(1)

Review. 12/03/14.

Problem 1. (a) Find an orthogonal basis $\{q_i\}$ of the subspace V of $\mathbb{R}^3$ spanned by

$$\underset{\sim}{x_1} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \underset{\sim}{x_2} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}; \quad \underset{\sim}{x_3} = \begin{pmatrix} 0 \\ -8 \\ -2 \end{pmatrix}$$

(b) Determine the dimension of V, along with a basis of V in terms of $\{x_i\}_{i=1}^{3}$.

Solution: (a) We apply Gram-Schmidt orthogonalization

(i) $\underset{\sim}{q_1} = \underset{\sim}{x_1} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

(ii) $\underset{\sim}{q_2} = \underset{\sim}{x_2} - \alpha \underset{\sim}{q_1} \Rightarrow \langle \underset{\sim}{q_2}, \underset{\sim}{q_1} \rangle = 0 = \langle \underset{\sim}{x_2}, \underset{\sim}{q_1} \rangle - \alpha \langle \underset{\sim}{q_1}, \underset{\sim}{q_1} \rangle$

$$\Rightarrow \quad \alpha = \frac{\langle \underset{\sim}{x_2}, \underset{\sim}{q_1} \rangle}{\langle \underset{\sim}{q_1}, \underset{\sim}{q_1} \rangle} = \frac{1+5+6}{1+1+4} = \frac{12}{6} = 2.$$

$$\Rightarrow \underset{\sim}{q_2} = \underset{\sim}{x_2} - \alpha \underset{\sim}{q_1} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$$

(iii) $\underset{\sim}{q_3} = \underset{\sim}{x_3} - \alpha \underset{\sim}{q_1} - \beta \underset{\sim}{q_2}$

$\Rightarrow \langle \underset{\sim}{q_3}, \underset{\sim}{q_1} \rangle = 0 = \langle \underset{\sim}{x_3}, \underset{\sim}{q_1} \rangle - \alpha \langle \underset{\sim}{q_1}, \underset{\sim}{q_1} \rangle$

$$\Rightarrow \alpha = \frac{\langle \underset{\sim}{x_3}, \underset{\sim}{q_1} \rangle}{\langle \underset{\sim}{q_1}, \underset{\sim}{q_1} \rangle} = \frac{0-8-4}{6} = \frac{-12}{6} = -2.$$

$$\langle \underline{q}_3, \underline{q}_2 \rangle = 0 = \langle \underline{x}_3, \underline{q}_2 \rangle - \beta \langle \underline{q}_2, \underline{q}_2 \rangle$$

$$\Rightarrow \beta = \frac{\langle x_3, \underline{q}_2 \rangle}{\langle \underline{q}_2, \underline{q}_2 \rangle} = \frac{-22}{11} = -2$$

$$\Rightarrow \underline{q}_3 = \begin{pmatrix} 0 \\ -8 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow V = \text{Span}\{\underline{q}_1, \underline{q}_2\} = \text{Span}\{(1,1,2)(-1,3,-1)^T\}.$$

(b) $\dim V = 2$, $V = \text{Span}\{\underline{x}_1, \underline{x}_2\} = \text{Span}\{(1,1,2)^T, (1,5,3)^T\}$.

Problem 2. (a) Prove that the polynomials

$$p_0(t) = 1, \quad p_1(t) = t - \frac{2}{3}$$

form an orthogonal basis of the space of polynomials of degree ≤ 1 with respect to $\langle f, g \rangle = \int_0^1 f(t) g(t)\, t\, dt$.

(b) Find the closest polynomial $p(t) \in \mathbb{P}_1$ to the function $f(t) = t^2$ with respect to the inner product of (a).

