Solutions Quiz # 3.

Problem 1.

(a) $U = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ pivot columns of A are 1 and 2. Then,

$$\operatorname{rgn} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Clearly these two vectors are l.i. $\Rightarrow$ basis of rgn A.

(b) To determine Ker A we identify basic variables $\rightarrow x_1, x_2$ and free variables $\rightarrow x_3, x_4$. We then proceed as follows:

$$A\underset{\sim}{x} = 0 \Longleftrightarrow U\underset{\sim}{x} = 0$$

$$U\underset{\sim}{x} = 0 \Longleftrightarrow \begin{aligned} x_1 - x_2 &= -2x_3 - x_4 \\ 3x_2 &= 5x_3 + 2x_4 \end{aligned}$$

$$\text{Then, } \operatorname{Ker}(A) = \{ x \in \mathbb{R}^4 : A\underset{\sim}{x} = 0 \} = \{ x \in \mathbb{R}^4 : U\underset{\sim}{x} = 0 \}$$

$$= \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : \begin{array}{l} x_2 = 5/3 x_3 + 2/3 x_4 \\ x_1 = -1/3 x_3 - 1/3 x_4 \end{array} \right\}$$

$$= \operatorname{Span}\left\{ \begin{bmatrix} -1/3 \\ 5/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 2/3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Clearly these two vectors are l.i $\Rightarrow$ basis of Ker (A).

(c) $\dim \operatorname{Ker} A = 2$

$\dim \operatorname{rgn} A = 2.$ (# of pivots of A)

Problem 2.

(a) $W = \{ f \in C[0,1] \quad \int_0^1 f\,dx = 0 \}$

(i) $0 \in W$ because $0 \in C[0,1]$ and

$$\int_0^1 0\,dx = 0$$

(ii) Given $\alpha \in \mathbb{R}$ and $f, g \in W$

$$\int_0^1 (\alpha f + g)\,dx = \alpha \int_0^1 f\,dx + \int_0^1 g\,dx = \alpha \cdot 0 + 0 = 0$$

(b) $P_2 = \operatorname{Span}\{1, x, x^2\} \Rightarrow \dim P_2 = 3$. Since we have 2 polynomials, they can not be a basis of P_2.

(c) Since $P_1 = \operatorname{Span}\{1, x\}$ and we have 2 polynomials it suffices to check that they are l.i. In fact

$c_1(2x+1) + c_2(-x-2) = 0 \quad \forall x$

$$\Rightarrow \begin{cases} c_1 - 2c_2 = 0 \\ 2c_1 - c_2 = 0 \end{cases} \Rightarrow c_1 = c_2 = 0 \Rightarrow \{2x+1, -x-2\} \text{ basis of } P_1.$$