**LECTURE 5** 09/12/14.

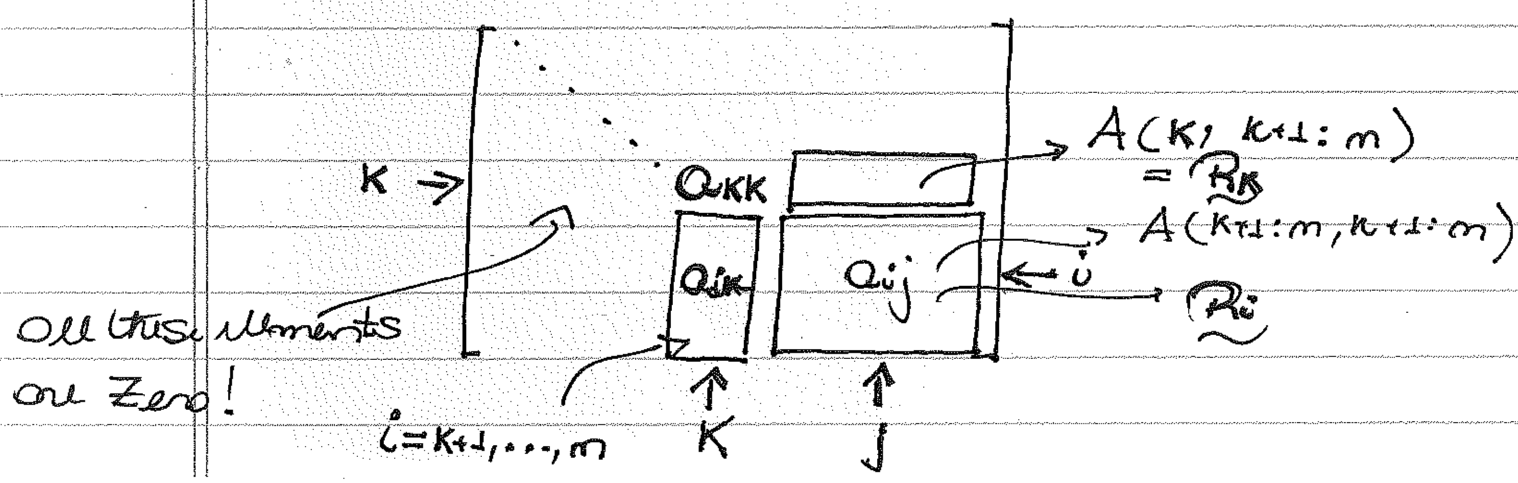
LU decomposition: Main idea

$$A = \begin{pmatrix} \overset{\neq 0}{x} & x & x \\ x & x & x \\ x & x & x \end{pmatrix} \to \begin{pmatrix} x & x & x \\ 0 & \underset{\neq 0}{x} & x \\ 0 & x & x \end{pmatrix} \to \underbrace{\begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{pmatrix}}_{U\text{ - upper triangular matrix.}}$$

General case:

$$\text{Let } A = (a_{ij})_{i,j=1}^{m} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix}$$

We want to compute LU decomposition of $A$.
Suppose we have performed $k-1$ steps for $k \geq 1$ and want to do $k^{th}$ step:



Assume $A$ is regular $\Rightarrow a_{kk} \neq 0$.

1) Choose $a_{kk}$ as the $k^{th}$ pivot and keep row $k$.
2) Def $l_{ik} := \dfrac{a_{ik}}{a_{kk}}$

$$a_{ik} \leftarrow a_{ik} - l_{ik} a_{kk}$$
$$= a_{ik} - \frac{a_{ik}}{\cancel{a_{kk}}} \cancel{a_{kk}} = 0.$$

Think in rows: $\underset{\sim}{R_i} = \underset{\sim}{R_i} - l_{ik} \underset{\sim}{R_k}$, $i = k+1, \dots, m$

3) Update lower $(m-k) \times (m-k)$ block
$$a_{ij} \leftarrow a_{ij} - l_{ik} a_{kj}$$

Algorithm (Matlab) $A \in \mathbb{R}^{m \times m}$, LU decomp.
$A$ is regular.

```
for k = 1, ..., m-1
    for i = k+1:m
        A(i,k) = A(i,k) / A(k,k) ;
        for j = k+1:m
            A(i,j) = A(i,j) - A(i,k)*A(k,j) ;
        end
    end
end
```

Computational cost

→ How many operations are required to compute the algorithm: multiplications / additions.

j-loop: # operations

$$\approx 2(m-(k+1)+1)$$
$$= 2(m-k).$$

i-loop: # operations.

$$\approx \sum_{i=k+1}^{m} [2(m-k)+1] = (2(m-k)+1)(m-k)$$
$$= 2(m-k)^2 + (m-k)$$
$$\underset{\uparrow}{\approx} 2(m-k)^2$$

m big!

k-loop: # operations.

$$\approx \sum_{k=1}^{m-1} 2(m-k)^2 \underset{\uparrow}{=} \sum_{l=1}^{m-1} 2l^2 \approx \frac{2}{3}m^3.$$

$l = m-k$; $k=1 \Rightarrow l = m-1$
$k = m-1 \Rightarrow l = 1$

Alternatively:

$$\# \text{ operations} \approx \int_0^m \int_k^m \int_k^m 2 \, dj \, di \, dk$$
$$= \frac{2}{3} m^3$$

In fact $\int_0^m \int_k^m \int_k^m 2 \, dj \, di \, dk = \int_0^m \int_k^m 2(m-k) \, di \, dk$
$$= \int_0^m 2(m-k)^2 \, dk$$
$$= -\frac{2}{3}(m-k)^3 \Big|_0^m = \frac{2}{3} m^3.$$

So, is LU a $\underbrace{\text{useful}}_{\text{efficient}}$ method to solve linear system?

$$A^{-1} = \frac{1}{\det A} (\text{Cof} A)^T \rightarrow \# \text{ oper} \approx m!$$

$$A = LU \rightarrow \frac{2}{3} m^3$$

Take $m = 30$ and a computer making $10^{12}$ #oper / sec

$$A^{-1} \rightarrow \frac{30! \, \# \text{oper}}{10^{12} \, \# \text{oper/sec}} \approx 8.4 \times 10^{12} \text{ years.}$$

$$LU \rightarrow \frac{2/3 \, (30)^3 \, \# \text{oper}}{10^{12} \, \# \text{oper/sec}} \approx 2 \times 10^{-8} \text{ sec.}$$