LECTURE 25 | 11/03/14

Def (Norm) A norm on a vector space V is a function

$$\|\cdot\| : V \longrightarrow \mathbb{R}$$
$$\underset{\sim}{v} \longmapsto \|\underset{\sim}{v}\|$$

s.t

(a) $\|\underset{\sim}{v}\| \geq 0 \ \forall v \in V$ and $\|\underset{\sim}{v}\| = 0 \Leftrightarrow \underset{\sim}{v} = 0$.

(b) $\|c\underset{\sim}{v}\| = |c|\,\|\underset{\sim}{v}\| \ \forall v \in V,\ c \in \mathbb{R}$.

(c) $\|\underset{\sim}{v} + \underset{\sim}{w}\| \leq \|\underset{\sim}{v}\| + \|\underset{\sim}{w}\| \ \forall v, w \in V$.

Examples

1) $V = \mathbb{R}^n$; $\underset{\sim}{v} = (v_1, v_2, \ldots, v_n)^T$

(a) 1-norm: $\|\underset{\sim}{v}\|_1 = |v_1| + |v_2| + \cdots + |v_n|$.

Properties of norm.

(a) $\|\underset{\sim}{v}\|_1 \geq 0$. If $\underset{\sim}{v} = 0 \Rightarrow \|\underset{\sim}{v}\|_1 = 0$, and if $\|\underset{\sim}{v}\|_1 = 0 \Rightarrow \underset{\sim}{v} = 0$

(b) $\|c\underset{\sim}{v}\|_1 = |cv_1| + |cv_2| + \cdots + |cv_n|$

$= |c||v_1| + |c||v_2| + \cdots + |c||v_n|$

$= |c|\,\|\underset{\sim}{v}\|_1$

(c) $\|\underset{\sim}{v} + \underset{\sim}{w}\|_1 = |v_1 + w_1| + \cdots + |v_n + w_n|$

$\leq |v_1| + |w_1| + \cdots + |v_n| + |w_n|$

$= \|\underset{\sim}{v}\|_1 + \|\underset{\sim}{w}\|_1$.

(b) ∞-norm: $\|\underset{\sim}{v}\|_\infty = \max\{|v_1|, |v_2|, \ldots, |v_n|\}$

Check properties (a), (b), (c).

(2)

(c) p-norm $\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}$

Here, properties (a) and (b) are not difficult to prove but triangle ineq is not trivial
$\Rightarrow$ Minkowski's ineq

2) $V = C[a,b]$ Given $f \in C[a,b]$

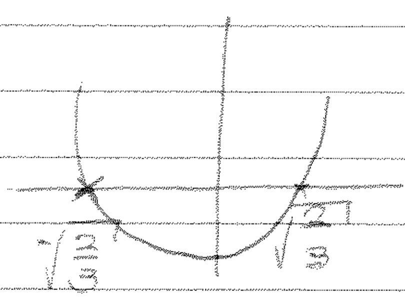
(a) $\|f\|_1 = \int_a^b |f(x)|\,dx$

(b) $\|f\|_\infty = \max |f(x)|$

(c) $\|f\|_p = \left(\int_a^b |f(x)|^p\,dx\right)^{1/p}$

Example. $V = C[a,b]$ where $a=-1$ and $b=1$
$\Rightarrow V = C[-1,1]$; and $p(x) = 3x^2-2$.

$$\|p\|_1 = \int_{-1}^{1} |3x^2-2| = \int_{-1}^{-\sqrt{2/3}} (3x^2-2)\,dx + \int_{-\sqrt{2/3}}^{\sqrt{2/3}} (2-3x^2)\,dx + \int_{\sqrt{2}/\sqrt{3}}^{1} (3x^2-2)\,dx$$



$$= 16/3\,\sqrt{2}/\sqrt{3} - 2 \simeq 2.3546$$

$$\|p\|_\infty = \max\{|3x^2-2|;\ -1 \le x \le 1\}$$
$$= 2.$$

$$\|p\|_2 = \left(\int_{-1}^{1} (3x^2-2)\,dx\right)^{1/2} = \sqrt{\frac{18}{5}} \approx 1.8974$$

Unit vectors

Let V be a vector space, the elements $u \in V$ s.t. $\|u\| = 1$ play a special role and they are called unit vectors.

