LECTURE 22 | 10/27/14.

§ 3.1. Inner products

Most basic example of inner product is the dot product

$$\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = \underset{\sim}{v} \cdot \underset{\sim}{w} = v_1 w_1 + \cdots + v_m w_m = \sum_{i=1}^{m} v_i w_i$$

between two vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$ in $\mathbb{R}^m$.

Observation: $\underset{\sim}{v} \cdot \underset{\sim}{w} = \underset{\sim}{v}^T w = (v_1 \cdots v_m) \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$

Now, if $\underset{\sim}{v} = \underset{\sim}{w}$

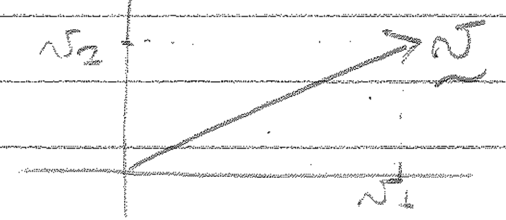
$$\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = \underset{\sim}{v} \cdot \underset{\sim}{v} = v_1^2 + v_2^2 + \cdots + v_m^2 \geq 0$$

and $\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = 0 \Leftrightarrow \underset{\sim}{v} = \underset{\sim}{0}$

The euclidean norm $\| \cdot \|$ is defined by

$$\| \underset{\sim}{v} \| = \sqrt{\underset{\sim}{v} \cdot \underset{\sim}{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_m^2}$$

Geometrical interpretation. $V = \mathbb{R}^2$



$\| \underset{\sim}{v} \| = \sqrt{v_1^2 + v_2^2}$
$=$ length of vector $\underset{\sim}{v}$

Notice that $\cdot \; \|\underset{\sim}{v}\| > 0$ if $\underset{\sim}{v} \neq 0$

$\cdot \; \|\underset{\sim}{v}\| = 0 \Leftrightarrow \underset{\sim}{v} = 0$

Definition: An inner product on a real vector space V is a function

$$\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{R}$$
$$(\underset{\sim}{v}, \underset{\sim}{w}) \longmapsto \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle$$

where $\langle \cdot , \cdot \rangle$ satisfies the following properties
Given $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V$ and $\alpha, \beta \in \mathbb{R}$

(i) Bilinearity:

$$\langle \alpha \underset{\sim}{u} + \beta \underset{\sim}{v}, \underset{\sim}{w} \rangle = \alpha \langle \underset{\sim}{u}, \underset{\sim}{w} \rangle + \beta \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle$$
$$\langle \underset{\sim}{u}, \alpha \underset{\sim}{v} + \beta \underset{\sim}{w} \rangle = \alpha \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle + \beta \langle \underset{\sim}{u}, \underset{\sim}{w} \rangle$$

(ii) Symmetry: $\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = \langle \underset{\sim}{w}, \underset{\sim}{v} \rangle$

(iii) Positivity: $\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle > 0$ whenever $\underset{\sim}{v} \neq 0$
while $\langle \underset{\sim}{0}, \underset{\sim}{0} \rangle = 0$.

$\Rightarrow$ A vector space + inner product = inner product space
$\Rightarrow$ A vector space can admit many different inner products

We define the associated norm $\| \cdot \|$ to $\langle \cdot , \cdot \rangle$ by

$$\|\underset{\sim}{v}\| = \sqrt{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle}.$$

Positivity(ii) $\Rightarrow$ $\|\underset{\sim}{v}\| \geq 0$ and $\|\underset{\sim}{v}\| = 0$ if and only if $\underset{\sim}{v} = 0$.

Example. Consider $V = \mathbb{R}^2$

1) $\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = v_1 w_1 + v_2 w_2$

Exercise: Verify properties (i), (ii) and (iii)

2) $\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = 2v_1 w_1 + 5 v_2 w_2$ (Weighted inner product).

