LECTURE 34 | 11/24/14.

§5. Orthogonality.

V - vector space and $\underset{\sim}{u}, \underset{\sim}{v} \in V$. We say $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are orthogonal if

$$\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = 0.$$

Def (orthogonal/orthonormal basis) A basis $\{\underset{\sim}{u_1}, \dots, \underset{\sim}{u_m}\}$ of V is called orthogonal if

$$\langle u_i, u_j \rangle = 0 \quad \forall i, j \in \{1, \dots, m\}, i \neq j$$

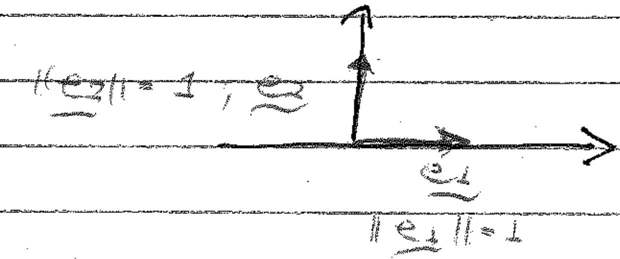
In addition, the basis is called orthonormal if

$$\|u_i\| = 1 \qquad \forall i = 1, \dots, m.$$

Examples.

1) $V = \mathbb{R}^2$, $\langle \cdot, \cdot \rangle$ = dot product

$\{\underline{e_1}, \underline{e_2}\}$ is an orthonormal basis



2) $V = \mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ = dot product

$\{\underline{e_1}, \underline{e_2}, \dots, e_n\}$ is an orthonormal basis.

3) Normal basis. $V = \mathbb{R}^4$, $\langle \cdot, \cdot \rangle$ = dot product.

$$\underset{\sim}{u_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix};\ \underset{\sim}{u_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix};\ \underset{\sim}{u_3} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix};\ \underset{\sim}{u_4} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

The basis $\{\underset{\sim}{u_1}, \underset{\sim}{u_2}, \underset{\sim}{u_3}, \underset{\sim}{u_4}\}$ is orthogonal in $\mathbb{R}^4$.

If $\{v_1, \dots, v_m\}$ is an orthogonal basis of V, then $u_i = v_i / \|v_i\|$ defines a orthonormal basis of V $\{u_1, \dots, u_m\}$.

Proposition. If $v_1, \dots, v_k \in V$ are mutually orthogonal in V and $v_i \neq 0$, then they are l.i.

Proof. Suppose

$$c_1 \underset{\sim}{v_1} + \cdots + c_k \underset{\sim}{v_k} = 0$$

Take $\langle \cdot, \cdot \rangle$ with v_i, then

$$c_1 \langle v_1, v_i \rangle + \cdots + c_i \langle v_i, v_i \rangle + \cdots + c_k \langle v_k, v_i \rangle = 0$$

$$\Rightarrow \quad c_i \langle v_i, v_i \rangle = 0.$$

$$\Rightarrow \quad c_i \|v_i\|^2 = 0.$$

Since $v_i \neq 0 \Rightarrow c_i = 0 \ \forall i \in \{1, \dots, k\}$

$\Rightarrow \{v_1, \dots, v_k\}$ is l.i. ▨

Coordinates.

THM Let $\underset{\sim}{u}_1, \dots, u_m$ be an orthonormal basis for V. Then, $\underset{\sim}{v} \in V$ can be written as:

$$\underset{\sim}{v} = c_1 u_1 + \cdots + c_m \underset{\sim}{u}_m$$

where $c_i = \langle \underset{\sim}{v}, \underset{\sim}{u}_i \rangle$. Moreover.

$$\| \underset{\sim}{v} \| = \sqrt{c_1^2 + \cdots + c_m^2} = \left(\sum_{i=1}^{m} \langle \underset{\sim}{v}, u_i \rangle^2 \right)^{1/2}$$

Proof. Compute

$$\langle \underset{\sim}{v}, \underset{\sim}{u}_i \rangle = \langle \sum_{i=1}^{m} c_j \underset{\sim}{u}_j, \underset{\sim}{u}_i \rangle$$

$$= \sum_{i=1}^{m} c_j \langle \underset{\sim}{u}_j, \underset{\sim}{u}_i \rangle$$

$$= c_i \| \underset{\sim}{u}_i \|^2 = c_i$$

$$\Rightarrow \quad c_i = \langle \underset{\sim}{v}, \underset{\sim}{u}_i \rangle$$

Finally,

$$\| \underset{\sim}{v} \|^2 = \langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = \langle \sum_{j=1}^{m} c_j u_j, \sum_{i=1}^{m} c_i u_i \rangle$$

$$= \sum_{i=1}^{m} c_i^2 \| u_i \|^2 = \sum_{i=1}^{m} c_i^2 \qquad \blacksquare$$

If $\{u_1, \dots, u_m\}$ is orthogonal, then

$$\underset{\sim}{v} = c_1 \underset{\sim}{u_1} + \cdots + c_m \underset{\sim}{u_m}$$

where,

$$c_i = \frac{\langle \underset{\sim}{v}, u_i \rangle}{\|u_i\|^2}$$

The norm,

$$\|\underset{\sim}{v}\|^2 = \langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = \sum_{i=1}^{m} c_i^2 \|u_i\|^2$$

$$= \sum \left(\frac{\langle \underset{\sim}{v}, u_i \rangle}{\|u_i\|} \right)^2$$

Example. $V = \mathbb{R}^4$, $\langle \cdot, \cdot \rangle$ = dot product. Let's consider the wavelet basis

$$\underset{\sim}{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \quad \underset{\sim}{v_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}; \quad \underset{\sim}{v_3} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}; \quad \underset{\sim}{v_4} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \|\underset{\sim}{v_1}\| = 2, \quad \|\underset{\sim}{v_2}\| = 2; \quad \|\underset{\sim}{v_3}\| = \sqrt{2}; \quad \|\underset{\sim}{v_4}\| = \sqrt{2}$$

Now, $\underset{\sim}{v} = \begin{pmatrix} 4 \\ -2 \\ 1 \\ 5 \end{pmatrix}$; $\underset{\sim}{v} = c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + c_3 \underset{\sim}{v_3} + c_4 \underset{\sim}{v_4}$

We compute the coordinates.

$$c_1 = \frac{\langle \underset{\sim}{v}, \underset{\sim}{v_1} \rangle}{\|\underset{\sim}{v_1}\|^2} = 2, \quad c_2 = \frac{\langle \underset{\sim}{v}, \underset{\sim}{v_2} \rangle}{\|\underset{\sim}{v_2}\|^2} = -1$$

$$c_3 = \frac{\langle \underset{\sim}{v}, \underset{\sim}{v_3} \rangle}{\|\underset{\sim}{v_3}\|^2} = 3; \quad c_4 = \frac{\langle \underset{\sim}{v}, \underset{\sim}{v_4} \rangle}{\|\underset{\sim}{v_4}\|} = -2.$$