LECTURE 26 | 11/05/14

Equivalence of norms: Given two norms $\|\cdot\|$ and $|||\cdot|||$, we say that they are equivalent if $\exists\, c, C$ s.t. that

$$c\,|||\underset{\sim}{v}||| \le \|\underset{\sim}{v}\| \le C\,|||\underset{\sim}{v}|||$$

Example. $V = \mathbb{R}^m$: 2-norm is equivalent to the ∞-norm. In fact

$$\|\underset{\sim}{v}\|_2^2 = \sum_{i=1}^{m} |v_i|^2 \le \sum_{i=1}^{m} \max_{i=1,\dots,m} |v_i|^2$$

$$= \max_{i=1,\dots,m} |v_i|^2 \sum_{i=1}^{m} 1$$

$$= \|\underset{\sim}{v}\|_\infty^2 \cdot m \quad \forall\, \underset{\sim}{v} \in \mathbb{R}^m$$

On the other hand,

$$\|\underset{\sim}{v}\|_2^2 = \sum_{i=1}^{m} |v_i|^2 \ge |v_k|^2 = \|\underset{\sim}{v}\|_\infty \quad \forall\, v \in \mathbb{R}^m$$

$$|v_k| = \max_{i=1,\dots,m} |v_i| = \|\underset{\sim}{v}\|_\infty$$

$$\Rightarrow \frac{1}{\sqrt{m}} \|\underset{\sim}{v}\|_2 \le \|\underset{\sim}{v}\|_\infty \le \|\underset{\sim}{v}\|_2$$

THM (Equivalence of norms) All the norms in $\mathbb{R}^n$ are equivalent.

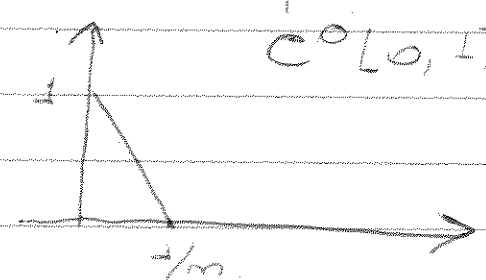
As a consequence: convergence of sequences in $\mathbb{R}^n$ is independent of the norm.

$\Rightarrow$ all norms in $\mathbb{R}^n$ induce same topology.

$\Rightarrow$ notions of open, closed sets and convergence is the same.

This result is not true in infinite-dimensional spaces.

Example. $V = \underset{\substack{\uparrow \\ C^0[0,1]}}{L^2}[0,1]$, and $f_n(x) = \begin{cases} 1-nx, & x \in [0,\frac{1}{n}] \\ 0, & [\frac{1}{n},1] \end{cases}$



Clearly $\|f_n\|_\infty = 1 \ \forall n \in \mathbb{N}$, and

$$\|f_n\|_2^2 = \int_0^{1/n} (1-nx)^2 = \frac{1}{3n} \Rightarrow \|f_n\|_2 = \frac{1}{\sqrt{3n}} \xrightarrow[n\to\infty]{} 0$$

Then, there is no constant C s.t.

$$\|f\|_\infty \le C\|f\|_2$$

$\forall f \in L^2[0,1]$

§ 34 Positive definite matrices

$V = \mathbb{R}^m$, $\underset{\sim}{x} = (x_1, \dots, x_m)^T$

$\underset{\sim}{y} = (x_1, \dots, x_m)^T$

$$\underset{\sim}{x} = x_1 \underset{\sim}{e_1} + \dots + x_m \underset{\sim}{e_m} = \sum_{i=1}^{m} x^i \underset{\sim}{e_i}$$

$$\underset{\sim}{y} = y^1 \underset{\sim}{e_1} + \dots + y^m \underset{\sim}{e_m} = \sum_{i=1}^{m} y^i \underset{\sim}{e_i}$$

