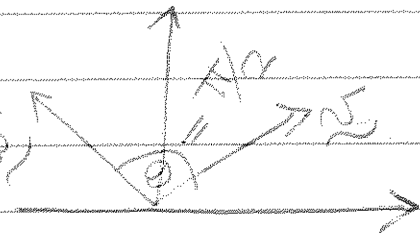
LECTURE 24

Orthogonal vectors.

$V = \mathbb{R}^2$



$\underline{v}$ and $\underline{w}$ are ortho-gonals if $\theta = \pi/2$

Definition (orthogonality). Two elements $\underline{v}$ and $\underline{w}$ of an inner product space V are called orthogonal if $\langle \underline{v}, \underline{w} \rangle = 0$.

Example. The polynomials $p(x) = x$ and $q(x) = x^2 - 1/2$ are orthogonal with respect to

$$\langle p, q \rangle = \int_0^1 p(x) q(x)\, dx.$$

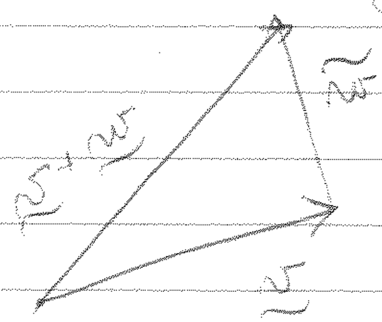
In fact,

$$\langle x, x^2 - 1/2 \rangle = \int_0^1 x(x^2 - 1/2)\, dx = \int_0^1 x^3\, dx - \frac{1}{2}\int_0^1 x$$

$$= 1/4 - 1/2 \cdot 1/2 = 0.$$

$\Rightarrow$ x and $x^2 - 1/2$ are orthogonal with respect to $\langle \cdot, \cdot \rangle$.

Triangle Inequality



length$(\underset{\sim}{v}+\underset{\sim}{w}) \leq$ length$(\underset{\sim}{v})$ + length$(\underset{\sim}{w})$

THM (triangle inequality) The norm associated with an inner product satisfies the triangle inequality

$$\|\underset{\sim}{v}+\underset{\sim}{w}\| \leq \|\underset{\sim}{v}\| + \|\underset{\sim}{w}\| \quad \forall \underset{\sim}{v}, \underset{\sim}{w} \in V$$

Proof. We compute

$$\|\underset{\sim}{v}+\underset{\sim}{w}\|^2 = \langle \underset{\sim}{v}+\underset{\sim}{w}, \underset{\sim}{v}+\underset{\sim}{w} \rangle = \|\underset{\sim}{v}\|^2 + \underbrace{2\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle}_{\leq 2\|\underset{\sim}{v}\|\,\|\underset{\sim}{w}\|} + \|\underset{\sim}{w}\|^2$$

$$\leq \|\underset{\sim}{v}\|^2 + 2\|\underset{\sim}{v}\|\,\|\underset{\sim}{w}\| + \|\underset{\sim}{w}\|^2$$

$$= (\|\underset{\sim}{v}\| + \|\underset{\sim}{w}\|)^2$$

$$\Rightarrow \|\underset{\sim}{v}+\underset{\sim}{w}\| \leq \|\underset{\sim}{v}\| + \|\underset{\sim}{w}\|$$

$\boxtimes$

Example. $V = L^2(0,1)$. Use the L^2 norm to verify the triangle inequality with $f(x) = x-1$, and $g(x) = x^2+1$.

$$\|f\| = \sqrt{\int_0^1 (x-1)^2\,dx} = \sqrt{\frac{1}{3}}$$

$$\|g\| = \sqrt{\int_0^1 (x^2+1)^2\,dx} = \sqrt{\frac{28}{15}}$$

$$\|f+g\| = \sqrt{\int_0^1 (x^2+x)^2\,dx} = \sqrt{\frac{31}{30}}$$

and $\sqrt{31/30} \le \sqrt{1/3} + \sqrt{28/15}$.

