LECTURE 16 | 10/08/14.

Last class Review

Definition of linear independence. A set $\{\underset{\sim}{v_i}\}_{i=1}^{k} \subset V$, where V is a vector space, is linearly indep. (l.i.) if

$$\alpha_1 \underset{\sim}{v_1} + \alpha_2 \underset{\sim}{v_2} + \cdots + \alpha_k \underset{\sim}{v_k} = \underset{\sim}{0}$$

$$\Rightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0.$$

Otherwise, $\{v_i\}_{i=1}^{k}$ is linearly dependent (l.d)

Examples.

(a) The polynomials $p_1(x) = x-2$; $p_2(x) = x^2-5x+4$, $p_3(x) = 3x^2-4x$, and $p_4(x) = x^2-1$ are linearly dep., since

$$\begin{aligned} & p_1(x) + p_2(x) - p_3(x) + 2p_4(x) \\ = \; & x-2 + (x^2-5x+4) - (3x^2-4x) + 2(x^2-1) \\ = \; & 0x^2 + 0 \cdot x + 0 \cdot 1 = \underset{\sim}{0} \end{aligned}$$

(b) Any collection $\underset{\sim}{v_1}, \ldots, \underset{\sim}{v_k}$ that includes $\underset{\sim}{0}$, say $\underset{\sim}{v_1} = 0$ is l.d because

$$\underset{\sim}{0} = 1 \cdot \underset{\sim}{0} + 0 \cdot \underset{\sim}{v_2} + \cdots + 0 \cdot \underset{\sim}{v_k}$$

(c) The set of functions

$1,\ \cos x,\ \sin x,\ \cos^2 x,\ \cos x \sin x,\ \sin^2 x$

is l.d, since

$$0 = -1 + 0\cdot\cos x + 0\,\sin + 1\cdot\cos^2 x + 0\cdot\cos x\sin + 1\cdot\sin^2 x$$
$$= -1 + \cos^2 x + \sin^2 x = -1 + 1 = 0$$

Let us now focus in the case $V = \mathbb{R}^m$ and study the linear dependence/independence of $\{\underset{\sim}{v_1}, \dots, \underset{\sim}{v_k}\} \in \mathbb{R}^m$.

Define

$$A = [\,\underset{\sim}{v_1} \cdots \quad \underset{\sim}{v_k}\,]$$

$$\Rightarrow A\underset{\sim}{c} = c_1\underset{\sim}{v_1} + c_2\underset{\sim}{v_2} + \cdots + c_k\underset{\sim}{v_k}$$

Example. $\underset{\sim}{v_1} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$; $\underset{\sim}{v_2} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$; $\underset{\sim}{v_3} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 4 & 1 & -2 \end{pmatrix}$; Now $A\underset{\sim}{c} = \begin{pmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$$= \begin{pmatrix} c_1 + 3c_2 + 0c_3 \\ -c_1 + 2c_2 + c_3 \\ 4c_1 + c_2 + 2c_3 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

THM. Let $\{\underset{\sim}{v}_i\}_{i=1}^{k} \in \mathbb{R}^m$ and $A = [\underset{\sim}{v}_1 \cdots \underset{\sim}{v}_k] \in \mathbb{R}^{m\times k}$

(a) The vectors $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k$ are l.d if and only if $\exists$ a non-zero solution to the system $A\underset{\sim}{x} = 0$

(b) The vectors $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k$ are l.i if and only if the system $A\underset{\sim}{c} = 0$ has the unique solution $x=0$.

(c) a vector $\underset{\sim}{b}$ lies in the span $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k\}$ if and only if the system $A\underset{\sim}{c} = \underset{\sim}{b}$ is compatible, i.e., has at least one solution

Proof. (a) If $\exists\ \underset{\sim}{x} \neq 0$ s.t $A\underset{\sim}{x} = 0$

$$\Rightarrow x_1 \underset{\sim}{v}_1 + x_2 \underset{\sim}{v}_2 + \cdots + x_k \underset{\sim}{v}_k = 0$$

where at least one $x_i \neq 0$

$$\Rightarrow \{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k\} \text{ is l.d.}$$

(b) $A\underset{\sim}{c} = 0$ has the unique solution $\underset{\sim}{c} = 0$.

$$\Rightarrow c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \cdots + c_k \underset{\sim}{v}_k = 0$$
$$\Leftrightarrow c_1 = c_2 = \cdots = c_k = 0$$
$$\Rightarrow \{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k\} \text{ is l.i.}$$

(c) Exercise!

Example. Determine whether the vectors

$$\underset{\sim}{v_1}=\begin{pmatrix}1\\2\\-1\end{pmatrix},\ \underset{\sim}{v_2}=\begin{pmatrix}3\\0\\4\end{pmatrix},\ \underset{\sim}{v_3}=\begin{pmatrix}1\\-4\\6\end{pmatrix} \text{ and } \underset{\sim}{v_4}=\begin{pmatrix}4\\2\\3\end{pmatrix}$$

are l.i or l.d. We construct

$$A=\begin{pmatrix}1&3&1&4\\2&0&-4&2\\-1&4&6&3\end{pmatrix}\rightarrow\begin{pmatrix}\boxed{1}&3&1&4\\0&\boxed{-6}&-6&-6\\0&0&0&0\end{pmatrix}=U.$$

free variables.

We need to study $U\underset{\sim}{c}=0$. The general solution is

$$\underset{\sim}{c}=(2c_3-c_4,\ -c_3-c_4,\ c_3,\ c_4)$$

THM (a) $\Rightarrow$ vectors $\{\underset{\sim}{v_1},\underset{\sim}{v_2},\underset{\sim}{v_3},\underset{\sim}{v_4}\}$ are l.d since

$$(2c_3-c_4)\underset{\sim}{v_1}+(-c_3-c_4)\underset{\sim}{v_2}+c_3\underset{\sim}{v_3}+c_4\underset{\sim}{v_4}=0$$

Lemma. Any collection of $k>n$ vectors in $\mathbb{R}^n$ is linearly dependent.

Example $V=\mathbb{R}^2$; $\underset{\sim}{v_1}=\begin{pmatrix}1\\0\end{pmatrix},\ \underset{\sim}{v_2}=\begin{pmatrix}0\\1\end{pmatrix},\ \underset{\sim}{v_3}=\begin{pmatrix}1\\1\end{pmatrix}$ are l.d

Prop. a set of k vectors in $\mathbb{R}^n$ is l.i $\Longleftrightarrow$ $A=[\underset{\sim}{v_1},\underset{\sim}{v_2}\cdots\underset{\sim}{v_k}]$ has rank k. In particular this requires $k\le n$.

Example. $\underset{\sim}{v_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\underset{\sim}{v_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $\underset{\sim}{v_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \end{pmatrix} = U.$$

pivots $= 3$; rank $A = 3$ = # of vectors

$\Rightarrow \{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \underset{\sim}{v_3}\}$ is l.i.

Prop. A collection of vectors $\{\underset{\sim}{v_1}, \ldots, \underset{\sim}{v_k}\}$ span $\mathbb{R}^n \Longleftrightarrow$
$A = [\underset{\sim}{v_1} \cdots \underset{\sim}{v_k}]$ has rank n

$\boxtimes$

Example. $\underset{\sim}{v_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\underset{\sim}{v_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\underset{\sim}{v_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

A has rank 3

$\Rightarrow \{v_1, v_2, v_3\}$ span $\mathbb{R}^3$.