Let us verify properties (i), (ii) and (iii).

$$
\begin{aligned}
\text{(i)} \ \langle \alpha \underset{\sim}{u} + \beta \underset{\sim}{v}, \underset{\sim}{w} \rangle &= 2(\alpha u_1 + \beta v_1) w_1 \\
&\quad + 5(\alpha u_2 + \beta v_2) w_2 \\
&= \alpha (2u_1 w_1 + 5 u_2 w_2) \\
&\quad + \beta (2 v_1 w_1 + 5 v_2 w_2) \\
&= \alpha \langle \underset{\sim}{u}, \underset{\sim}{w} \rangle + \beta \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle
\end{aligned}
$$

Check second property.

(iii) $\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = 2v_1^2 + 5 v_2^2 > 0$.

$\langle \underset{\sim}{0}, \underset{\sim}{0} \rangle = 2 \cdot 0^2 + 5 \cdot 0^2 = 0$.

$$
\begin{aligned}
\text{(ii)} \ \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle &= 2v_1 w_1 + 5 v_2 w_2 \\
&= 2 w_1 v_1 + 5 w_2 v_2 = \langle \underset{\sim}{w}, \underset{\sim}{v} \rangle
\end{aligned}
$$

$\Rightarrow$ $\langle \cdot, \cdot \rangle$ is an inner product in $\mathbb{R}^2$, and we have a weighted norm

$$\|\underset{\sim}{v}\| = \sqrt{2v_1^2 + 5v_2^2}$$

Example $V = \mathbb{R}^n$. Let $c_1, c_2, \ldots, c_n > 0$.
The corresponding weighted inner product and weighted norm on $\mathbb{R}^n$ are defined by

$$\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = \sum_{i=1}^{n} c_i v_i w_i, \quad \| \underset{\sim}{v} \| = \sqrt{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} = \sqrt{\sum_{i=1}^{n} c_i v_i^2}.$$

The numbers c_i are the weights. $\to$ data fitting!

§ 3.2 Inner products on function spaces.

Example. Let $[a, b] \subset \mathbb{R}$ closed interval.
Consider $C^0[a, b] = \{ f : f \text{ is continuous on } [a, b] \}$.
We define

$$\langle \underset{\sim}{f}, \underset{\sim}{g} \rangle = \int_a^b f(x)\, g(x)\, dx$$

$\langle \cdot, \cdot \rangle$ defines an inner product. In fact, we check properties (i), (ii) and (iii). Given, $f, g, h \in C^0[a, b]$ and $\alpha, \beta \in \mathbb{R}$

$$\text{(i)} \quad \langle \alpha f + \beta g, h \rangle = \int_a^b (\alpha f + \beta g) h\, dx$$

$$= \alpha \int_a^b f h\, dx + \beta \int_a^b g h\, dx$$

$$= \alpha \langle f, h \rangle + \beta \langle g, h \rangle$$

Second property follows similarly.

(ii) $\langle f, g\rangle = \int_a^b f g \, dx = \int_a^b g f \, dx = \langle g, f\rangle$

(iii) $\langle f, f\rangle = \int_a^b f^2 > 0$ if $f \not\equiv 0$

$\langle 0, 0\rangle = \int_a^b 0^2 = 0$

$\Rightarrow$ $\langle \cdot, \cdot\rangle$ defines an inner product on $C^0[a,b]$

We define the associated norm

$$\| f \| = \sqrt{\int_a^b f^2(x)\,dx}.$$

We observe $\|f\|^2 = \int_a^b f^2(x)\,dx \geq 0$ since $f^2(x) \geq 0$ on $[a,b]$. However since f^2 is continuous and nonnegative

$$\int_a^b f^2(x)\,dx = 0 \iff f \equiv 0.$$

The norm $\|f\|$ is called the norm L^2 of a function f. In fact, we define

$$L^2(a,b) = \{ f : \int_a^b f^2\,dx < \infty \}$$

This is a particular case of Hilbert spaces

Warning: Consider $L^2(-1,1)$ and

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\|f\|^2 = \int_{-1}^{1} f^2 \, dx = 0 \Rightarrow f \in L^2(-1,1)$$

but f is clearly not continuous.