LECTURE 27 | 11/07/14

Review: Positive definite matrix.

$V = \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ an inner product in V.
Take $\underset{\sim}{x}, y \in \mathbb{R}^n$:

$$\underset{\sim}{x} = x_1 \underset{\sim}{e_1} + \cdots + x_n \underset{\sim}{e_n}$$
$$y = y_1 \underset{\sim}{e_1} + \cdots + y_n \underset{\sim}{e_n}$$

Then

$$\langle x, y \rangle = \sum_{i,j=1}^{n} x_i y_j \underbrace{\langle \underset{\sim}{e_i}, \underset{\sim}{e_j} \rangle}_{K_{ij}}$$

We define $K_{ij} := \langle \underset{\sim}{e_i}, \underset{\sim}{e_j} \rangle$, then

$$\langle x, y \rangle = \sum_{i,j=1}^{n} K_{ij} x_i y_j = x^T K y \in \mathbb{R}$$

$K \in \mathbb{R}^{n \times n}$,
$x \in \mathbb{R}^n, y \in \mathbb{R}^n$

$\Rightarrow$ any inner product can be expressed as a bilinear form $x^T K y$ where $K \in \mathbb{R}^{n \times n}$

Now,

1) $\langle \cdot, \cdot \rangle$ is symmetric

$\Rightarrow$ $K_{ij} = \langle \underset{\sim}{e_i}, e_j \rangle = \langle \underset{\sim}{e_j}, e_i \rangle = K_{ji}$

$\Rightarrow$ K is symmetric, i.e., $K = K^T$

2) $\|\underset{\sim}{x}\|^2 = \langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = \sum_{i,j=1}^{n} x_i K_{ij} x_j \geq 0 \ \forall x \in \mathbb{R}^n$

$$= \underset{\sim}{x}^T K \underset{\sim}{x}$$

However $\|\underset{\sim}{x}\|^2 = 0 \iff \underset{\sim}{x} = 0$

These steps motivate the following definition

Def (positive definite matrix) $K \in \mathbb{R}^{n\times n}$ is called positive definite if it is symmetric, $K = K^T$ and

$$\underset{\sim}{x}^T K \underset{\sim}{x} > 0 \ \forall x \in \mathbb{R}^n, x \neq 0.$$

Notation $K > 0$

Warning: $K > 0$ does not mean that all entries are positive!

THM. Every inner product on $\mathbb{R}^n$ is given by

$$\langle x, y \rangle = \underset{\sim}{x}^T K \underset{\sim}{y}, \ \forall \underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n$$

where K is a symmetric, positive definite matrix.

Definition (Quadratic form). Given K symmetric, we define the quadratic form

$$q(\underset{\sim}{x}) = \underset{\sim}{x}^T K \underset{\sim}{x} = \sum_{i,j=1}^{n} K_{ij} x_i x_j$$

Def (positive semi definite) K is called semi-positive definite if it is symmetric and

$$\underset{\sim}{x}^T K \underset{\sim}{x} \geq 0 \ \forall\ \underset{\sim}{x} \in \mathbb{R}^n$$

Gram matrices $\Rightarrow$ Matrices whose entries are given by inner products of elements.

Def (Gram matrix) Let V be an inner space, and let $\underset{\sim}{v_1}, \ldots, \underset{\sim}{v_n} \in V$. The associated Gram matrix is def by

$$K = \begin{pmatrix} \langle \underset{\sim}{v_1}, \underset{\sim}{v_1} \rangle & \langle \underset{\sim}{v_1}, \underset{\sim}{v_2} \rangle & \cdots & \langle \underset{\sim}{v_1}, \underset{\sim}{v_n} \rangle \\ \langle \underset{\sim}{v_2}, \underset{\sim}{v_1} \rangle & \langle \underset{\sim}{v_2}, \underset{\sim}{v_2} \rangle & \cdots & \langle \underset{\sim}{v_2}, \underset{\sim}{v_n} \rangle \\ \vdots & & & \vdots \\ \langle \underset{\sim}{v_n}, \underset{\sim}{v_1} \rangle & \langle \underset{\sim}{v_n}, \underset{\sim}{v_2} \rangle & \cdots & \langle \underset{\sim}{v_n}, \underset{\sim}{v_n} \rangle \end{pmatrix}$$

Symmetry of $\langle \cdot, \cdot \rangle \Rightarrow$

$$\begin{aligned} K_{ij} &= \langle \underset{\sim}{v_i}, \underset{\sim}{v_j} \rangle \\ &= \langle \underset{\sim}{v_j}, \underset{\sim}{v_i} \rangle \\ &= K_{ji} \end{aligned}$$

$\Rightarrow$ Gram matrix is symmetric!

THM All Gram matrices are semi-positive definite.
The Gram matrix is positive def $\Leftrightarrow$ $v_1, \ldots, v_n$ are l.i.

Proof. Prove semi-definitness of K.

$$q(x) = \underset{\sim}{x}^T K \underset{\sim}{x} = \sum_{i,j=1}^{n} K_{ij}\, x_i x_j = \sum_{i,j=1}^{n} \langle \underset{\sim}{v_i}, \underset{\sim}{v_j} \rangle x_i x_j$$

The quadratic form is positive if $q(\underset{\sim}{x}) > 0 \; \forall x \in \mathbb{R}^n$, $\underset{\sim}{x} \neq 0$.

Example. Show that $K = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$ is a positive definite matrix. We compute the quadratic form

$$q(\underset{\sim}{x}) = \underset{\sim}{x}^T K \underset{\sim}{x} = (x_1 \; x_2) \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (x_1 \; x_2) \begin{pmatrix} 4x_1 - 2x_2 \\ -2x_1 + 3x_2 \end{pmatrix}$$

$$= 4x_1^2 - 2x_2 x_1 - 2x_1 x_2 + 3x_2^2$$

$$= (2x_1 - x_2)^2 + 2x_2^2 \geq 0$$

Moreover $q(x) = 0 \Longleftrightarrow \underset{\sim}{x} = 0$

$\Rightarrow$ K is positive definite.

Exercise $K = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is not positive definite matrix

In fact, $q(\underset{\sim}{x}) = \underset{\sim}{x}^T K \underset{\sim}{x} = (x_1 \; x_2) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$= (x_1 \; x_2) \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \end{pmatrix}$$

$$= x_1^2 + 2x_1 x_2 + 2x_1 x_2 + x_2^2$$

$$= x_1^2 + 4x_1 x_2 + x_2^2,$$

and $q((1, -1)^T) = -2 < 0$.

$\Rightarrow$ K is not positive definite.

$$\Rightarrow q(x) = \left\langle \sum_{i=1}^{m} x_i \underset{\sim}{v_i}, \sum_{j=1}^{m} x_j v_j \right\rangle$$

$$= \langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = \| \underset{\sim}{v} \|^2 \geq 0,$$

where $\underset{\sim}{v} = \sum_{i=1}^{m} x_i \underset{\sim}{v_i}$.

$\Rightarrow$ K is semi positive definite.

Moreover, $q(x) = \| \underset{\sim}{v} \|^2 > 0$ as long as $\underset{\sim}{v} \neq 0$

If $\{v_1, \ldots, v_m\}$ is l.i.

$$\Rightarrow \underset{\sim}{v} = x_1 v_1 + \cdots + x_m \underset{\sim}{v_m} = 0$$

$$\Leftrightarrow x_1 = \cdots = x_m = 0$$

$\Rightarrow q(\underset{\sim}{x}) = 0 \Leftrightarrow \underset{\sim}{x} = 0 \Rightarrow$ K is positive definite $\blacksquare$