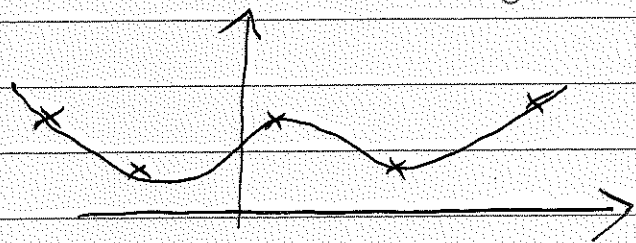
LECTURE 33

Data fitting and least squares: given the collection of points

$(t_1, y_1), \ldots, (t_m, y_m)$



and a family of functions $\{\phi_1(t), \ldots, \phi_m(t)\}$, we would like to compute $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ s.t.

$$\phi = \sum_{i=1}^{m} \alpha_i \phi_i$$

minimizes the error

$$\|\underset{\sim}{e}\| = \left(\sum_{i=1}^{m} (y_i - \phi(t_i))^2 \right)^{1/2}.$$

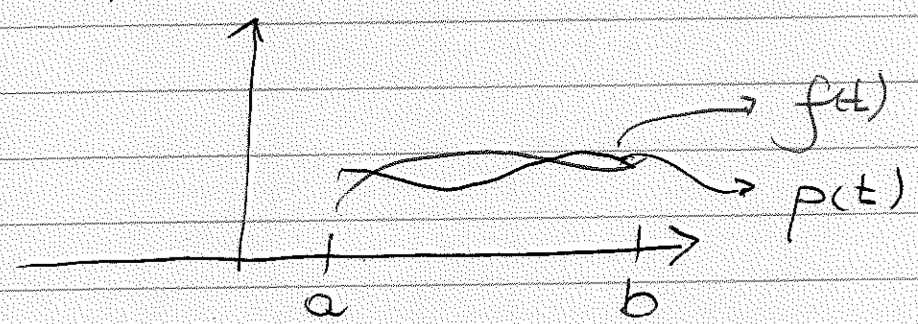
We now consider a different problem: Least Squares approximation in function spaces.

$$V = C^0[a,b], \quad \langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

$$\|f\| = \left(\int_a^b f(x)^2 \, dx \right)^{1/2}$$

Let $\mathbb{P}_m$ the space of polynomials of degree $\leq m$.
We consider the basis $\{1, t, t^2, \ldots, t^m\}$, and we approximate a function $f \in C^0[a,b]$ by a polynomial

$$p(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_m t^m$$



The error function $e(t) = f(t) - p(t)$ measures the discrepancy between f and p at each t.

$$\text{ERROR} = \| e \| = \| p - f \| = \left(\int_a^b (p(t) - f(t))^2 \, dt \right)^{1/2}$$

Now, we compute

$$\| e \|^2 = \| p - f \|^2 = \langle p - f, p - f \rangle$$
$$= \left\langle \sum_{i=1}^{m} \alpha_i t^i - f, \sum_{j=1}^{m} \alpha_j t^j - f \right\rangle$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j \langle t^i, t^j \rangle - 2 \sum_{i=1}^{m} \alpha_i \langle t^i, f \rangle + \| f \|^2$$

Define $\underset{\sim}{x} = (\alpha_1, \dots, \alpha_m)^T$.

$K = (k_{ij})_{i,j=1}^{m}$, where $k_{ij} = \langle t^i, t^j \rangle$.

