LECTURE 4 | 09/10/14 (1)

Last class: LU decomposition and solving linear systems using LU decomposition.

$$A\underset{\sim}{x} = \underset{\sim}{b} \iff LU\underset{\sim}{x} = \underset{\sim}{b}$$
$$\iff L\underset{\sim}{y} = b \; ; \quad U\underset{\sim}{x} = y$$

U - upper triangular matrix
L - lower triangular matrix with 1's on the diagonal.

Today: Forward Substitution to solve

$$L\underset{\sim}{y} = \underset{\sim}{b}$$

$$\begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & l_{32} & 1 & & \\ \vdots & & & & \\ l_{m1} & l_{m2} & \cdots & & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix}$$

How do we solve this system?

$$\Rightarrow \quad y_1 = b_1$$
$$\Rightarrow \quad y_2 = b_2 - l_{21} y_1$$
$$\Rightarrow \quad y_3 = b_3 - l_{31} y_1 - l_{32} y_2$$

In general, $y_i = b_i - \sum_{j=1}^{i-1} l_{ij} y_j$

for $j = 1, \dots, n$.

Algorithm (Matlab).

```
for i = 1 : n
    y(i) = b(i);
    for j = 1 : i-1
        y(i) = y(i) - L(i,j) · y(j);
    end
end
```

Backward Substitution to solve

$$U\underset{\sim}{x} = \underset{\sim}{y}$$

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ & u_{22} & u_{23} & \cdots & u_{2n} \\ & & & \ddots & \vdots \\ & & & u_{n-1,n-1} & u_{n-1,n} \\ & & & & u_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ y_n \end{pmatrix}$$

$$\Rightarrow x_n = 1/u_{nn}\, y_n$$
$$\Rightarrow x_{n-1} = 1/u_{n-1,n-1}\,(y_n - u_{n-1,n}\, x_n)$$

In general $x_n = y_n / u_{nn}$

$$x_i = \frac{1}{u_{ii}}\left(y_i - \sum_{j=i+1}^{n} u_{ij}\, x_j\right), \quad i = n-1, \dots 1$$

Algorithm (Matlab).

```
for i = n : -1 : 1      ~> n, n-1, n-2, ... 1.
    x(i) = y(i);
    for j = i+1 : n
        x(i) = x(i) - U(i,j) x(j);
    end
    x(i) = x(i) / U(i,i);
end.
```

Computational cost.

How many operations are required to complete an algorithm?

1) $x^T y$. $\rightsquigarrow$ n multiplications and n sums
   $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ $\approx n$ operations!

2) $A \in \mathbb{R}^{n \times n}$, and $b \in \mathbb{R}^n$

$Ax \rightsquigarrow$ n^2 multiplications
n^2 sums
$\approx n^2$ operations!

3) $A^{-1} = \frac{1}{\det A}$ (adjoint of A)

$= \frac{1}{\det A}$ (Cofactor of $A)^T$

Computing det A

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\det A = a_{11}a_{22} - a_{12}a_{21}$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

det A involves $\sim n(n-1)\cdots 1 = n!$

Computing A^{-1} via the formula above involves $n!$ operations.

Suppose we have a fast computer that computes 10^{12} # oper / sec.

Take $n = 30$, then

$$\text{time} \simeq \frac{30! \text{ \# oper}}{10^{12} \text{ \# oper/sec}} = 8.4 \times 10^{12} \text{ years!}$$ ∎