LECTURE 9 | 09/22/14.

§1.6 Special Matrices.

§1.6.1 Symmetric matrices

Def (Transpose of a matrix). We define the transpose $A^T \in \mathbb{R}^{n\times m}$ of $A \in \mathbb{R}^{m\times n}$ as

$$A^T := (b_{ij})_{i,j}$$

where $b_{ij} = a_{ji}$, $1 \le i \le n$ and $1 \le j \le m$

Example.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 3} \Rightarrow A^T = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 1 \end{pmatrix}$$

Def (Symmetric matrix). A matrix $A \in \mathbb{R}^{n\times n}$ is symmetric if $A = A^T$, i.e.,

$$a_{ij} = a_{ji} \quad \forall i = 1, \dots, n.$$

Example.

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathbb{R}^{3\times 3} \Rightarrow A^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

Then, A is symmetric if $b = d$; $c = g$; and $h = f$.

General form in $n = 3$: $A = \begin{pmatrix} a & b & c \\ b & e & h \\ c & h & i \end{pmatrix}$, $a_{ij} \in \mathbb{R}$.

Properties Let $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times n}$. Then,

(1) $(A^T)^T = A$.

(2) $(A+B)^T = A^T + B^T$; $(\alpha A)^T = \alpha A^T$.

(3) $(AB)^T = B^T A^T$.

(4) $(A^{-1})^T = (A^T)^{-1}$ if A is invertible

Proof (1), (2) and (4) $\rightarrow$ Exercise!

(3) Let $C = AB$ and $D = (AB)^T$. Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$d_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki} = \sum_{k=1}^{n} b_{ki} a_{jk}$$

$$= \sum_{k=1}^{n} b^T_{ik} a^T_{kj} = (B^T A^T)_{ij}$$

$\blacksquare$

Def (Skew-Symmetric matrix). Let $A \in \mathbb{R}^{n\times n}$. A is skew symmetric if $A = -A^T$, i.e.,

$$a_{ij} = -a_{ji} \quad \forall i,j = 1,\ldots,n.$$

Example

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow -A^T = \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix}$$

Then, A is skew-symmetric if and only if

$$a = d = 0 \text{ and } c = -b$$

$$\Rightarrow A^T = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad b \in \mathbb{R}.$$

Any matrix $A \in \mathbb{R}^{n \times n}$ can be decomposed as

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{Symmetric part}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{Skew-symmetric part.}}$$

THM (LDL factorization) A regular matrix $A \in \mathbb{R}^{n \times n}$ and symmetric can be factorized as

$$A = LDL^T,$$

where D is a diagonal matrix and $\underbrace{L \text{ is lower triangular}}_{\text{with } l_{ii} = 1}$

Proof. Recall LDV factorization.

$$A = LDV$$

If A is symmetric

$$A = LDV = \underline{A^T = V^T D^T L^T}$$

$$\Rightarrow \quad \boxed{A = V^T D L^T}$$

V^T is lower triangular with $v_{ii} = 1$ and L^T is upper triangular with $l_{ii} = 1$.

Uniqueness of LDV factorization

$$\Rightarrow \quad L = V^T \text{ and } V = L^T$$

$$\Rightarrow \quad A = LDL^T \quad \blacksquare$$

§ 1.6.2 Triangular matrices.

$$A = \begin{pmatrix} q_1 & r_1 & & & \\ p_1 & q_2 & r_2 & & \\ & p_2 & q_3 & r_3 & \\ & & \ddots & \ddots & \ddots \\ & & p_{n-2} & q_{n-1} & r_{n-1} \\ & & & p_{n-1} & q_n \end{pmatrix}$$

Triangular matrices appear, for instance, in the numerical solution of partial/differential equations.

Let's explore a LU factorization

$$A = LU = \begin{pmatrix} 1 & & & & \\ l_1 & 1 & & & \\ & l_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & l_m & 1 \end{pmatrix} \begin{pmatrix} d_1 & u_1 & & & \\ & d_2 & u_2 & & \\ & & \ddots & \ddots & \\ & & & d_{m-1} & u_{m-1} \\ & & & & d_m \end{pmatrix}$$

Then, $d_1 = q_1; \quad u_1 = r_1; \quad l_1 d_1 = p_1$

$l_1 u_1 + d_2 = q_2; \quad u_2 = r_2; \quad l_2 d_2 = p_2$

$\vdots$

$\boxed{l_{j-1} u_{j-1} + d_j = q_j}; \quad \boxed{u_j = r_j}; \quad \boxed{l_j d_j = p_j}$

$l_{m-1} u_{m-1} + d_m = q_m.$

Computational Cost. Suppose we know $u_{j-1}, l_{j-1}, d_{j-1}$

$\rightarrow \quad u_j = r_j \;, \quad i = 1, \dots, m$

$\rightarrow \quad d_j = q_j - l_{j-1} \cdot u_{j-1}, \quad i = 1, \dots, m$

$\rightarrow \quad l_j = p_j / d_j \;, \quad i = 1, \dots, m$

operations $= 3m$.