LECTURE 35. 12/01/14.

Last class review.

§5. Orthogonality.

§5.1 Orthogonal basis. V - vector space, $\langle \cdot , \cdot \rangle$ - inner prod.

Recall that two elements $\underset{\sim}{u}, \underset{\sim}{v} \in V$ are orthogonal if

$$\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = 0$$

Def. A basis $\{ \underline{u_i} \}_{i=1}^{m}$ of V is called orthogonal if

$$\langle \underline{u_i}, \underline{u_j} \rangle = 0 \ \forall i \neq j, \ i, j \in \{1, \dots, m\}.$$

The basis is called orthonormal if, in addition,

$$\| u_i \| = 1 \ \forall i \in \{1, \dots, m\}.$$

Example: $V = \mathbb{P}_2$ and $\langle p, q \rangle = \int_0^1 p(x) q(x) \, dx$.

Then,

$$\{ 1, x - 1/2, x^2 - x + 1/6 \}$$

is an orthonormal basis for $\mathbb{P}_2$.

What are the advantage of orthogonal and orthonormal basis? V - vector space, $\langle \cdot , \cdot \rangle$ inner product on V.

If $\{ v_1, \dots, v_m \}$ is a basis of V, then

$$V \ni v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m.$$

How do we find the coefficients α_i, $i = 1, \dots, m$?

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In general, we need to solve a linear system. But, if $\{ \underset{\sim}{v_1}, \dots, \underset{\sim}{v_m} \}$ is orthonormal, then

$$\alpha_i = \langle \underset{\sim}{v}, \underset{\sim}{v_i} \rangle, \quad i = 1, \cdots, m$$

If $\{ \underset{\sim}{v_1}, \dots, \underset{\sim}{v_m} \}$ is orthogonal, then

$$\alpha_i = \frac{\langle \underset{\sim}{v}, \underset{\sim}{v_i} \rangle}{\| \underset{\sim}{v_i} \|^2}, \quad i = 1, \cdots, m.$$

§ 5.2 Gram - Schmidt process.

How can we construct an orthogonal / orthonormal basis?

V - finite dimensional vector space; $\dim V = m$ and $\langle \cdot, \cdot \rangle$ - inner product on V. Let's consider

$$\{ \underset{\sim}{w_1}, \dots, \underset{\sim}{w_m} \}$$

to be a basis of V.

Goal: construct an orthogonal basis $\{ \underset{\sim}{v_1}, \dots, \underset{\sim}{v_m} \}$

Key idea: $\mathrm{span} \{ w_i \}_{i=1}^{K} = \mathrm{span} \{ v_i \}_{i=1}^{K} \; \forall K \in \{1, \dots, m\}$.

Step 1: Take $\underset{\sim}{v_1} = \underset{\sim}{w_1}$.

Step 2: Construct $\underset{\sim}{v_2} = \underset{\sim}{w_2} - \alpha \underset{\sim}{v_1}$ s.t

$$\langle \underset{\sim}{v_2}, \underset{\sim}{v_1} \rangle = 0.$$

Now,

$$0 = \langle \underset{\sim}{v_2}, \underset{\sim}{v_1} \rangle = \langle \underset{\sim}{w_2} - \alpha \underset{\sim}{v_1}, \underset{\sim}{v_1} \rangle = \langle \underset{\sim}{w_2}, \underset{\sim}{v_1} \rangle - \alpha \langle \underset{\sim}{v_1}, \underset{\sim}{v_1} \rangle$$

$$\Rightarrow \alpha = \frac{\langle \underset{\sim}{w_2}, \underset{\sim}{v_1} \rangle}{\| \underset{\sim}{v_1} \|^2}$$

Then, $\underset{\sim}{v_2} = \underset{\sim}{w_2} - \dfrac{\langle \underset{\sim}{w_2}, \underset{\sim}{v_1} \rangle}{\| \underset{\sim}{v_1} \|^2} \underset{\sim}{v_1}$.

Note that $v_2 \neq 0$ since $\{\overset{w_1}{\overset{\parallel}{v_1}}, w_2\}$ is l.i.

