LECTURE 29 | Review 11/12/14.

§3.1.

Inner product. V vector space, an inner product is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$
$$(\underset{\sim}{u}, \underset{\sim}{v}) \mapsto \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle$$

s.t.

(i) $\langle c\underset{\sim}{u} + d\underset{\sim}{v}, \underset{\sim}{w} \rangle = c\langle \underset{\sim}{u}, \underset{\sim}{w} \rangle + d\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle$

$\langle \underset{\sim}{u}, c\underset{\sim}{v} + d\underset{\sim}{w} \rangle = c\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle + d\langle \underset{\sim}{u}, \underset{\sim}{w} \rangle$

(ii) $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = \langle \underset{\sim}{v}, \underset{\sim}{u} \rangle$

(iii) $\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle > 0 \ \forall v \neq 0$ and

$\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = 0 \iff \underset{\sim}{v} = 0$.

Examples.

1) $V = \mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ = dot product

2) $V = \mathbb{R}^n$, $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = \sum_i a_i u_i v_i$, where $a_i > 0$.

3) $V = \mathbb{R}^n$, $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = \underset{\sim}{u}^T A \underset{\sim}{v}$ where A is positive definite.

4) $V = C^0[a,b]$, $\langle f, g \rangle = \int_a^b f(x) g(x)$ L^2 inner product.

§3.2.

Cauchy-Schwarz ineq. Every inner product $\langle \cdot, \cdot \rangle$ on a vector space V satisfies

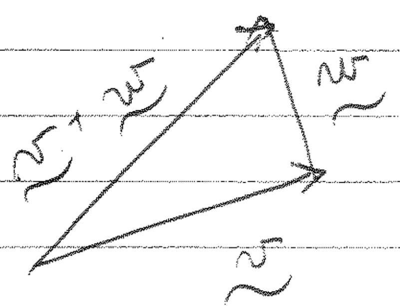
$$|\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle| \leq \|\underset{\sim}{v}\| \, \|\underset{\sim}{w}\|$$

Where $\|\cdot\|$ is the associated norm to $\langle \cdot, \cdot \rangle$ and is defined by

$$\|\underset{\sim}{v}\| = \sqrt{\langle v, \underset{\sim}{v} \rangle}$$

Triangle inequality: Given a vector space V and any inner product $\langle \cdot, \cdot \rangle$

$$\|\underset{\sim}{v} + \underset{\sim}{w}\| \leq \|\underset{\sim}{v}\| + \|\underset{\sim}{w}\|$$



orthogonal vectors. Two vectors are orthogonal if and only if

$$\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = 0.$$

In particular, $\{f_n\} \subset C^0[a,b]$, we say that the family $\{f_n\}$ is orthogonal if $\langle f_n, f_m \rangle = 0 \quad \forall n \neq m.$

§ 3.3. Norms. V vector space, a norm is a function

$$\| \; \| : V \longrightarrow [0, \infty)$$
$$\underset{\sim}{v} \longmapsto \|\underset{\sim}{v}\|$$

s.t. 1) $\|\underset{\sim}{v}\| \geq 0$ and $\|\underset{\sim}{v}\| = 0 \iff \underset{\sim}{v} = 0$

2) $\|c\,\underset{\sim}{v}\| = |c|\,\|\underset{\sim}{v}\|$

3) $\|\underset{\sim}{v} + \underset{\sim}{w}\| \leq \|\underset{\sim}{v}\| + \|\underset{\sim}{w}\|$.

§ 3.4. Positive definite matrix.

$A \in \mathbb{R}^{n \times n}$ is called positive definite if it is symmetric and

$$\underset{\sim}{x}^T A \underset{\sim}{x} > 0 \quad \forall x \neq 0,\ x \in \mathbb{R}^n$$

Notation $K > 0$.

Quadratic form (associated with A)

$$q(\underset{\sim}{x}) = \underset{\sim}{x}^T A \underset{\sim}{x}$$

q is positive if $q(\underset{\sim}{x}) > 0 \quad \forall x \in \mathbb{R}^n, x \neq 0$.

$A \in \mathbb{R}^{n \times n}$ is called semi-positive definite if

$$q(\underset{\sim}{x}) = \underset{\sim}{x}^T K \underset{\sim}{x} \geq 0 \; \forall x \in \mathbb{R}^n$$

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Gram matrices: V, $\langle \cdot, \cdot \rangle$, $\{v_1, \ldots, v_n\}$

$$K_{ij} = \langle \underset{\sim}{v_i}, \underset{\sim}{v_j} \rangle$$

Main result. 1) Gram matrices are semi-positive definite
2) Gram matrices are positive definite if and only if $\{v_1, \ldots, v_n\}$ is l.i.