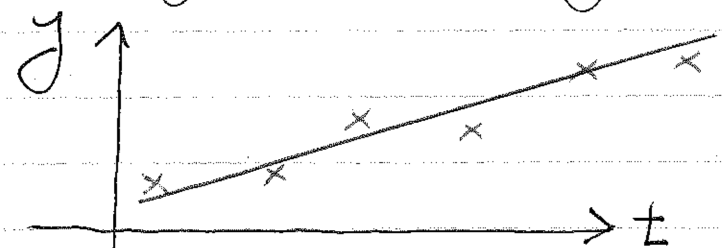
LECTURE 32) 11/19/14. (1)

§ 4.4 Data fitting and interpolation.

Suppose we have
$$(t_1, y_1), \dots, (t_m, y_m)$$



We are looking for the best straight line that fits the data. We propose $y = \alpha + \beta t$. We need to find α and β.

The error $e_i = y_i - (\alpha + \beta t_i)$

$$\Rightarrow \quad \underset{\sim}{e} = \underset{\sim}{y} - A\underset{\sim}{x}$$

$e = (e_1, \cdots, e_m)^T$; $y = (y_1, \dots, y_m)^T$

$$A = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{pmatrix}; \text{ and } \underset{\sim}{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

We want to find α, β s.t
$$\text{ERROR} = \|\underset{\sim}{e}\| = \|\underset{\sim}{y} - A\underset{\sim}{x}\|$$
is smallest as possible. $\Rightarrow$ best approximation
$\Leftrightarrow$ Normal equations.

$\Rightarrow$ $\underset{\sim}{x}$ satisfies $A^T A \underset{\sim}{x} = A^T \underset{\sim}{y}$

Now,

$$A^T A = \begin{pmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{pmatrix} \begin{pmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{pmatrix} = \begin{pmatrix} m & \sum_i t_i \\ \sum_i t_i & \sum_i t_i^2 \end{pmatrix}$$

$$= m \begin{pmatrix} 1 & \overline{t} \\ \overline{t} & \overline{t^2} \end{pmatrix}$$

where $\overline{t} = \frac{1}{m} \sum_{i=1}^{m} t_i$.

$$A^T y = \begin{pmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} \sum_i y_i \\ \sum_i t_i y_i \end{pmatrix} = m \begin{pmatrix} \overline{y} \\ \overline{ty} \end{pmatrix}$$

$$\Rightarrow \cancel{m} \begin{pmatrix} 1 & \overline{t} \\ \overline{t} & \overline{t^2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \cancel{m} \begin{pmatrix} \overline{y} \\ \overline{ty} \end{pmatrix}$$

$$\Rightarrow \begin{cases} \alpha = \overline{y} - \overline{t}\beta \\ \beta = \dfrac{\overline{ty} - \overline{t}\,\overline{y}}{\overline{t^2} - (\overline{t})^2} \end{cases}$$

$$\Rightarrow \quad y = \beta(t - \overline{t}) + \overline{y}$$

best straight line that fits the given data

Example.

t_i	0	1	3	6
y_i	2	3	7	12

; $m = 4$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 6 \end{pmatrix}; \quad A^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 6 \end{pmatrix}$$

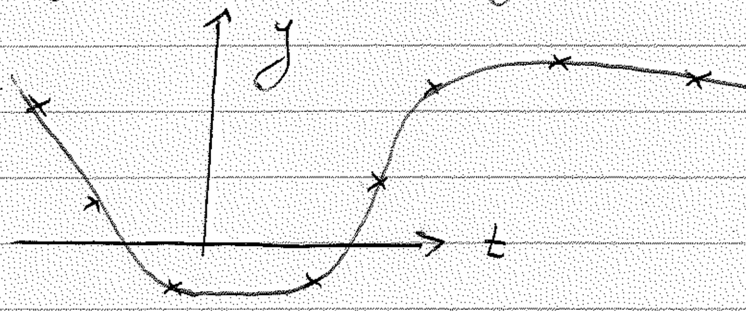
$$\Rightarrow A^T A = \begin{pmatrix} 4 & 10 \\ 10 & 46 \end{pmatrix}$$

and $A^T y = \begin{pmatrix} 24 \\ 96 \end{pmatrix}$

Normal eqns:
$$4\alpha + 10\beta = 24$$
$$10\alpha + 46\beta = 96$$
$$\Rightarrow \alpha = 12/7, \quad \beta = 12/7$$
$$\Rightarrow y = 12/7 + 12/7\, t.$$

Consider now a more general situation:

$$(t_1, y_1), \ldots, (t_m, y_m)$$



Suppose we have a family of functions $\{\phi_1, \ldots, \phi_m\}$
We would like to find $\alpha = (\alpha_1, \ldots, \alpha_m)^T$ s.t.

$$\phi = \sum_{i=1}^{m} \alpha_i \phi_i$$

is the best curve that fits the data in the following sense:

$$\min \sum_{i=1}^{m} |\phi(t_i) - y_i|^2, \quad \text{i.e.,} \quad \min \|\underset{\sim}{e}\|^2$$

where $\|\underset{\sim}{e}\|^2 = \sum_{i=1}^{m} |\phi(t_i) - y_i|^2 = \|A\underset{\sim}{x} - \underset{\sim}{y}\|^2$

where $A = \begin{pmatrix} \phi_1(t_1) & \phi_2(t_1) & \cdots & \phi_n(t_1) \\ \vdots & & & \\ \phi_1(t_m) & \phi_2(t_m) & \cdots & \phi_n(t_m) \end{pmatrix}$

$\Rightarrow A \in \mathbb{R}^{m \times n}$

Then, we solve the normal equations

$$A^T A \underset{\sim}{x} = A^T y$$

and we get $x = (\alpha_1, \dots, \alpha_n)^T$, and then

$$\bar{\phi} = \sum_{i=1}^{n} \alpha_i \phi_i .$$

Application: polynomial interpolation

Take $\phi_i(t) = t^{i-1} \Rightarrow \phi_1(t) = 1$

$\phi_2(t) = t$

$\phi_n(t) = t^{n-1}$

and $m = n$. So $A \in \mathbb{R}^{n \times n}$, and

$$A = \begin{pmatrix} 1 & t_1 & t_1^2 \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 \cdots & t_2^{n-1} \\ \vdots & & & \\ 1 & t_n & t_n^2 \cdots & t_n^{n-1} \end{pmatrix}$$

If $t_i \neq t_j, i \neq j \Rightarrow \det A \neq 0$

$\Rightarrow A$ is invertible

$\Rightarrow$ System $Ax = y$ can be solved exactly.

Example: Approx. $f(t) = e^t$ on $[0,1]$ by interpolating with a quadratic polynomial

$$p(t) = a + bt + ct^2$$

Choose 3 points.

$$t_1 = 0 \;, \quad t_2 = 1/2 \;, \quad t_3 = 1$$
$$y_1 = 1 \;; \quad y_2 \approx 1.65 \;; \quad y_3 \approx 2.72$$

Linear system

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1/2 & 1/4 \\ 1 & 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\underset{\sim}{x}} = \underbrace{\begin{pmatrix} 1 \\ 1.65 \\ 2.72 \end{pmatrix}}_{\underset{\sim}{y}}$$

$$\Rightarrow \underset{\sim}{x} = \begin{pmatrix} 1 \\ 0.87 \\ 0.84 \end{pmatrix}$$

$$\Rightarrow p(t) = 1 + 0.87t + 0.84t^2$$