LECTURE 2 | 09/05/14

Last class: Introduction to Gaussian elimination and def. of a matrix.

Today: Matrix arithmetic, augmented matrix and introduce LU decomposition.

Def. ($A=B$). If $A, B \in \mathbb{R}^{m\times n}$, we say that $A=B$ if

$$a_{ij} = b_{ij}, \quad i = 1, \dots, m \\ j = 1, \dots, n.$$

General linear system

$$(1.3) \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{12}x_1 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$(1.3) \iff A\underset{\sim}{x} = \underset{\sim}{b}$$

where $A = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$; $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$; $\underset{\sim}{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

Matrix analysis is fundamental to solve linear systems of eqns!

Def. (Oddition of Motrices) If $A, B \in \mathbb{R}^{m \times n}$ then $C = A + B \in \mathbb{R}^{m \times n}$ and

$$c_{ij} = a_{ij} + b_{ij} \quad \forall i, \forall j.$$

Ekomple.
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}.$$

Properties of Oddition: If $A, B, C \in \mathbb{R}^{m \times n}$, then

i) $A + B = B + A$

ii) $(A + B) + C = A + (B + C)$.

Why? Proof of ii) $(a_{ij} + b_{ij}) + c_{ij}$
$$= a_{ij} + (b_{ij} + c_{ij}) \quad \boxtimes$$

Multiblication of Matrices.

$$\underset{3 \times \cancel{2}}{\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 3 \end{pmatrix}} \underset{\cancel{2} \times 2}{\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 6 \\ 2 & 11 \\ 0 & 3 \end{pmatrix}$$

Def $(A \cdot B)$ If $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, then $C = AB \in \mathbb{R}^{m \times p}$ and

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, p$$

Property (Associativity). Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$ and $C \in \mathbb{R}^{p \times q}$.
Then $A(BC) = (AB)C$.

Proof. $(BC)_{ij} = \sum_k b_{ik} c_{kj}$ Then

$$A(BC) = \sum_k a_{ik} (BC)_{kj} = \sum_k a_{ik} \sum_\ell b_{k\ell} c_{\ell j}$$
$$= \sum_k a_{ik} b_{k\ell} \sum_\ell c_{\ell j} = \sum_\ell (AB)_{i\ell} c_{\ell j} = (AB)C.$$ (3)

Bad news: $AB \neq BA$. For example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 4 & 11 \end{pmatrix}$$

and
$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} \quad \boxtimes$$

Two special matrices.

Def (Identity matrix) We define the identity matrix $I_n \in \mathbb{R}^{n \times n}$ as

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Property. If $A \in \mathbb{R}^{n \times n}$, then $A \cdot I_n = I_n \cdot A = A$

Def (Diagonal matrix) We define a diagonal matrix $D \in \mathbb{R}^{n \times n}$ by

$$d_{ii} \neq 0 \quad i = 1, \dots, n.$$
$$d_{ij} = 0, \quad i \neq j.$$

Example:
$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathbb{R}^{3 \times 3}.$$

§ 1.3. Gaussian elimination - regular case.

$$(1.1)\quad \begin{cases} x + 2y + z = 2 \\ 2x + 6y + z = 7 \\ x + y + 4z = 3 \end{cases} \iff A\underset{\sim}{x} = \underset{\sim}{b}$$

We define the augmented matrix to be:

$$M = (A \mid \underset{\sim}{b}) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 6 & 1 & 7 \\ 1 & 1 & 4 & 3 \end{array}\right) \xrightarrow[-R_1+R_2]{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 0 & -1 & 3 & 1 \end{array}\right)$$

first pivot $\neq 0$.

second pivot $\neq 0$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 5/2 & 5/2 \end{array}\right) = N = (U \mid c)$$

third pivot

U is an upper triangular matrix, i.e,

$$u_{ij} = 0 \quad \text{if} \quad i > j$$

$$A\underset{\sim}{x} = \underset{\sim}{b} \iff U\underset{\sim}{x} = \underset{\sim}{c}$$

Def (Regular matrix). $A \in \mathbb{R}^{n\times n}$ is regular if it can be reduced to an upper triangular matrix $U \in \mathbb{R}^{n\times n}$ s.t all pivots $\neq 0$.