LECTURE 18 10/13/14

§2.5 The fundamental matrix Subspaces

Let us return to the linear System

$$A\underset{\sim}{x} = \underset{\sim}{b},$$

where $A \in \mathbb{R}^{m\times n}$ $\Rightarrow$ m number of equations

$\Rightarrow$ n number of unknowns.

and $\underset{\sim}{b} \in \mathbb{R}^m$.

We define the kernel and range of a matrix.

Def (Kernel) Let $A \in \mathbb{R}^{m\times n}$. The kernel of A is the subspace consisting of all vectors s.t. $A\cdot\underset{\sim}{z} = \underset{\sim}{0}$, i.e.,

$$\ker(A) = \{\underset{\sim}{z} \in \mathbb{R}^n : A\underset{\sim}{z} = 0\} \subset \mathbb{R}^n.$$

Def (range) Let $A \in \mathbb{R}^{m\times n}$ The range of A is the subspace $\operatorname{rng} A \subset \mathbb{R}^m$ spanned by its columns, i.e.,

$$A = [\underset{\sim}{v_1} \cdots \underset{\sim}{v_n}] \Rightarrow \operatorname{rng}(A) = \operatorname{Span}\{\underset{\sim}{v_1}, \ldots, \underset{\sim}{v_n}\}.$$

Remarks.

1. The range is also known as the column space a the image of A. If $\underset{\sim}{b} \in \operatorname{rng} A$

$$\Rightarrow \underset{\sim}{b} = x_1\underset{\sim}{v_1} + x_2\underset{\sim}{v_2} + \cdots + x_n\underset{\sim}{v_n}$$

$$\Rightarrow \underset{\sim}{b} = A\cdot\underset{\sim}{x}$$

$$A = [\underset{\sim}{v_1} \cdots \underset{\sim}{v_n}]$$

Then,

$$\text{rgn}(A) = \{ A\underline{x} \, , \, \underline{x} \in \mathbb{R}^n \} \subset \mathbb{R}^m$$

2. $\underline{b}$ lies on the rgn(A) if the linear system $A\underline{x} = \underline{b}$ has a solution.

Exercise. Prove that $\ker(A)$ is a subspace of $\mathbb{R}^n$. Here $A \in \mathbb{R}^{m \times n}$

Solution. i) $0 \in \ker(A)$, because $A \cdot \underline{0} = \underline{0}$
ii) Let $\underline{v}, \underline{w} \in \ker(A)$ and $\alpha \in \mathbb{R}$. Then,

$$A(\alpha \underline{v} + \underline{w}) = \alpha A \underline{v} + A \underline{w}$$
$$= \alpha \cdot 0 + 0 = \underline{0}$$

$\Rightarrow \alpha \underline{v} + \underline{w} \in \ker(A)$.

THM. If $\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_k$ are individual solutions to $A\underline{z} = 0$, then any linear combination

$$\alpha_1 \underline{z}_1 + \alpha_2 \underline{z}_2 + \cdots + \alpha_k \underline{z}_k$$

is also a solution of $A\underline{z} = 0$.

Exercise. Let us compute the kernel of

$$A = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & -3 & -1 & -4 \\ 3 & -5 & -1 & -1 \end{pmatrix}$$

Solution. We first perform row operations:

$$A = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & -3 & -1 & -4 \\ 3 & -5 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & -1 & -10 \\ 0 & 1 & -1 & -10 \end{pmatrix}$$

$$\to \begin{pmatrix} \boxed{1} & -2 & 0 & 3 \\ 0 & \boxed{1} & -1 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

$$\Longleftrightarrow \begin{cases} x - 2y + 3w = 0 \\ y - z - 10w = 0 \end{cases}$$

free variables: z, w $\Rightarrow$ we solve for x and y.

2nd eqn: $\boxed{y = z + 10w}$

1st eqn: $x = 2y - 3w = 2z + 20w - 3w$

$\boxed{x = 2z + 17w}$

General solution:

$$\underset{\sim}{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2z + 17w \\ z + 10w \\ z \\ w \end{pmatrix} = z \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 17 \\ 10 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \text{Ker}(\underset{\sim}{A}) = \text{Span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 17 \\ 10 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \dim(\text{Ker}(A)) = 2$$

Since we know the solution of $A\underset{\sim}{z} = 0$, now we need to study the solutions to $A\underset{\sim}{x} = \underset{\sim}{b}$

THM The linear system $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$ has a solution x^* $\Longleftrightarrow$ $\underset{\sim}{b} \in \text{rgn}(\underset{\sim}{A})$. If this occurs, then x is a solution to $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$ $\Longleftrightarrow$

$$\underset{\sim}{x} = \underset{\sim}{x}^* + \underset{\sim}{z}$$

where $\underset{\sim}{z} \in \ker(A)$

Consequently, to construct the most general solution to $A\underset{\sim}{x} = \underset{\sim}{b}$ we only know

→ one particular solution $\underset{\sim}{x}^*$

→ general solution $\underset{\sim}{z} \in \ker(A)$.

Example. Consider the linear system $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$, where

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{pmatrix}; \quad \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad \underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Solution: We consider the augmented matrix:

$$M = (A|b) = \left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 1 & -2 & b_2 \\ 1 & -2 & 3 & b_3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 1 & -2 & b_2 \\ 0 & -2 & 4 & b_3 - b_1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & b_1 \\ 0 & \boxed{1} & -2 & b_2 \\ 0 & 0 & 0 & b_3 + 2b_2 - b_1 \end{array}\right)$$

free variable

Therefore, $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$ has a solution $\Longleftrightarrow$ compatibility condition

$$b_3 + 2b_2 - b_1 = 0$$

holds.

Take $b_3 = -2b_2 + b_1 \Rightarrow$ Comp. Cond ✓

$$\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ -2b_2 + b_1 \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$\Rightarrow \mathrm{rng}(A) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}\right\}$; $\dim(\mathrm{rng}(A)) = 2$

Solutions to $A\underline{z} = 0$. Consider $\underline{b} = 0$.

Then $A\underline{z} = 0 \Leftrightarrow \begin{cases} z_1 \quad - z_3 = 0 \\ z_2 - 2z_3 = 0 \end{cases}$

$z_3 \to$ free variable

$\Rightarrow$ We solve for z_1 and z_2, and

$$z_1 = z_3$$
$$z_2 = 2z_3$$

General Solution to $A\underline{z} = 0$

$$\underline{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = z_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$\Rightarrow \mathrm{Ker}(A) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right\}$ $\dim(\mathrm{Ker}(A)) = 1$

Now, take $\underline{b} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$; Compatibility condition holds!

General solution to $A\underline{x} = \underline{b}$

$$\underline{x} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \underline{x}^* + \underline{z}$$

$\hookrightarrow$ particular solution.

In fact, if $\underset{\sim}{b} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

$$\underset{\sim}{A}x = \underset{\sim}{b} \iff \begin{cases} x_1 \qquad - x_3 = 3 \\ \qquad x_2 - 2x_3 = 1. \end{cases}$$

$$\Rightarrow \quad x_1 = 3 + x_3$$
$$\qquad x_2 = 2x_3 + 1$$

$$\Rightarrow \quad \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 + \underset{\sim}{x_3} \\ 2x_3 + 1 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}}_{\text{particular solution}} + x_3 \underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}_{\underline{z} \in \ker(A)} \qquad \blacksquare$$