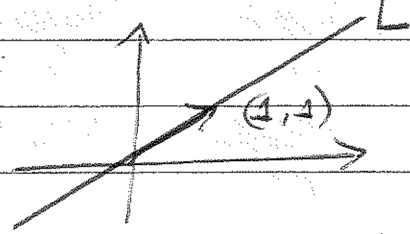
LECTURE 15 | 10/06/14

(1)

§2.3 Span and linear independence

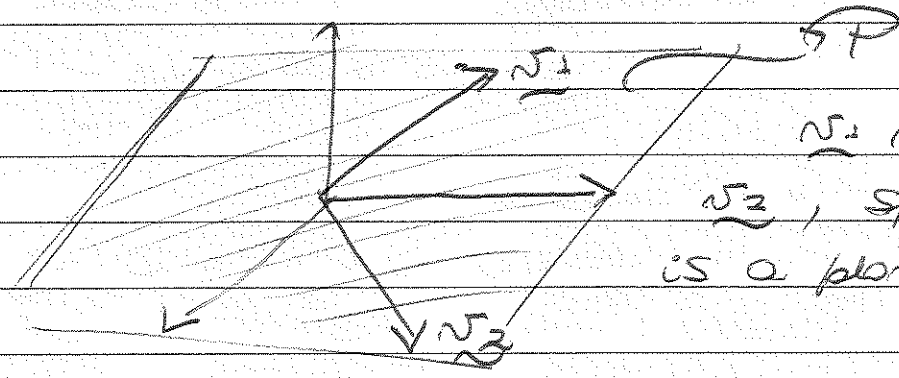
Motivation:



$V = \mathbb{R}^2$

$W = \{ \alpha(1,1) \,;\, \alpha \in \mathbb{R} \}$ = line consisting of all vectors parallel to $(1,1)$ = $\mathrm{Span}\{(1,1)\}$

$\mathbb{R}^3$



$v_1, v_2 \neq 0$. $\underset{\sim}{v_1}$ not parallel to $\underset{\sim}{v_2}$, $\mathrm{Span}\{\underset{\sim}{v_1}, \underset{\sim}{v_2}\}$ is a plane in $\mathbb{R}^3$.

Definition (Span) Let $\underset{\sim}{v_1}, \dots, \underset{\sim}{v_k} \in V$ and V a vector space. A sum of the form

$$c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + \dots + c_k \underset{\sim}{v_k} = \sum_{i=1}^{k} c_i \underset{\sim}{v_i}$$

where $c_1, c_2, \dots, c_k \in \mathbb{R}$ is known as a linear combination of $\underset{\sim}{v_1}, \dots, \underset{\sim}{v_k}$. Their span is

$$W = \mathrm{Span}\{\underset{\sim}{v_1}, \dots, \underset{\sim}{v_k}\} \subset V.$$

Example. $V = \mathbb{R}^3$. If $\underset{\sim}{v_1} \neq 0$, then

1) $\mathrm{Span}\{\underset{\sim}{v_1}\}$ = line of all vectors parallel to $\underset{\sim}{v_1}$
2) $\underset{\sim}{v_1}$ is not parallel to $\underset{\sim}{v_2} \Rightarrow \mathrm{Span}\{\underset{\sim}{v_1}, \underset{\sim}{v_2}\}$ is a plane in $\mathbb{R}^3$. If $\underset{\sim}{v_1}$ parallel to $\underset{\sim}{v_2}$ $\Rightarrow \mathrm{Span}\{\underset{\sim}{v_1}, \underset{\sim}{v_2}\} = \mathrm{Span}\{\underset{\sim}{v_1}\} = \mathrm{Span}\{\underset{\sim}{v_2}\}$.

3) What is the span of $\{(1,0,0), (0,1,0), (0,0,1)\}$
All possible combination of these elements are

$$x(1,0,0) + y(0,1,0) + z(0,0,1), \quad x,y,z \in \mathbb{R}$$
$$(x,y,z)$$
$$\Rightarrow \text{Span}\{(1,0,0), (0,1,0), (0,0,1)\} = \mathbb{R}^3$$

Prop (Span is a Subspace of V). The Span $W = \text{Span}\{\underset{\sim}{v_1}, \dots, \underset{\sim}{v_n}\}$ is a subspace of V.

