LECTURE 7 | 09/16/14.

We continue with §1.4 Pivoting and Permutations.

Consider
$$\begin{cases} + 2y + z = 2 \\ 2x + 6y + z = 7 \\ x + y + 4z = 3 \end{cases}$$

first pivot $= 0 \Rightarrow$ LU does not apply!

Recall. Linear system operation #1. Add a multiple of one eq. to another eq.

Now, we introduce
Linear system operation #2. Interchange two equations.

$$M = (A|b) = \left(\begin{array}{ccc|c} 0 & 2 & 1 & 2 \\ 2 & 6 & 1 & 7 \\ 1 & 1 & 4 & 3 \end{array}\right) \xrightarrow[\text{oper \#2.}]{} \underbrace{\left(\begin{array}{ccc|c} 2 & 6 & 1 & 7 \\ 0 & 2 & 1 & 2 \\ 1 & 1 & 4 & 3 \end{array}\right)}_{\tilde{A}_1}$$

In matrices, $P_1 A = \tilde{A}_1$, where $P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Second Step:

(2)

$$\xrightarrow[\text{Opu \#1}]{-\frac{1}{2}R_1+R_3} \underbrace{\left(\begin{array}{ccc|c} 2 & 6 & 1 & 7 \\ 0 & 2 & 1 & 2 \\ 0 & -2 & +\frac{7}{2} & -\frac{1}{2} \end{array}\right)}_{A_1}$$

In matrices: $A_1 = E_1 \tilde{A}_1$, where $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}$

$= E_1 P_1 A$

$\Rightarrow \boxed{L_1 A_1 = P_1 A}$

Recall $E_1^{-1} = L_1$

Next Step:

$$\xrightarrow[\text{Opu \#1}]{+R_2+R_3} \underbrace{\left(\begin{array}{ccc|c} 2 & 6 & 1 & 7 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 9/2 & 3/2 \end{array}\right)}_{A_2}$$

In matrices: $A_2 = E_2 A_1$, where $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

Then, $U \underset{\text{definition}}{:=} A_2 = E_2 A_1$

$\Rightarrow L_2 U = A_1$

$L_2 = E_2^{-1}$

$\Rightarrow L_1 L_2 U = L_1 A_1$

$= P_1 A.$

Define $L = L_1 L_2$, $U = A_2$, $P = P_1$

$\Rightarrow \boxed{LU = PA}$

Permuted LU decomposition

Def (nonsingular matrix) $A \in \mathbb{R}^{n \times n}$ is nonsingular if it can be reduced to an upper triangular matrix $U \in \mathbb{R}^{n \times n}$ s.t $u_{ii} \neq 0$ with operations of Type 1 and 2.

Example $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix}$ is nonsingular.

THM. Let $A \in \mathbb{R}^{n \times n}$. Then, the following conditions are equivalent.

i) A is nonsingular

ii) A has n nonzero pivots.

iii) A admits a permuted LU factorization

$$\boxed{PA = LU}$$

For numerical stability it is desirable to pivot even when $x_{kk} \neq 0$ if there is a larger element available.

Example. $A = \begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix}$

$|a_{11}| > |a_{21}|$; $|a_{11}| > |a_{31}|$

Example 1

$$A = \begin{pmatrix} 5 & -1 & 5 \\ -3 & 2 & 6 \\ 10 & -7 & 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{P_1}\underbrace{\begin{pmatrix} 5 & -1 & 5 \\ -3 & 2 & 6 \\ 10 & -7 & 0 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix}}_{\tilde{A}_1}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3/10 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix}}_{E_1}\underbrace{\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix}}_{\tilde{A}_1} = \underbrace{\begin{pmatrix} 10 & -7 & 0 \\ 0 & -1/10 & 6 \\ 0 & 5/2 & 5 \end{pmatrix}}_{A_1}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{P_2}\underbrace{\begin{pmatrix} 10 & -7 & 0 \\ 0 & -1/10 & 6 \\ 0 & 5/2 & 5 \end{pmatrix}}_{A_1} = \underbrace{\begin{pmatrix} 10 & -7 & 0 \\ 0 & 5/2 & 5 \\ 0 & -1/10 & 6 \end{pmatrix}}_{\tilde{A}_2}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/25 & 1 \end{pmatrix}}_{E_2}\underbrace{\begin{pmatrix} 10 & -7 & 0 \\ 0 & 5/2 & 5 \\ 0 & -1/10 & 6 \end{pmatrix}}_{\tilde{A}_2} = \underbrace{\begin{pmatrix} 10 & -7 & 0 \\ 0 & 5/2 & 5 \\ 0 & 0 & 31/5 \end{pmatrix}}_{A_2 = U}$$

In matrices:

$$\tilde{A}_1 = P_1 A$$
$$A_1 = E_1 \tilde{A}_1$$
$$\Rightarrow E_1^{-1} A_1 = \tilde{A}_1$$
$$L_1 A_1 = \tilde{A}_1 = P_1 A$$

$$\Rightarrow \boxed{L_1 A_1 = P_1 \cdot A} \qquad (1)$$

$$\tilde{A}_2 = P_2 A_1$$

$$A_2 = \cancel{E_2} \tilde{A}_2 \Rightarrow L_2 A_2 = \tilde{A}_2 = P_2 A_1$$

$$\Rightarrow \boxed{L_2 A_2 = P_2 A_1} \quad (2)$$

Multiply (1) by P_2

$$P_2 P_1 A = P_2 L_1 A_1$$

$$= P_2 L_1 P_2^{-1} \underbrace{P_2 A_1}_{L_2 A_2}$$

$$= P_2 L_1 P_2^{-1} L_2 A_2$$

$$= L_1' L_2 A_2 = L_1' L_2 U$$

$P_2 L_1 P_2^{-1} = L_1' \rightarrow$ lower triangular matrix

$$L := L_1' L_2$$

$$P := P_2 P_1$$

$$U := A_2$$

$$\Rightarrow \boxed{PA = L \cdot U}$$

Compute $P_2 L_1 P_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$

$$P_2 L_1 P_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3/10 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/10 & 1 & 0 \end{pmatrix} P_2^{-1}$$

$$P_2 L_1 P_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/10 & 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/10 & 0 & 1 \end{pmatrix} \quad \blacksquare$$