Solutions QUIZ #2.

Problem 1.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{pmatrix} \xrightarrow[-bR_1+R_3]{-aR_1+R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & b & a-b \end{pmatrix}$$

If $a \neq 0$, we proceed as follows:

$$\xrightarrow[-\frac{b}{a}R_2+R_3]{} \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{pmatrix}$$

Then, LU decomposition exits for $a \in \mathbb{R} \setminus \{0\}$ ($a \in \mathbb{R}$ s.t $a \neq 0$) and $b \in \mathbb{R}$. Moreover,

$$U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{pmatrix} \text{ and } L = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & b/a & 1 \end{pmatrix}$$

Problem 2.

Let $A \in \mathbb{R}^{2\times 2}$ be given by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\Longrightarrow A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Then,

$$A \cdot A^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix}$$

$$A^T \cdot A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix}$$

$$A \cdot A^T = A^T A \iff \begin{cases} a^2+b^2 = a^2+c^2 \\ ac+bd = ab+d \end{cases}$$

$$a^2+b^2 = a^2+c^2 \Rightarrow b^2 = c^2$$
$$\Rightarrow b = \pm c.$$

If $b = c \Rightarrow ac + cd = ac + cd$ ✓

If $b = -c \Rightarrow ac - cd = -ac + cd$

$$\Rightarrow 2(ac - cd) = 0.$$
$$\Rightarrow a = d \quad \text{if } c \neq 0.$$

Then, $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ or $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$