LECTURE 3 | 09/08/14.

Last class: Matrix arithmetic (§1.2) and introduction to LU decomposition (§1.3).

$$(1.1)\quad \begin{cases} x+2y+z=2 \\ 2x+6y+z=7 \\ x+y+4z=3 \end{cases} \iff A\underset{\sim}{x}=\underset{\sim}{b}$$

$$M := (A\,|\,b) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 6 & 1 & 7 \\ 1 & 1 & 4 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 5/2 & \frac{5}{2} \end{array}\right) = \underset{(U|\underset{\sim}{c})}{N}$$

U: upper triangular matrix.

Today: Elementary Matrices.

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}\underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix}}_{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix}$$

What do we consider here? $\rightarrow$ E_1 : lower triangular matrix
$e_{ij}=0$ if $j>i$

Def (Elementary Matrix). Matrix associated with an elementary row operation. It is obtained by applying the row operation to I_n.

Example:
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-2R_1+R_2} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_1$$

Then, we have

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}}_{E_3} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}}_{E_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1} \underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix}}_{A} = U = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 5/2 \end{pmatrix}$$

Exercise: Check this computation!

Conclusion: The matrix EA, where $E = E_3 E_2 E_1$ is the same as the one obtained via <u>row operations</u>.

<u>Inverse Elementary Matrix</u>.

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ; \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is s.t. $E_1 \cdot E_1^{-1} = E_1^{-1} \cdot E_1 = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$L_1 := E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{lower triangular matrix!}$$

$\Rightarrow L_1 E_1 = E_1 L_1 = I_3$

$$L_2 := E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad L_3 = E_3^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

$\Rightarrow L_2 E_2 = E_2 L_2 = I_3$

$\Downarrow$

$L_3 E_3 = E_3 L_3 = I_3$

Prop. (Multiplication of lower triangular matrices)
If L and $\hat{L}$ are lower triangular matrices in $\mathbb{R}^{m\times m}$ $\Rightarrow$ $L\hat{L}$ is lower triangular as well.

Proof. L is lower triang $\Rightarrow$ $l_{ik} = 0$ for $k > i$
$\hat{L}$ is lower triang $\Rightarrow$ $\hat{l}_{kj} = 0$ for $j > k$

$C := L\hat{L}$

C is lower triang if $c_{ij} = 0$ for $j > i$

$$\underset{i<j}{c_{ij}} = \sum_{k=1}^{m} l_{ik}\hat{l}_{kj} = \sum_{k>i} \cancel{l_{ik}}^{\,0}\hat{l}_{kj} + \sum_{k\le i<j} l_{ik}\cancel{\hat{l}_{kj}}^{\,0} = 0$$

$\blacksquare$

LU factorization.

Define $L = L_1 L_2 L_3$ and recall that
$$EA = \underbrace{E_3 E_2 E_1}_{E} A = U$$

Then, $LU = L_1 L_2 L_3 E_3 E_2 E_1 A$
$$= L_1 L_2 \underbrace{(L_3 E_3)}_{I_3} E_2 E_1 A$$
$$= L_1 \underbrace{(L_2 E_2)}_{I_3} E_1 \cdot A$$
$$= L_1 E_1 \cdot A = A \Rightarrow \boxed{LU = A}$$

THM. A matrix A is regular (all pivots $\neq 0$) if and only if it can be factored

$$\boxed{A = LU}$$

where L is lower triangular having 1's on the diagonal and U is upper triangular.

Solving $A\underset{\sim}{x} = \underset{\sim}{b}$ via LU decomposition.

Suppose $A = LU$, then

$$A\underset{\sim}{x} = \underset{\sim}{b} \iff LU\underset{\sim}{x} = \underset{\sim}{b}$$

Define $\underset{\substack{\uparrow \\ \text{upper triangular}}}{U}\underset{\sim}{x} = \underset{\sim}{y} \Rightarrow \underset{\substack{\uparrow \\ \text{lower triangular.}}}{L}\underset{\sim}{y} = \underset{\sim}{b}$

These systems can be easily solved.

Example

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= LU$$

⑤

Solve $A\underset{\sim}{x} = \underset{\sim}{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

First, we solve $L\underset{\sim}{y} = b$

$$\begin{cases} y_1 & = 0 \\ 2y_1 + y_2 & = 0 \\ y_1 - y_2 + y_3 & = 1 \end{cases}$$

$$\Rightarrow y_1 = 0 ; y_2 = 0 ; y_3 = 1.$$

Now, we solve $U\underset{\sim}{x} = \underset{\sim}{y}$

$$\begin{cases} 2x_1 + x_2 + x_3 & = 0 \\ x_2 & = 0 \\ -x_3 & = 1 \end{cases}$$

$$\Rightarrow \boxed{x_3 = -1 ; \; x_2 = 0 ; \; x_1 = 1/2}.$$