LECTURE 231 10/29/14.

V - vector space. An inner product $\langle \cdot , \cdot \rangle$ is a function

$$\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{R}$$
$$(\underset{\sim}{v}, \underset{\sim}{w}) \longmapsto \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle$$

Satisfying:

(i) $\langle \alpha \underset{\sim}{u} + \beta \underset{\sim}{v}, \underset{\sim}{w} \rangle = \alpha \langle \underset{\sim}{u}, \underset{\sim}{w} \rangle + \beta \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle$

$\langle \underset{\sim}{u}, \alpha \underset{\sim}{v} + \beta \underset{\sim}{w} \rangle = \alpha \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle + \beta \langle \underset{\sim}{u}, \underset{\sim}{w} \rangle$

(ii) $\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = \langle \underset{\sim}{w}, \underset{\sim}{v} \rangle$

(iii) $\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle \geq 0 \ \forall \underset{\sim}{v}$ and $\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = 0 \Leftrightarrow \underset{\sim}{v} = 0$

Associated norm: $\| \underset{\sim}{v} \| = \sqrt{\langle v, v \rangle}$.

Examples

1. $V = \mathbb{R}^n$, $\langle \cdot , \cdot \rangle =$ dot product.
2. $V = \mathbb{R}^n$, $\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = \sum_{i=1}^{n} c_i v_i w_i$, where $c_i > 0$.
3. $V = C[a,b] = \{ f : f \text{ is continuous on } [a,b] \}$

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx.$$

$$\| f \| = \left(\int_a^b f^2(x) \, dx \right)^{1/2}.$$

4. $V = L^2(a,b) = \{ f : \| f \|_{L^2(a,b)} < \infty \}$.

particular case of Hilbert space.

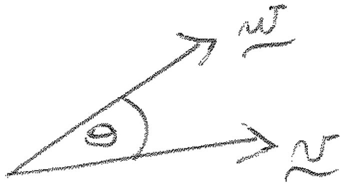
Warning. Consider $L^2(-1,1)$ and $f(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$

$$\| f \|^2 = \int_{-1}^{1} f^2 \, dx = 0 \Rightarrow f \in L^2(-1,1)$$

but f is not continuous.

§3.3 Inequalities: Cauchy-Schwarz ineq and triangle inequality.

$V = \mathbb{R}^m$



$\underset{\sim}{v} \cdot \underset{\sim}{w} = \|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\| \cos\theta$, where θ is the angle between $\underset{\sim}{v}$ and $\underset{\sim}{w}$

Since $|\cos\theta| \le 1$

$$\Rightarrow \quad |\underset{\sim}{v} \cdot \underset{\sim}{w}| = \|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\| \, |\cos\theta|$$

$$\boxed{|\underset{\sim}{v} \cdot \underset{\sim}{w}| \le \|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\|}$$

Simplest form of Cauchy-Schwarz ineq.

THM Every inner product satisfies the Cauchy-Schwarz inequality.

$$|\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle| \le \|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\| \quad \forall v, w \in V$$

where $\|\cdot\|$ is the associated norm, $|\cdot|$ denotes the absolute value. Equality holds $\Leftrightarrow$ v and w are parallel.

Proof. If $\underset{\sim}{w} = 0$ the inequality is trivial.
Assume $w \neq 0$: let $t \in \mathbb{R}$

$$0 \le \|\underset{\sim}{v} + t\underset{\sim}{w}\|^2 = \langle \underset{\sim}{v} + t\underset{\sim}{w}, \underset{\sim}{v} + t\underset{\sim}{w} \rangle$$

$$= \langle \underset{\sim}{v}, \underset{\sim}{v} \rangle + \underbrace{t\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle + t\langle \underset{\sim}{w}, \underset{\sim}{v} \rangle}_{2t\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle} + t^2\langle \underset{\sim}{w}, \underset{\sim}{w} \rangle$$

$$= \|\underset{\sim}{v}\|^2 + 2t\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle + t^2\|\underset{\sim}{w}\|^2$$

Fix $\underset{\sim}{v}$ and $\underset{\sim}{w}$. $\quad p(t) = t^2\|\underset{\sim}{w}\|^2 + 2t\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle + \|\underset{\sim}{v}\|^2$

$$p(t) = at^2 + 2bt + c$$

where $a = \|w\|^2$, $b = \langle v, w\rangle$, $c = \|v\|^2$

Let's get the maximum of p:

$$p'(t) = 2at + 2b = 0 \Rightarrow t = -\frac{b}{a} = \frac{-\langle v, w\rangle}{\|w\|^2}$$

In particular

$$0 \leq \|v + tw\|^2 = t^2\|w\|^2 + 2t\langle v, w\rangle + \|v\|^2$$

$$= \frac{(\langle v, w\rangle)^2}{\|w\|^{\cancel{4}\,2}} \cancel{\|w\|^2} - 2\frac{\langle v, w\rangle\langle v, w\rangle}{\|w\|^2} + \|v\|^2$$

$$= -\frac{\langle v, w\rangle^2}{\|w\|^2} + \|v\|^2$$

$$\Rightarrow \quad \langle v, w\rangle^2 \leq \|v\|^2\|w\|^2$$

$$\Rightarrow \quad |\langle v, w\rangle| \leq \|v\|\,\|w\|.$$

If v and w are parallel: Exercise!

Now, as a consequence of Cauchy-Schwarz inequality, we have

$$\frac{|\langle v, w\rangle|}{\|v\|\,\|w\|} \leq 1$$

$$\Rightarrow -1 \leq \frac{\langle v, w\rangle}{\|v\|\,\|w\|} \leq 1$$

Since $\cos\theta \in [-1, 1]$ for $\theta \in [0, \pi]$, we may write

$$\cos\theta = \frac{\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle}{\|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\|}$$

Then $\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle$ gives information about the angle between the vectors (functions, polynomials) $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

<u>Example.</u> 1) $\underset{\sim}{v} = (1, 0, 1)^T$, $\underset{\sim}{w} = (0, 1, 1)^T$.

$$\Rightarrow \underset{\sim}{v} \cdot \underset{\sim}{w} = 1$$

$$\Rightarrow \|\underset{\sim}{v}\| = \sqrt{2} = \|\underset{\sim}{w}\|$$

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

2) L^2 inner product on $[0, 1]$.

$$p(x) = x \quad \text{and} \quad q(x) = x^2$$

$$\cos\theta = \frac{\langle x, x^2 \rangle}{\|x\| \, \|x^2\|} = \frac{\int_0^1 x^3 dx}{\sqrt{\int_0^1 x^2 dx}\sqrt{\int_0^1 x^4}} = \frac{\frac{1}{4}}{\sqrt{\frac{1}{3}}\sqrt{\frac{1}{5}}} = \sqrt{\frac{15}{16}}$$

$$\Rightarrow \theta \approx 45.5. \quad \blacksquare$$