Step 3: $v_3 = w_3 - \alpha_1 v_1 - \alpha_2 v_2$ s.t.
$\langle v_3, v_1 \rangle = 0$ and $\langle v_3, v_2 \rangle = 0$.

$$0 = \langle v_3, v_1 \rangle = \langle w_3, v_1 \rangle - \alpha_1 \| v_1 \|^2 - \alpha_2 \cancelto{0}{\langle v_2, v_1 \rangle}$$

$$\Rightarrow \quad \alpha_1 = \frac{\langle w_3, v_1 \rangle}{\| v_1 \|^2}$$

$$0 = \langle v_3, v_2 \rangle = \langle w_3, v_2 \rangle - \alpha_2 \| v_2 \|^2 - \alpha_1 \cancelto{0}{\langle v_1, v_2 \rangle}$$

$$\Rightarrow \quad \alpha_2 = \frac{\langle w_3, v_2 \rangle}{\| v_2 \|^2}$$

$$\Rightarrow \quad v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\| v_1 \|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\| v_2 \|^2} v_2$$

Step k: Compute $v_k = w_k - \sum_{i=1}^{k-1} \alpha_i v_i$. What are the values of α_i?
Imposing orthogonality, we derive

$$\underset{\substack{\uparrow \\ j<k}}{0} = \langle v_k, v_j \rangle = \langle w_k, v_j \rangle - \alpha_j \| v_j \|^2$$

$$\Rightarrow \quad \alpha_j = \frac{\langle w_k, v_j \rangle}{\| v_j \|^2}, \quad j = 1, \ldots, k-1.$$

Then, we have the Gram-Schmidt formula.

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\| v_j \|^2} v_j, \quad k = 1, \ldots, m.$$

Remark. Gram-Schmidt process defines an orthogonal basis. To derive an orthonormal basis, we define

$$u_k = \frac{v_k}{\|v_k\|}, \quad k = 1, \dots, n.$$

Example: Compute an orthogonal basis $\{q_1, q_2, q_3\}$ of P_2 with respect to

$$\langle p, q\rangle = \int_0^1 p(x)\, q(x)\, dx.$$

Solution: We consider a basis of P_2 : $\{1, x, x^2\}$.

Now, we compute $\{q_1, q_2, q_3\}$ as follows

1. $q_1 = 1$.
2. $q_2 = x - \alpha\, q_1$, where $\alpha = \dfrac{\langle x, 1\rangle}{\|1\|^2}$

Now,

$$\langle x, 1\rangle = \int_0^1 x\, dx = \frac{1}{2}; \quad \|1\|^2 = \int_0^1 dx = 1$$

$$\Rightarrow \quad \alpha = 1/2$$

$$\Rightarrow \quad q_2 = x - 1/2.$$

3. $q_3 = x^2 - \alpha \cdot 1 - \beta\,(x - 1/2)$, where

$$\alpha = \frac{\langle x^2, 1\rangle}{\|1\|^2}; \quad \beta = \frac{\langle x^2, x - 1/2\rangle}{\|x - 1/2\|^2}$$

Now,

$$\langle x^2, 1\rangle = \int_0^1 x^2\, dx = \frac{1}{3}; \quad \langle x^2, x - 1/2\rangle = \int_0^1 x^2 (x - 1/2)\, dx = \frac{1}{12}$$

$$\|x - 1/2\|^2 = \int_0^1 (x - 1/2)^2\, dx = \frac{1}{12}$$

$$\Rightarrow \quad q_3 = x^2 - 1/3 \cdot 1 - (x - 1/2) = x^2 - x + 1/6. \quad \blacksquare$$

§ 5.5. Orthogonal matrices.

Def. A matrix $Q \in \mathbb{R}^{m \times m}$ is orthogonal if
$$Q Q^T = Q^T Q = I_m .$$

Properties.

1. $Q^{-1} = Q^T$

2. $Q = [\, \underline{q}_1 \ \underline{q}_2 \ \cdots \ \underline{q}_m \,]$; $Q^T = \begin{bmatrix} \underline{q}_1^T \\ \underline{q}_2^T \\ \vdots \\ \underline{q}_m^T \end{bmatrix}$

Then, $Q Q^T = [\, \underline{q}_1 \ \underline{q}_2 \cdots \underline{q}_m \,] \begin{bmatrix} \underline{q}_1^T \\ \underline{q}_2^T \\ \vdots \\ \underline{q}_m^T \end{bmatrix} = I_m$

$$\Rightarrow \begin{cases} \underline{q}_i \underline{q}_j^T = 0 & \text{for } i \neq j \ , \ i, j \in \{1, \dots, m\} \\ \underline{q}_i \underline{q}_i^T = 1 & \text{for } i \in \{1, \dots, m\}. \end{cases}$$

$\Rightarrow \{ \underline{q}_i \}_{i=1}^{m}$ is an orthonormal basis of $\mathbb{R}^m$.

3. $\| Q \underline{x} \|^2 = \langle Q\underline{x}, Q\underline{x} \rangle = \langle Q^T Q \underline{x}, \underline{x} \rangle$
$= \langle \underline{x}, \underline{x} \rangle = \| \underline{x} \|^2$

$\Rightarrow \| Q \underline{x} \| = \| \underline{x} \|.$

4. $1 = \det(I_m) = \det(Q Q^T) = \det(Q) \cdot \det(Q^T)$
$= \det(Q)^2$

$\Rightarrow \det(Q) = \pm 1 .$