$\underline{f} = (\langle 1, f \rangle, \langle t, f \rangle, \dots, \langle t^m, f \rangle)^T$

Then, $\|e\|^2 = \underset{\sim}{x}^T K \underset{\sim}{x} - 2x^T f + c$,

where $c = \|f\|^2$

Minimize the error:

$$g(e) = \|e\|^2 = \underset{\sim}{x}^T K \underset{\sim}{x} - 2\underset{\sim}{x}^T \underline{f} + c$$

$$\nabla g(e) = 0 \iff K\underset{\sim}{x} = \underline{f}$$

<u>Example!</u> $f(t) = e^t$. Consider $V = C[0,1]$, and

$$p(t) = \alpha + \beta t + \gamma t^2$$

We need to find α, β, γ s.t. $\|e\|$ is minimized.

$$k_{ij} = \langle t^i, t^j \rangle = \int_0^1 t^i t^j \, dt = \frac{1}{i+j+1}$$

$$f_i = \langle f, t^i \rangle = \int_0^1 e^t t^i$$

Then, the system $K\underline{x} = f$ reads:

$$\begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} e-1 \\ 1 \\ e-2 \end{pmatrix}$$

$$\Rightarrow \quad \alpha \approx 1.01, \ \beta \approx 0.85; \ \gamma \approx 0.83$$

$$\Rightarrow \quad p(t) = 1.01 t^2 + 0.85 t + 0.83,$$

and

$$\|e^t - p(t)\| \simeq 0.005. \quad \blacksquare$$

§5. Orthogonality. V- Vector space and $\langle \cdot, \cdot \rangle$ an inner product

Def (orthogonal basis). A basis $\{\underset{\sim}{u_i}\}_{i=1}^{m}$ is called orthogonal if

$$\langle u_i, u_j \rangle = 0 \quad \forall i,j \in \{1, \dots, m\}, \; i \neq j$$

The basis is called orthonormal if

$$\| u_i \| = 1 \quad \forall i \in \{1, \dots, m\}.$$

Example!

$V = \mathbb{R}^2$: $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}\}$ is an orthonormal basis

$V = \mathbb{R}^3$: $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_3}\}$ is orthonormal basis

Result: if $\{\underset{\sim}{u_i}\}_{i=1}^{m}$ is an orthogonal basis of V. Then $\{v_i\}_{i=1}^{m}$ with $v_i = u_i / \|u_i\|$ is an orthonormal basis of V

Proposition. If $v_1, \dots, v_k \in V$ are nonzero, and mutually orthogonal: $\langle v_i, v_j \rangle = 0 \; \forall i \neq j$, then they are linearly independent.

Proof. Suppose

$$c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + \dots + c_k \underset{\sim}{v_k} = 0.$$

Take the inner product with v_i, then

$$c_1 \underbrace{\cancel{\langle v_1, v_i \rangle}}_{0} + c_2 \underbrace{\cancel{\langle v_2, v_i \rangle}}_{0} + \dots + c_i \langle v_i, v_i \rangle + \dots + c_k \underbrace{\cancel{\langle v_k, v_i \rangle}}_{0} = \underset{\sim}{0}$$

$\Rightarrow c_i \|v_i\|^2 = 0$

$\Rightarrow c_i = 0$ because $v_i \neq 0 \quad \forall i \in \{1, \dots, m\}$

$\Rightarrow \{v_1, \dots, v_k\}$ is l.i. ▨

Example) $V = C[0,1]$ and $\mathcal{P}_2$ = polynomials of degree ≤ 2.

$$\langle p, q \rangle = \int_0^1 p(x)\, q(x)\, dx$$

$$\|p\| = \sqrt{\langle p, p \rangle} = \left(\int_0^1 p(x)^2 dx \right)^{1/2}$$

The set $\{1, x, x^2\}$ is not orthogonal:

$$\langle 1, x \rangle = \int_0^1 x\, dx = \frac{1}{2} \neq 0$$

$$\langle 1, x^2 \rangle = \int_0^1 x^2\, dx = \frac{1}{3} \neq 0$$

$$\langle x, x^2 \rangle = \int_0^1 x^3\, dx = \frac{1}{4} \neq 0$$

But, the set $\{p_1, p_2, p_3\}$, where $p_1(x) = 1$

$p_2(x) = x - 1/2$

$p_3(x) = x^2 - x + 1/6$

is orthogonal. Exercise!