Now, compute

$$\langle x, y \rangle = \left\langle \sum_{i=1}^{m} x^i e_i, \sum_{i=1}^{m} y^i \underset{\sim}{e_i} \right\rangle = \sum_{i,j=1}^{m} x^i y^j \langle \underset{\sim}{e_i}, \underset{\sim}{e_j} \rangle$$

Denote $K_{ij} = \langle \underset{\sim}{e_i}, \underset{\sim}{e_j} \rangle$, $i, j = 1, \dots, m$

$$\Rightarrow \langle x, y \rangle = \sum_{i,j=1}^{m} x^i y^j K_{ij} = \sum_{i,j=1} x_i K_{ij} y_j$$

$$= \sum_{i=1}^{m} x_i \sum_{j=1}^{m} K_{ij} y_j = \underset{\sim}{x}^T K \underset{\sim}{y}$$

Conclusion: Any inner product can be expressed in the form $\langle x, y \rangle = \underset{\sim}{x}^T K \underset{\sim}{y}$: "bilinear form"

However, $\langle \cdot, \cdot \rangle$ is symmetric

$\Leftrightarrow \quad \langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = \langle \underset{\sim}{y}, \underset{\sim}{x} \rangle \quad \forall \underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^m$

In particular,

$$k_{ij} = \langle e_i, e_j \rangle = \langle e_j, e_i \rangle = k_{ji} \quad \forall i, j = 1, \dots, n$$
$$\Rightarrow K = K^T \Rightarrow K \text{ is symmetric.}$$

Symmetry of $K \Rightarrow$ Symmetry of bilinear form.

$$\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = \underset{\sim}{x}^T K \underset{\sim}{y} = (\underset{\sim}{x}^T K \underset{\sim}{y})^T$$
$$= \underset{\sim}{y}^T K^T \underset{\sim}{x}$$
$$= \underset{\sim}{y}^T K \underset{\sim}{x}$$
$$\langle \underset{\sim}{y}, \underset{\sim}{x} \rangle$$

Finally, $\|x\|^2 = \langle x, x \rangle \geq 0$, Then

$$\|\underset{\sim}{x}\|^2 = \langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = \underset{\sim}{x}^T K \underset{\sim}{x} = \sum_{i,j=1}^{n} k_{ij} x_i x_j \geq 0 \quad \forall x \in \mathbb{R}^n$$

Equality holds $\Leftrightarrow \underset{\sim}{x} = 0$

Def (positive definite) $A \in \mathbb{R}^{n \times n}$ is called positive definite if it is symmetric, $A = A^T$, and

$$\underset{\sim}{x}^T A \underset{\sim}{x} > 0 \quad \forall x \in \mathbb{R}^n, \underset{\sim}{x} \neq 0$$

Warning A positive definite $\not\Rightarrow a_{ij} > 0$

THM. Every inner product on $\mathbb{R}^n$ is given by

$$\langle x, y \rangle = \underset{\sim}{x}^T K y \quad \forall \underset{\sim}{x}, y \in \mathbb{R}^n$$

where K is symmetric and positive definite.

Given K symmetric, we define

$$q(x) = x^T K x \Rightarrow \text{quadratic form.}$$

Example. Consider $K = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$, and

Compute

$$q(x) = x^T K x = (x_1 \;\; x_2)\begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (4x_1 - 2x_2 \quad -2x_1 + 3x_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (4x_1 - 2x_2)x_1 - (2x_1 + 3x_2)x_2$$

$$= 4x_1^2 - 2x_1x_2 - 2x_1x_2 + 3x_2^2$$

$$= (2x_1 - x_2)^2 + 2x_2^2 \geq 0$$

$$\Rightarrow x^T K x \geq 0$$

Now, if $x = 0 \Rightarrow x^T K x = 0$, and if $x^T K x = 0 \Rightarrow 2(x_1 - x_2)^2 + 2x_2^2 = 0$

$$\Rightarrow x_1 = x_2 = 0$$

Finally K is symmetric.

In conclusion K is positive definite.