LECTURE 8 | 09/19/14

Last class: Permuted LU decomposition

Solving linear systems with $PA = LU$

$$A\underset{\sim}{x} = \underset{\sim}{b} \quad \text{and} \quad PA = LU$$

Then, $A\underset{\sim}{x} = \underset{\sim}{b} \iff PA\underset{\sim}{x} = LU\underset{\sim}{x} = \underbrace{P\underset{\sim}{b}}_{\underset{\sim}{c}}$

$\Rightarrow$ Solving $U\underset{\sim}{x} = y$ and $Ly = \underset{\sim}{c}$,
which can be solved via backward and forward substitution.

§1.5 Matrix Inverses

Def (A^{-1}) Let $A \in \mathbb{R}^{n\times n}$. $X \in \mathbb{R}^{n\times n}$ is called the inverse of A if

$$A \cdot X = X \cdot A = I_n,$$

where I_n is the identity matrix.

Notation: $X = A^{-1}$

Inverses of rectangular matrices. Let $A \in \mathbb{R}^{m\times n}$

$$\overset{n}{\underset{A\in\mathbb{R}^{m\times n}}{\underbrace{m\ \boxed{\ A\ }}}}\ \underset{X\in\mathbb{R}^{n\times m}}{\underbrace{\boxed{\ X\ }}} = I_m$$

X is called the left inverse of A.

In a similar way you can define the right inverse of $A \in \mathbb{R}^{m\times n}$.

Properties (matrix inverses) Let $A, B \in \mathbb{R}^{n\times n}$. Then

i) The inverse is unique

ii) $(A^{-1})^{-1} = I_n$

iii) $(AB)^{-1} = B^{-1}A^{-1}$

Proof. i) Suppose exist $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times n}$ s.t. $X\cdot A = I_n$ and $A\cdot Y = I_n$

$$\Rightarrow X = X\cdot I_n = X\cdot A\cdot Y = (X\cdot A)\cdot Y = I_n\cdot Y = Y$$

$\Rightarrow X = Y.$

$\Rightarrow$ the inverse of A is unique.

ii) A^{-1} satisfies

$$A \cdot A^{-1} = A^{-1} \cdot A = I_n$$

$$\Rightarrow (A^{-1})^{-1} = A.$$

iii) Let $X = B^{-1}A^{-1}$, then

$$X(AB) = B^{-1}\underbrace{A^{-1} \cdot A}_{I_n} \cdot B = B^{-1}B = I_n$$

$$(AB)X = A\underbrace{B \cdot B^{-1}}_{I_n}A^{-1} = \underbrace{A \cdot A^{-1}}_{I_n} = I_n.$$

∎

In general property (iii) can be extended to

$$(A_1 \cdots A_k)^{-1} = A_k^{-1} \cdots A_1^{-1}.$$

THM. $A \in \mathbb{R}^{n \times n}$ has an inverse $\Leftrightarrow$ it is non singular.

LEMMA. $A \in \mathbb{R}^{n \times n}$ is invertible $\Leftrightarrow$ $A\underset{\sim}{x} = \underset{\sim}{b}$ has a unique solution for any $b \in \mathbb{R}^n$.

Proof ($\Rightarrow$) A is invertible, then

$$A\underset{\sim}{x} = \underset{\sim}{b} \quad / A^{-1}$$

$$A^{-1}A\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$$

$$\underset{\sim}{x} = A^{-1}\underset{\sim}{b} \rightarrow \text{unique solution.}$$

$(\Leftarrow)$ Consider $A \underset{\sim}{x_i} = \underset{\sim}{e_i}$, where $\underset{\sim}{e_i} = \begin{array}{r} \\ i\text{th entry} \rightarrow \\ \\ \end{array} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$

$\underset{\sim}{x_i}$ exists and is unique.
Since $A\underset{\sim}{x} = \underset{\sim}{b}$ has a unique solution.

Let $B = [\underset{\sim}{x_1}\ \underset{\sim}{x_2} \cdots \underset{\sim}{x_n}] \in \mathbb{R}^{n\times n}$, then

$$
\begin{aligned}
AB &= A\,[\underset{\sim}{x_1}\ \ \underset{\sim}{x_2}\ \cdots\ \underset{\sim}{x_n}] \\
&= [A\underset{\sim}{x_1}\ \ A\underset{\sim}{x_2}\ \cdots\ A\underset{\sim}{x_n}] \\
&= [\underset{\sim}{e_1}\ \ \underset{\sim}{e_2}\ \cdots\ \underset{\sim}{e_n}] = I_n
\end{aligned}
$$

$\Rightarrow$ B is the inverse of A $\blacksquare$

Gauss-Jordan elimination.

Based on the last proof we have a technique to find A^{-1} based on LU decomposition

$$
\left[\ \underbrace{A}_{A \in \mathbb{R}^{n\times n}} \ \middle|\ I_n \ \right] \xrightarrow[\text{row operation}]{\text{elementary}} \left[\ I_n \ \middle|\ X \ \right]
$$

$$X = A^{-1}.$$

Please check example 1.23 in the book.

LDV factorization

Example $A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 5 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 8 \end{pmatrix}}_{U}$

$D := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ then $U = D \cdot \underbrace{D^{-1} U}_{V}$

$= D \cdot V$

Then, if $U = DV \Rightarrow V = D^{-1} U$

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow A = L D V$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1/2 & \\ & & 1/8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

THM. (LDV decomposition) Let $A \in \mathbb{R}^{n\times n}$ be a regular matrix, then it admits a LDV decomposition.

Proof. Since A is regular, we have $A = LU$ with $u_{ii} \neq 0 \ \forall i = 1, \dots, n$.

Let $D = \text{diag}(u_{ii})$ and $V = D^{-1}U$

Then, $A = LU = LDV$ ▨

Remark. Since LU decomposition is unique $\rightarrow$ LDV decomposition is unique as well.