Consider $V = \mathbb{R}^3$

Cauchy-Schwarz ineq.

$$\underbrace{\left|\sum_{i=1}^{3} v_i w_i\right|}_{|\langle v, w\rangle|} \le \underbrace{\left(\sum_{i=1}^{3} v_i^2\right)^{1/2}\left(\sum_{i=1}^{3} w_i^2\right)^{1/2}}_{\|v\|\,\|w\|}$$

$|\langle \underline{v}, \underline{w}\rangle| \le \|\underline{v}\|\,\|\underline{w}\|$.

Triangle inequality:

$$\left(\sum_{i=1}^{3} (v_i + w_i)^2\right)^{1/2} \le \left(\sum_{i=1}^{3} v_i^2\right)^{1/2} + \left(\sum_{i=1}^{3} w_i^2\right)^{1/2}$$

$\|\underline{v} + \underline{w}\| \le \|\underline{v}\| + \|\underline{w}\|$.

$V = L^2(a,b)$

$$\left| \int_a^b f(x)\, g(x) \right| \le \left(\int_a^b f^2(x)\, dx \right)^{1/2} \left(\int_a^b g^2(x)\, dx \right)^{1/2}$$

$$\langle f, g \rangle \le \|f\| \, \|g\|$$

$$\left(\int_a^b (f+g)^2 dx \right)^{1/2} \le \left(\int_a^b f^2(x)\, dx \right)^{1/2} + \left(\int_a^b g(x)^2\, dx \right)^{1/2}.$$

§3.3 Norms

Every inner product $\rightarrow$ norm

$$\|\underset{\sim}{v}\| = \sqrt{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle}$$

However not every norm comes from an inner product

Definition (norm) A norm on a vector space V is a function

$$\| \ \| : V \longrightarrow \mathbb{R}$$
$$\underset{\sim}{v} \mapsto \|\underset{\sim}{v}\|$$

s.t.

(a) Positivity: $\|\underset{\sim}{v}\| \ge 0 \ \forall \underset{\sim}{v} \in V$ and $\|\underset{\sim}{v}\| = 0 \Leftrightarrow \underset{\sim}{v} = \underset{\sim}{0}$

(b) Homogeneity: $\|c\underset{\sim}{v}\| = |c| \, \|\underset{\sim}{v}\| \ \forall \underset{\sim}{v} \in V, \ c \in \mathbb{R}$

(c) Triangle Ineq: $\|\underset{\sim}{v} + \underset{\sim}{w}\| \le \|\underset{\sim}{v}\| + \|\underset{\sim}{w}\| \ \forall \underset{\sim}{v}, \underset{\sim}{w} \in V$

If $\|v\| = \sqrt{\langle v, v\rangle}$ with $\langle \cdot, \cdot \rangle$ being an inner product, we know that (a) and (c) ✓

What about b?

$$\|c\underset{\sim}{v}\| = \sqrt{\langle cv, cv\rangle} = \sqrt{c^2 \langle v, v\rangle}$$
$$= |c|\sqrt{\langle v, v\rangle}$$
$$= |c|\,\|v\|$$

then properties (a), (b) and (c) hold

More examples of norms

$V = \mathbb{R}^n$

1) 1-norm: $\underset{\sim}{v} = (v_1, v_2, \dots, v_n)^T$

$$\|\underset{\sim}{v}\|_1 = |v_1| + |v_2| + \cdots + |v_n|$$

2) ∞-norm: $\underset{\sim}{v} = (v_1, v_2, \dots, v_n)^T$

$$\|\underset{\sim}{v}\|_\infty = \max\{|v_1|, |v_2|, \dots, |v_n|\}$$

3) p-norm: $\underset{\sim}{v} = (v_1, v_2, \dots, v_n)^T$

$$\|\underset{\sim}{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}$$

Here (a), (b) are not difficult to prove. However triangle inequality is not trivial: Minkowski's ineq.

⑥

4) $V = C^0[a,b]$, Given $f \in C^0[a,b]$

we define

$$\|f\|_p = \left(\int_a^b |f(x)|^p\,dx\right)^{1/p}$$

where $1 \le p \le \infty$

In particular $\|f\|_1 = \int_a^b |f(x)|\,dx$

$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$$