Solution (a) We compute

$$\langle p_0, p_1 \rangle = \int_0^1 1 \cdot \left(t - \frac{2}{3}\right) t\, dt = \frac{1}{3} - \frac{\cancel{2}}{3}\frac{1}{\cancel{2}} = 0.$$

Since p_0, p_1 are elements of $\mathbb{P}_1$ and they are orthogonal $\Rightarrow \{p_0, p_1\}$ is linearly independent. Moreover $\dim \mathbb{P}_1 = 2$

$$\Rightarrow \mathbb{P}_1 = \text{Span}\{p_0(t), p_1(t)\}.$$

(b) The closest polynomial $p(t) \in \mathbb{P}_1$ to $f(t) = t^2$ can be written as

$$p(t) = \alpha_0 p_0(t) + \alpha_1 p_1(t)$$

To find α_0 and α_1, we impose

$$\langle f - p, p_0 \rangle = 0.$$
$$\langle f - p, p_1 \rangle = 0.$$

Then

$$\langle p, p_0 \rangle = \langle f, p_0 \rangle \Rightarrow \alpha_0 \langle p_0, p_0 \rangle = \langle f, p_0 \rangle$$
$$\langle p, p_1 \rangle = \langle f, p_1 \rangle \Rightarrow \alpha_1 \langle p_1, p_1 \rangle = \langle f, p_1 \rangle$$

$$\Rightarrow \alpha_0 = \frac{\langle f, p_0 \rangle}{\langle p_0, p_0 \rangle} = \frac{\int_0^1 t^2 \cdot 1 \cdot t \, dt}{\int_0^1 t \cdot 1 \cdot 1 \, dt} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$\alpha_1 = \frac{\langle f, p_1 \rangle}{\langle p_1, p_1 \rangle} = \frac{\int_0^1 t^2 (t - 2/3)\, t \, dt}{\int_0^1 \underbrace{(t-2/3)(t-2/3)}_{t^2 - \frac{4}{3}t + \frac{4}{9}} t \, dt} = \frac{1/5 - 2/3 \cdot 1/4}{1/4 - 4/3 \cdot 1/3 + 4/9 \cdot \frac{1}{2}} = \frac{6}{5}$$

$$\Rightarrow p(t) = \frac{1}{2} p_0(t) + \frac{6}{5} p_1(t) = \frac{1}{2} + \frac{6}{5}(t - 2/3)$$

$$= -\frac{3}{10} + \frac{6}{5} t.$$

Problem 3. Let $P \in \mathbb{R}^{m \times m}$ be non-singular and $D = \mathrm{diag}(d_1, \ldots, d_m)$, s.t $d_i > 0$, $i = 1, \ldots, m$. Prove that

$$A = P D P^T \text{ is SPD.}$$

Solution. First, A is symmetric $\Leftrightarrow A = A^T$.

In fact, $A^T = (P D P^T)^T = P D^T P^T \underset{\substack{\uparrow \\ D = D^T}}{=} P D P^T$

Now, $\underset{\sim}{x}^T A \underset{\sim}{x} = \underset{\sim}{x}^T P D P^T \underset{\sim}{x}$

$$= \underbrace{(P^T \underset{\sim}{x})^T}_{y} D \, P^T \underset{\sim}{x}$$

$$= \underset{\sim}{y}^T D \underset{\sim}{y} = \sum_{i=1}^{m} d_i y_i^2 > 0 \text{ if } \underset{\sim}{y} \neq 0$$

Alternatively, $\underset{\sim}{x}^T A \underset{\sim}{x} = \underset{\sim}{x}^T P D P^T \underset{\sim}{x}$

$$= \underset{\sim}{x}^T P D^{1/2} D^{1/2} P^T \underset{\sim}{x}$$

$$= (D^{1/2} P^T \underset{\sim}{x})^T (D^{1/2} P^T \underset{\sim}{x})$$

$$= \| D^{1/2} P^T \underset{\sim}{x} \|^2 > 0 \text{ if } \underset{\sim}{x} \neq 0$$

Example. $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 5 & 2 \\ 2 & 2 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = U$

$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ pivots of U are all positive $\Rightarrow A$ is SPD.

5

$$A = LDL^T \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$D^{1/2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = CC^T \; ; \; C = LD^{1/2}$$