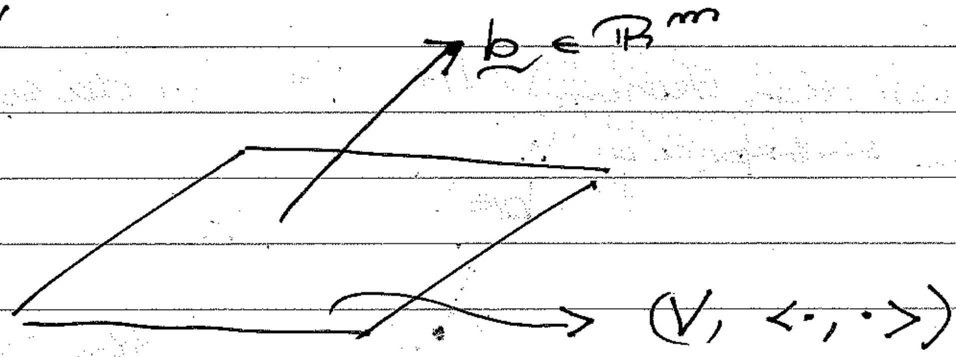
LECTURE 30 | 11/14/14.

§1 Minimization and Least Squares Approximation.

Applications: a) Least squares
b) Data fitting.
c) Polynomial approximation.

Motivation.
Let V a vector space and $\langle \cdot, \cdot \rangle$ an inner product on V



Question: Find the $v^* \in V$ that is closest to $b \in \mathbb{R}^m$. In other words, we seek to minimize the distance $d(v, b) = \| v - b \|$ over all possibles $v \in V$.

Let $\{v_1, \dots, v_n\}$ be a basis of V, then if $v \in V$

$$v = x_1 v_1 + \dots + x_n v_n$$
$$= \underbrace{[v_1 \dots v_n]}_{A} x = Ax$$

We identify $V = \text{rng}(A)$. Then, the closest point in V to $\underset{\sim}{b}$ is found by minimizing

$$\|\underset{\sim}{v} - \underset{\sim}{b}\|^2 = \|A\underset{\sim}{x} - \underset{\sim}{b}\|^2 \quad \forall\, x \in \mathbb{R}^n ?$$

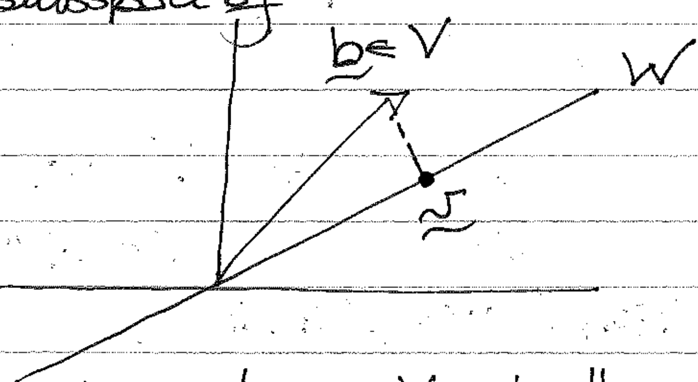
Least squares problem. $\Rightarrow \|A\underset{\sim}{x} - \underset{\sim}{b}\|^2 = 0.$

If $\underset{\sim}{b} \in \text{rng}(A) \Rightarrow$ System $A\underset{\sim}{x} = b$ is solvable.

If $\underset{\sim}{b} \notin \text{rng}(A) \Rightarrow$ System $A\underset{\sim}{x} = b$ is not solvable

$\Rightarrow \|A\underset{\sim}{x} - \underset{\sim}{b}\| \neq 0.$

THM (projection theorem) Let V a vector space, $\underset{\sim}{b} \in V$ and W a subspace of V



$\dim W < \dim V$

What is the element $\underset{\sim}{v} \in V$ (∈W) s.t $\|\underset{\sim}{v} - \underset{\sim}{b}\|$ is minimized?

The following two conditions are equivalent:

(1) $\underset{\sim}{v}$ is the closest element to $\underset{\sim}{b} \in V$

$$\|\underset{\sim}{b} - \underset{\sim}{v}\| \leq \|\underset{\sim}{b} - w\| \quad \forall\, w \in W$$

(2) $\underset{\sim}{v}$ is the orthogonal projection of $\underset{\sim}{b}$ onto W

$$\langle \underset{\sim}{b} - \underset{\sim}{v}, \underset{\sim}{w} \rangle = 0 \quad \forall\, \underset{\sim}{w} \in W.$$

Proof.

