LECTURE 28 | 11/10/14

Def (Gram matrices) Let V be an inner product, and let $\underset{\sim}{v_1}, \ldots, \underset{\sim}{v_m} \in V$. The associated Gram matrix is def. by

$$K = \begin{pmatrix} \langle \underset{\sim}{v_1}, \underset{\sim}{v_1} \rangle & \langle \underset{\sim}{v_1}, \underset{\sim}{v_2} \rangle & \cdots & \langle \underset{\sim}{v_1}, \underset{\sim}{v_m} \rangle \\ \langle \underset{\sim}{v_2}, \underset{\sim}{v_1} \rangle & \langle \underset{\sim}{v_2}, \underset{\sim}{v_2} \rangle & \cdots & \langle \underset{\sim}{v_2}, \underset{\sim}{v_m} \rangle \\ \vdots & & & \\ \langle \underset{\sim}{v_m}, \underset{\sim}{v_1} \rangle & \langle \underset{\sim}{v_m}, \underset{\sim}{v_2} \rangle & \cdots & \langle \underset{\sim}{v_m}, \underset{\sim}{v_m} \rangle \end{pmatrix}$$

Property: K is symmetric, i.e. $K = K^T$

THM. All Gram matrices are semi-positive definite.
A Gram matrix is positive def $\iff \{v_1, \ldots, v_m\}$ is l.i.

Proof. Prove of semi-definiteness of K.

$$q(\underset{\sim}{x}) = \underset{\sim}{x}^T K \underset{\sim}{x} = \sum_{i,j=1}^{m} K_{ij} x_i x_j = \sum_{i,j=1}^{m} \langle \underset{\sim}{v_i}, \underset{\sim}{v_j} \rangle x_i x_j$$

$$= \Big\langle \underbrace{\sum_{i=1}^{m} x_i \underset{\sim}{v_i}}_{\underset{\sim}{v}}, \sum_{j=1}^{m} x_j \underset{\sim}{v_j} \Big\rangle = \langle \underset{\sim}{v}, \underset{\sim}{v} \rangle$$

$$= \|\underset{\sim}{v}\|^2,$$

where $\underset{\sim}{v} = \sum_{i=1}^{m} x_i v_i$

$\Rightarrow$ K is semi positive definite.

Moreover $q(\underset{\sim}{x}) = \|v\|^2 > 0 \iff \underset{\sim}{v} \neq 0$.

If $\{v_1, \ldots, v_m\}$ is l.i

$$\Rightarrow \underset{\sim}{v} = x_1 \underset{\sim}{v_1} + \cdots + x_m \underset{\sim}{v_m} = 0$$

$$\Rightarrow x_1 = \cdots = x_m = 0$$

$$\Rightarrow q(\underset{\sim}{x}) = 0 \iff \underset{\sim}{x} = 0$$

$\Rightarrow$ K is positive definite ▨

Examples

1) $V = \mathbb{R}^3$; $\underset{\sim}{v_1} = (1 \;\; 2 \;\; -1)^T$; $\langle \cdot , \cdot \rangle$ = dot product

$\underset{\sim}{v_2} = (3 \;\; 0 \;\; 6)^T$

$$K = \begin{pmatrix} \underset{\sim}{v_1} \cdot \underset{\sim}{v_1} & \underset{\sim}{v_1} \cdot \underset{\sim}{v_2} \\ \underset{\sim}{v_2} \cdot \underset{\sim}{v_1} & \underset{\sim}{v_2} \cdot \underset{\sim}{v_2} \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -3 & 45 \end{pmatrix}$$

K is positive definite since $\{\underset{\sim}{v_1}, \underset{\sim}{v_2}\}$ is l.i

2) $V = C^0[0,1]$, $\langle \cdot , \cdot \rangle = L^2$ inner product

$\underset{\sim}{v_1} = 1$; $\underset{\sim}{v_2} = x$; $\underset{\sim}{v_3} = x^2$

$$K = \begin{pmatrix} \langle 1,1 \rangle & \langle 1,x \rangle & \langle 1,x^2 \rangle \\ \langle x,1 \rangle & \langle x,x \rangle & \langle x,x^2 \rangle \\ \langle x^2,1 \rangle & \langle x^2,x \rangle & \langle x^2,x^2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$$

$\langle 1,1 \rangle = \int_0^1 dx = 1$; $\langle x^2, x \rangle = \int_0^1 x^3 dx = \frac{1}{4}$

$\langle 1,x \rangle = \int_0^1 x \, dx = 1/2$

$\langle x,x \rangle = \int_0^1 x^2 dx = \frac{1}{3}$

K is positive definite since $\{1, x, x^2\}$ is l.i

3) $V = C^0[-\pi, \pi]$, $\langle \cdot, \cdot \rangle = L^2$ inner product

$\underset{\sim}{v_1} = 1$, $\underset{\sim}{v_2} = \cos x$, $\underset{\sim}{v_3} = \sin x$.

$$K = \begin{pmatrix} 2\pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix}$$ K is positive definite.

§3.5. Cholesky decomposition.

Assume A is SPD

$\Rightarrow$ A is non-singular.

In fact, $A\underset{\sim}{x} = 0$.

$\Rightarrow \underset{\sim}{x}^T A \underset{\sim}{x} = 0 \Rightarrow \underset{\sim}{x} = 0$.

Then $Ax = 0$ has a unique solution $\underset{\sim}{x} = 0$

$\Rightarrow$ A is non-singular.

We can also prove that A is regular.

Since A is regular,

$$A = LU = LDV$$

L - unit lower triangular matrix

U - upper triangular matrix

D - diagonal matrix

V - unit upper triangular matrix.

Now, since A is symmetric then $A = LDL^T$

and since A is SPD $\Rightarrow D = \text{diag}(d_1, \dots, d_n)$

where $d_i > 0$ $\forall i = 1, \dots, n$

Proof. Assume A is Symmetric.

$$A = A^T \Rightarrow A = LDV = V^T D^T L^T = A^T$$

$$\text{uniqueness} \Rightarrow \begin{matrix} L = V^T \\ V = L^T \end{matrix} \Rightarrow L = V^T$$

$$\Rightarrow A = LDL^T.$$

Now, if A is SPD

$$\underset{\sim}{x}^T A \underset{\sim}{x} > 0 \ \forall \underset{\sim}{x} \neq 0$$

$$\underset{\sim}{x}^T L D L^T \underset{\sim}{x} > 0 \ \forall \underset{\sim}{x} \neq 0$$

$$\underbrace{(L^T \underset{\sim}{x})}_{y^T} D \underbrace{(L^T \underset{\sim}{x})}_{y} > 0 \ \forall \underset{\sim}{x} \neq 0 \Leftrightarrow y^T D y > 0 \ \forall y > 0$$

Take $y = e_i \Rightarrow d_i > 0 \ \forall i = 1, \dots, n$. ▨

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}; \quad D^{1/2} = \begin{pmatrix} \sqrt{d_1} & & \\ & \sqrt{d_2} & \\ & & \sqrt{d_n} \end{pmatrix}$$

and $D^{1/2} \cdot D^{1/2} = D$

$$\text{Then,} \quad A = LDL^T = L D^{1/2} D^{1/2} L^T \ ; \ (D^{1/2})^T = D^{1/2}$$

$$= \underbrace{L D^{1/2}}_{G} \underbrace{(L D^{1/2})^T}_{G^T}$$

$$= G \cdot G^T$$

G is a lower triangular matrix. The decomposition $A = GG^T$ is called Cholesky decomposition ▨