LEMMA (Unit vector) If $\underset{\sim}{v} \neq 0$, then $\underset{\sim}{u} = \underset{\sim}{v}/\|\underset{\sim}{v}\|$ is a unit vector parallel to $\underset{\sim}{v}$

Proof. $\|\underset{\sim}{u}\| = \left\| \frac{\underset{\sim}{v}}{\|\underset{\sim}{v}\|} \right\| = \frac{1}{\|\underset{\sim}{v}\|} \|\underset{\sim}{v}\| = 1$ $\blacksquare$

Example. $V = C[0,1]$, $p(x) = x^2 - 1/2$.

$$\|p\|_2 = \sqrt{\int_0^1 (x^2 - 1/2)} = \sqrt{\frac{7}{60}}$$

Therefore, $u(x) = \frac{p(x)}{\|p\|_2} = \sqrt{\frac{60}{7}}\, x^2 - \sqrt{\frac{60}{7}}\, \frac{1}{2}$

is a "unit polynomial" parallel to $p(x)$.

Consider $\|\cdot\|_\infty$ norm: $\|p\|_\infty = \max\{x^2 - 1/2, \; 0 \leq x \leq 1\} = 1/2$

$$\Rightarrow \tilde{u}(x) = \frac{p}{\|p\|_\infty} = 2x^2 - 1 \text{ is a}$$

$\|\cdot\|_\infty$ "unit" polynomial

Def (Unit sphere) We define the unit sphere with respect to the norm $\|\cdot\|$ by

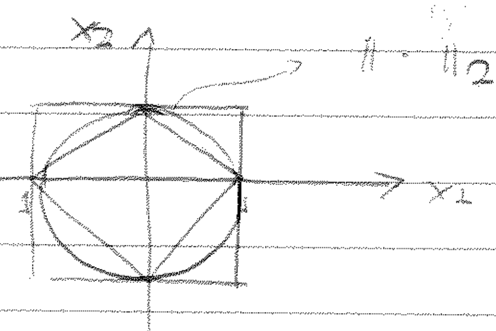
$$S = \{ \underset{\sim}{u} : \|\underset{\sim}{u}\| = 1 \}$$

Examples. $V = \mathbb{R}^2$

(a) $\|\cdot\|_2$: $S = \{ \underset{\sim}{x} : x_1^2 + x_2^2 = 1 \}$

(b) $\|\cdot\|_\infty$: $S = \{ \underset{\sim}{x} : \max\{|x_1|, |x_2|\} = 1 \}$

(c) $\|\cdot\|_1$: $S = \{ \underset{\sim}{x} : |x_1| + |x_2| = 1 \}$



Def (Unit ball) We define the unit ball with respect to the norm $\|\cdot\|$ by

$$B = \{ \underset{\sim}{u} : \|\underset{\sim}{u}\| \leq 1 \}$$

Remark If V is finite dimensional, B is closed and bounded $\Rightarrow$ compact. This is not true in infinite dimensional spaces.

Equivalence of norms.

Equivalence: Given two norms $\|\cdot\|$ and $|||\cdot|||$, we say that these norms are equivalent if there exist constants c and C s.t.

$$c\,|||\underset{\sim}{v}||| \le \|\underset{\sim}{v}\| \le C\,|||\underset{\sim}{v}|||.$$

Example. $V = \mathbb{R}^n$. 2-norm is equivalent to ∞-norm. In fact

$$\|\underset{\sim}{v}\|_2^2 = \sum_{i=1}^{n} |v_i|^2 \le \sum_{i=1}^{n} \max |v_i|^2$$

$$\le \underbrace{\max |v_i|^2}_{\|\underset{\sim}{v}\|_\infty^2} \sum_{i=1}^{n} 1 = n\|\underset{\sim}{v}\|_\infty^2 \quad \forall \underset{\sim}{v} \in \mathbb{R}^n$$

On the other hand,

$$\|\underset{\sim}{v}\|_2^2 = \sum_{i=1}^{n} |v_i|^2 \ge |v_k|^2 = \|\underset{\sim}{v}\|_\infty^2 \quad \forall \underset{\sim}{v} \in \mathbb{R}^n$$

$$|v_k| = \max_{1 \le i \le n} |v_i| = \|\underset{\sim}{v}\|_\infty$$

$$\Rightarrow \frac{1}{\sqrt{n}} \|\underset{\sim}{v}\|_2 \le \|\underset{\sim}{v}\|_\infty \le \|\underset{\sim}{v}\|_2$$

Th4 (Equivalence of norms) All the norms in $\mathbb{R}^n$ are equivalent.