Proof (i) $\underset{\sim}{0} \in W$ because
$$\underset{\sim}{0} = 0 \cdot \underset{\sim}{v_1} + 0 \cdot \underset{\sim}{v_2} + \dots + 0 \cdot \underset{\sim}{v_k}$$

(ii) If $\underset{\sim}{v}, \underset{\sim}{w} \in W$ then
$$\underset{\sim}{v} = \sum_{i=1}^{k} c_i \underset{\sim}{v_i} \text{ and } \underset{\sim}{w} = \sum_{i=1}^{k} d_i \underset{\sim}{v_i}$$

Consequently
$$\alpha \underset{\sim}{v} + \underset{\sim}{w} = \alpha \sum_{i=1}^{k} c_i \underset{\sim}{v_i} + \sum_{i=1}^{k} d_i \underset{\sim}{v_i}$$
$$= \sum_{i=1}^{k} (\alpha c_i + d_i) \underset{\sim}{v_i}$$
$$\Rightarrow \alpha \underset{\sim}{v} + \underset{\sim}{w} \in W$$

(i) + (ii) $\Rightarrow$ W is a subspace of V. ☒

Examples

(a) $V = \mathbb{P}_2$ – polynomial of degree ≤ 2

$\text{Span}\{1, x\} = \{p(x) : \alpha \cdot 1 + \beta \cdot x\}$

$= \text{polynomials of degree} \leq 1,$

and $\text{Span}\{1, x\} = \mathbb{P}_1$ is a subspace of $V = \mathbb{P}_2$

$\text{Span}\{1, x, x^2\} = \{p(x) : \alpha \cdot 1 + \beta \cdot x + \gamma \cdot x^2\}$

$= \mathbb{P}_2$ – polynomial of degree ≤ 2

$\text{Span}\{1, x, x^2\} = \mathbb{P}_2$ is a subspace of $V = \mathbb{P}_2$

(b) Consider the ODE

$$u'' + 2u' - 3u = 0. \qquad (1)$$

Solutions: $\lambda^2 + 2\lambda - 3 = 0$

$(\lambda - 1)(\lambda + 3) = 0$

$\Rightarrow u(x) = C_1 e^{x} + C_2 e^{-3x}$

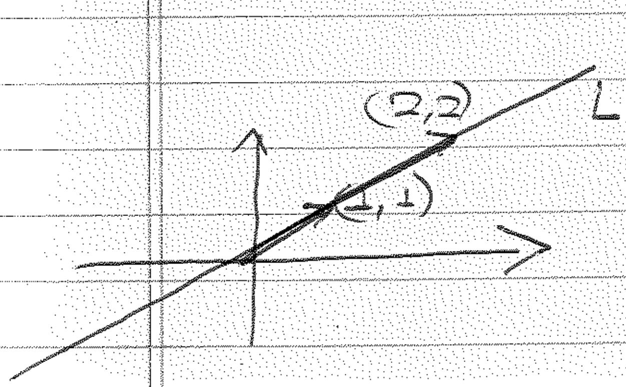
Solution of ODE $= \text{Span}\{e^{x}, e^{-3x}\}$ ☒

Linear independence and dependence.

Motivation $\mathbb{R}^2 = \text{Span}\{(1,0), (0,1)\}$

$= \text{Span}\{(1,0), (0,1), \underbrace{(1,1)}\}$

but $(1,1)$ is redundant.



$\text{Span}\{(1,1)\} = L$

$\text{Span}\{(2,2)\} = L$

$\text{Span}\{(1,1), \underbrace{(2,2)}_{\text{redundant}}\} = L$

Definition (linearly-(in)dependence). The vectors $\underset{\sim}{v_1}, \dots, \underset{\sim}{v_k} \in V$ are linearly dependent (l.d) if $\exists$ $c_1, c_2, \dots, c_k$ not all zero s.t.

$$c_1 \underset{\sim}{v_1} + \dots + c_k \underset{\sim}{v_k} = 0.$$

Vectors $\underset{\sim}{v_1}, \dots, \underset{\sim}{v_k}$ are linearly independent (l.i) if

$$c_1 \underset{\sim}{v_1} + \dots + c_k \underset{\sim}{v_k} = 0 \Rightarrow c_1 = c_2 = \dots = c_k = 0.$$

Example.

$V = \mathbb{R}^3$

(a) $\underset{\sim}{v_1} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\underset{\sim}{v_2} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$, $\underset{\sim}{v_3} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$

Notice that

$$\boxed{1} v_1 - \boxed{2} v_2 + \boxed{1} v_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = \underset{\sim}{0}$$

$\Rightarrow \{v_1, v_2, v_3\}$ is l.d.

(b) Any collection $\{\underset{\sim}{v_1}, \dots, \underset{\sim}{v_k}, \underset{\sim}{0}\}$ including the vector $\underset{\sim}{0}$ is l.d. In fact

$$\underset{\sim}{0} = \boxed{1}\, \underset{\sim}{0} + 0 \cdot \underset{\sim}{v_1} + \dots + 0 \cdot \underset{\sim}{v_k}.$$

(c) $V = \mathbb{R}^3$

$$\underset{\sim}{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underset{\sim}{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \underset{\sim}{v_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + c_3 \underset{\sim}{v_3} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

if and only if $c_1 = c_2 = c_3 = 0$.

(d) The set

$$1, \cos x, \sin, \cos^2 x, \cos x \sin x, \sin^2 x$$

is l.d because $\sin^2 x + \cos^2 x = 1$. In fact

$$0 = 1 + 0 \cdot \cos x + 0 \cdot \sin x - 1 \cdot \cos^2 x + 0 \cdot \cos x \sin x - 1 \cdot \sin^2 x. \quad \blacksquare$$