LECTURE 6 | 09/15/14.

Last class: LU decomposition (general case) and its computational cost.

Today: uniqueness of LU decomposition.

Let $A \in \mathbb{R}^{m\times m}$, we recall that $X \in \mathbb{R}^{m\times m}$ is called its inverse if

$$AX = XA = I_m.$$

Notation: $X = A^{-1}$.

Prop 1. If $L \in \mathbb{R}^{m\times m}$ is lower triangular with $l_{ii} \neq 0, i = 1, \dots, m$. Then L^{-1} exists and is also lower triangular.

In addition if $l_{ii} = 1 \Rightarrow l_{ii}^{-1} = 1$.

Prop 2 (Uniqueness of LU decomposition). If $A = LU$ is regular then L and U are uniquely determined.

Proof. Suppose $LU = \tilde{L}\tilde{U}$. Since $l_{ii} \neq 0$
$\Rightarrow \tilde{l}_{ii} \neq 0 \Rightarrow L$ and $\tilde{L}$ are invertible. (Prop 1). Then

$$\tilde{L}^{-1}LU = \tilde{L}^{-1}\tilde{L}\tilde{U} = \tilde{U}$$
$$\Rightarrow \tilde{L}^{-1}L\underbrace{UU^{-1}}_{I_m} = \tilde{U}\cdot U^{-1} = I_m$$

$u_{ii} \neq 0$ since A is regular $\Rightarrow u_{ii}^{-1} \neq 0$, $i = 1, \dots, n$

$$\Rightarrow \underbrace{\tilde{L}^{-1} L}_{\text{lower triangular}} = \underbrace{\tilde{U} \cdot U^{-1}}_{\text{upper triangular}}$$

$$\Rightarrow \tilde{L}^{-1} L = \tilde{U} \cdot U^{-1} = D \text{ - diagonal matrix}$$

But $l_{ii} = 1$ and $\tilde{l}_{ii}^{-1} = 1 \Rightarrow D = I_n$

$$\Rightarrow \tilde{L}^{-1} L = \tilde{U} \cdot U^{-1} = I_n$$

$$\Rightarrow \tilde{L} = L \text{ and } \tilde{U} = U \quad \blacksquare$$

Note: If A is regular, LU works perfectly! Yes, in theory!

Example. Let's consider

$$A = \begin{pmatrix} 10^{-20} & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 10^{-20} & 1 \\ 0 & 1 - 10^{20} \end{pmatrix}}_{U}.$$

Then, $$A = \begin{pmatrix} 1 & 0 \\ 10^{20} & 1 \end{pmatrix} \begin{pmatrix} 10^{-20} & 1 \\ 0 & 1 - 10^{20} \end{pmatrix}$$

Suppose these computations are performed in floating point arithmetic with $\varepsilon_{machine} \approx 10^{-16}$

$$\Rightarrow \hat{L} = \begin{pmatrix} 1 & 0 \\ 10^{20} & 1 \end{pmatrix} \text{ and } \tilde{U} = \begin{pmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{pmatrix}$$

$$\Rightarrow \tilde{L}\tilde{U} = \begin{pmatrix} 10^{-20} & 1 \\ 1 & 0 \end{pmatrix}$$

Now, consider $A\underset{\sim}{x} = \underset{\sim}{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Computer solution: $\tilde{L}\tilde{U}\underset{\sim}{x} = \underset{\sim}{b}$

$$\Rightarrow \underset{\sim}{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In fact, $\begin{pmatrix} 10^{-20} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Exact solution: $A\underset{\sim}{x} = \underset{\sim}{b} \Rightarrow \underset{\sim}{x} \approx \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

In fact, $\begin{pmatrix} 10^{-20} & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -10^{-20}+1 \\ 0 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ §1.4 Pivoting and Permutations.

Def (Permutation matrix) A permutation matrix is a matrix obtained from the identity by any combination of row interchanges.

Example.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad PA = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}$$