LECTURE 17 | 10/10/14

Last class review.

Let $\{v_1, \dots, v_k\}$ and let $A = [\underset{\sim}{v_1} \cdots \underset{\sim}{v_k}] \in \mathbb{R}^{n \times k}$
Then,
(a.) $\{\underset{\sim}{v_1}, \dots, \underset{\sim}{v_k}\}$ are l.d $\Leftrightarrow$ $A\underset{\sim}{c} = 0$ has a non-zero solution $\underset{\sim}{c} \neq 0$.

Example.

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\quad\ \ v_1 \quad v_2 \quad v_3$

free variable!

$$c_1 + c_2 + c_3 = 0 \quad \Rightarrow \quad c_2 = -c_3$$
$$c_2 + c_3 = 0 \qquad\quad c_1 = -c_2 - c_3 = 0$$

Example $\underset{\sim}{c} = (0, -1, 1)$
$\Rightarrow \{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \underset{\sim}{v_3}\}$ are l.d.

(b) $\{v_1, \dots, \underset{\sim}{v_n}\}$ are l.i $\Leftrightarrow$ $A\underset{\sim}{c} = 0$ has the unique solution $\underset{\sim}{c} = 0$.

Example.

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{r} c_1 + c_2 + c_3 = 0 \\ c_2 + c_3 = 0 \\ c_3 = 0 \end{array}$$

$$\Rightarrow c_1 = c_2 = c_3 = 0.$$

$A = [\underset{\sim}{v_1} \cdots \underset{\sim}{v_k}] \in \mathbb{R}^{k \times m}$

Case $k > m \Rightarrow \{\underset{\sim}{v_1}, \ldots, \underset{\sim}{v_k}\}$ is l.d.

more columns than rows

Prop. A set of k vectors in $\mathbb{R}^m$ is l.i $\Leftrightarrow$
$A = [\underset{\sim}{v_1}, \ldots, \underset{\sim}{v_k}] \in \mathbb{R}^{k \times m}$ has rank $= m$

Example. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \end{bmatrix}$

$\qquad\quad\ \ v_1 \quad v_2 \quad v_3$

rank $A = 3 \Rightarrow \{v_1, v_2, v_3\}$ is l.i

§ 2.4 Bases and dimension

$V = \mathbb{R}^2 \qquad \underset{\sim}{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \underset{\sim}{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \underset{\sim}{v_3} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Span $\{v_1, \underset{\sim}{v_2}\} = \mathbb{R}^2$

Span $\{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \underset{\sim}{v_3}\} = \mathbb{R}^2$

Span $\{v_1, v_2\} = \{\alpha \cdot (1,0)^T + \beta(0,1)^T = (\alpha, \beta)^T\} = \mathbb{R}^2$

Span $\{v_1, v_2, v_3\} = \{\alpha(1,0)^T + \beta(0,1)^T + \gamma(1,1)^T\}$

$= \{(\alpha+\gamma, \beta+\gamma)^T\} = \mathbb{R}^2$

Span $\{v_1\} = \{\alpha v_1, \ \alpha \in \mathbb{R}\}$.

Definition (basis) A basis of a vector space V is a finite collection of elements $\{v_1, \dots, v_m\} \in V$ s.t

(a) $\text{Span}\{v_1, \dots, v_m\} = V$

(b) $\{v_1, \dots, v_m\}$ is l.i

Examples.

1) $V = \mathbb{R}^n$ and $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$; $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$; $e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$\text{Span}\{e_1, \dots, e_n\} = \mathbb{R}^n$

$\{e_1, e_2, \dots, e_n\}$ is l.i

$\Longrightarrow \{e_1, \dots, e_n\}$ is a basis of $V = \mathbb{R}^n$.

2) $V = \mathbb{P}_2$ = polynomials of degree 2

$p_1(x) = 1$; $p_2(x) = x$; $p_3(x) = x^2$

$\text{Span}\{p_1(x), p_2(x), p_3(x)\} = \mathbb{P}_2$

$\{p_1(x), p_2(x), p_3(x)\}$ is l.i

$\Longrightarrow \{p_1(x), p_2(x), p_3(x)\}$ is a basis for $V = \mathbb{P}_2$

3) $V = \mathbb{R}^{2\times 2}$

$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\text{Span}\{A_1, A_2, A_3, A_4\} = \mathbb{R}^{2\times 2}$

$\{A_1, A_2, A_3, A_4\}$ is l.i

$\Longrightarrow \{A_1, A_2, A_3, A_4\}$ is a basis of $V = \mathbb{R}^{2\times 2}$

THM. Every basis of $\mathbb{R}^m$ consists of exactly m vectors. In addition, $\{v_1, \dots, v_m\}$ is a basis if and only if

$$A = [\, \underline{v_1} \ \dots \ v_m \,]$$

has rank $= m$ $\iff$ A is non singular
$\iff$ A is invertible
$\iff$ $\det A \neq 0$.

Suppose a vector space V has a basis $\{v_1, \dots, v_m\}$. Then every other basis of V has the same number of elements in it. This number is called the dimension of the space:

$$\dim V = m$$

Examples
(a) $\dim \mathbb{R}^m = m$.
(b) $\dim \mathbb{P}^2 = 3$.
(c) $\dim \mathbb{R}^{2\times 2} = 4$.

Summary. V is a m-dimensional vector space ($\dim V = m$). Then

(a) Every set of more than m elements of V is l.d.
Example $V = \mathbb{R}^3$ the set $\{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \underset{\sim}{v_3}, \underset{\sim}{v_4}\}$ where

$$\underset{\sim}{v_1} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \underset{\sim}{v_2} = \begin{pmatrix}1\\0\\1\end{pmatrix}; \quad v_3 = \begin{pmatrix}0\\1\\0\end{pmatrix}; \quad \underset{\sim}{v_4} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

is l.d.

(b) No set of less than n elements span V

Example: $V = \mathbb{R}^3$; Span $S_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \neq \mathbb{R}^3$

(c) A set of n elements form a basis $\Leftrightarrow$ it spans V.

(d) A set of n elements forms a basis $\Leftrightarrow$ it is l.i.

Example $V = \mathbb{R}^{2\times 2}$

$$W = \{ A \in \mathbb{R}^{2\times 2} : A = A^T \}$$

Find a basis for W and its dimension

Solution $A = A^T \Longleftrightarrow A = \begin{pmatrix} a & \boxed{b} \\ \boxed{c} & d \end{pmatrix} = \begin{pmatrix} a & \boxed{c} \\ \boxed{b} & d \end{pmatrix} = A^T$

$$\Rightarrow A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

$$W = \{ A \in \mathbb{R}^{2\times 2} : A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \}$$

$$= \left\{ A \in \mathbb{R}^{2\times 2} : A = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$= \text{Span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

This set is l.i. In fact

$$\alpha\begin{pmatrix}1&0\\0&0\end{pmatrix}+\beta\begin{pmatrix}0&1\\1&0\end{pmatrix}+\gamma\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\alpha&\beta\\\beta&\gamma\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}\Rightarrow \alpha=\beta=\gamma=0$$

Then $\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ is a basis for V

and $\dim W = 3$. ☒