(2) $\Rightarrow$ (1)

$$\|\underset{\sim}{b}-\underset{\sim}{w}\|^2 = \langle \underset{\sim}{b}-\underset{\sim}{w}, \underset{\sim}{b}-\underset{\sim}{w}\rangle$$
$$= \langle (\underset{\sim}{b}-\underset{\sim}{v})-(\underset{\sim}{w}-\underset{\sim}{v}), (\underset{\sim}{b}-\underset{\sim}{v})-(\underset{\sim}{w}-\underset{\sim}{v})\rangle$$
$$= \langle \underset{\sim}{b}-\underset{\sim}{v}, \underset{\sim}{b}-\underset{\sim}{v}\rangle - 2\langle \underset{\sim}{w}-\underset{\sim}{v}, \underset{\sim}{b}-\underset{\sim}{v}\rangle + \langle \underset{\sim}{w}-\underset{\sim}{v}, \underset{\sim}{w}-\underset{\sim}{v}\rangle$$
$$= \|\underset{\sim}{b}-\underset{\sim}{v}\|^2 - 2\langle \underset{\sim}{w}-\underset{\sim}{v}, \underset{\sim}{b}-\underset{\sim}{v}\rangle + \|\underset{\sim}{w}-\underset{\sim}{v}\|^2$$

Now, $\langle \underset{\sim}{w}-\underset{\sim}{v}, \underset{\sim}{b}-\underset{\sim}{v}\rangle = 0$ (because of (2))

$\underset{\sim}{w}, \underset{\sim}{v} \in W$ $\quad$ $b \in V$, $\underset{\sim}{v} \in W \Rightarrow \underset{\sim}{b}-\underset{\sim}{v} \in V$

$\Rightarrow \underset{\sim}{w}-\underset{\sim}{v} \in W$

$$\Rightarrow \|\underset{\sim}{b}-\underset{\sim}{w}\|^2 = \|\underset{\sim}{b}-\underset{\sim}{v}\|^2 + \|\underset{\sim}{w}-\underset{\sim}{v}\|^2 \geq \|\underset{\sim}{b}-\underset{\sim}{v}\|^2$$

$$\Rightarrow \|\underset{\sim}{b}-\underset{\sim}{v}\|^2 \leq \|b-\underset{\sim}{w}\|^2$$
$$\Rightarrow \|\underset{\sim}{b}-\underset{\sim}{v}\| \leq \|b-\underset{\sim}{w}\|.$$

(1) $\Rightarrow$ (2). Consider the function

$$\varphi(t) = \|\underset{\sim}{b}-\underset{\sim}{v}-t\underset{\sim}{w}\|^2$$

for $w \in W$, $t \in \mathbb{R}$.

We have

(i) $\psi(0) = \| \underset{\sim}{b} - \underset{\sim}{v} \|^2 \underset{(1)}{\leq} \| \underset{\sim}{b} - \underset{\sim}{v} - t\underset{\sim}{w} \|^2 = \psi(t)$

$\Rightarrow$ $\psi(t)$ has a minimum at $t = 0$.

(ii) $\psi(t) = \| \underset{\sim}{b} - \underset{\sim}{v} - t\underset{\sim}{w} \|^2$

$$= \langle b - v - t\underset{\sim}{w}, b - \underset{\sim}{v} - t\underset{\sim}{w} \rangle$$

$$= \| \underset{\sim}{b} - \underset{\sim}{v} \|^2 - 2t \langle \underset{\sim}{b} - \underset{\sim}{v}, \underset{\sim}{w} \rangle + t^2 \| \underset{\sim}{w} \|^2$$

Since ψ has a minimum at $t = 0$, we need $\psi'(0) = 0$.

$\psi'(t) = - \langle \underset{\sim}{b} - \underset{\sim}{v}, \underset{\sim}{w} \rangle + 2t \| \underset{\sim}{w} \|^2$

$\Rightarrow \psi'(0) = - \langle \underset{\sim}{b} - \underset{\sim}{v}, \underset{\sim}{w} \rangle = 0.$

$\Rightarrow$ $\langle \underset{\sim}{b} - \underset{\sim}{v}, \underset{\sim}{w} \rangle = 0$ ▨

Normal equations.

$V = \mathbb{R}^m$, $W = \text{span}\{\underset{\sim}{a_i}\}_{i=1}^n$, $\underset{\sim}{a_i} \in V$ and $n < m$.

Consider $\langle \cdot, \cdot \rangle$ s.t $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = \underset{\sim}{u}^T C \underset{\sim}{v}$, where C is SPD. ...