Math 401 Section 0401: Quiz 4

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Complete problems 1–2. Each of these problems is worth 5 points. Explain your steps carefully. If you use a *well known* theorem, make clear which theorem you are using and justify its use.

Problem 1: (5 pts). Let $\omega(x) > 0$ for $a \leq x \leq b$ be a weight function. Prove that

$$\|f\|_{1,\omega} := \int_a^b |f(x)|\omega(x)dx,$$

defines a norm on $C^0[a,b]$, called the *weighted* L^1 norm.

Solution:

(a) Positivity. Given $f \in C^0[a,b]$:

$\|f\|_{1,\omega} = \int_a^b |f(x)|\,\omega(x)\,dx \geq 0$ because $\omega(x) > 0$ $\forall x \in [a,b]$

However, if $f = 0 \Rightarrow \|f\|_{1,\omega} = 0$.

If $\|f\|_{1,\omega} = \int_a^b |f(x)|\,\omega(x)\,dx = 0 \Rightarrow f \equiv 0$ $\forall x \in [a,b]$

Since $f \in C^0[a,b]$ and $\omega(x) > 0$ $\forall x \in [a,b]$.

(b) Homogenity: Given $f \in C^0[a,b]$ and $c \in \mathbb{R}$

$\|cf\|_{1,\omega} = \int_a^b |cf(x)|\,\omega(x)\,dx = |c| \int_a^b |f(x)|\,\omega(x)\,dx$

$= |c|\,\|f\|_{1,\omega}$

(d) Triangle ineq: Given $f, g \in C^0[a,b]$

$\|f+g\|_{1,\omega} = \int_a^b |(f+g)(x)|\,\omega(x)\,dx \leq \int_a^b (|f(x)| + |g(x)|)\,\omega(x)$

$= \|f\|_{1,\omega} + \|g\|_{1,\omega}$ $\boxtimes$

Problem 2: (5 pts). Use the L^2 inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$

to answer the following:

1. Find the *angle* between the function 1 and x. Are they orthogonal?
2. Find all quadratic polynomials $p(x) = a + bx + cx^2$ that are orthogonal to both functions 1 and x.

Solution.

1. $\cos\theta = \dfrac{\langle 1, x\rangle}{\|1\|\,\|x\|}$, where

$$\langle 1, x\rangle = \int_{-1}^{1} 1\cdot x\,dx = 0, \text{ and}$$

$$\|1\|^2 = \int_{-1}^{1} 1\,dx = 2, \quad \|x\|^2 = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

$$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \pi/2$$

$\Rightarrow$ vectors are orthogonal!

2. We compute

$$0 = \langle 1, p\rangle = \int_{-1}^{1} (a + bx + cx^2)dx = 2a + \frac{2}{3}c$$

$$\Rightarrow \quad 2a + \frac{2}{3}c = 0 \qquad (1)$$

$$0 = \langle x, p\rangle = \int_{-1}^{1} (ax + bx^2 + cx^3)dx = \frac{2}{3}b$$

$$\Rightarrow \frac{2}{3}b = 0 \Rightarrow b = 0. \quad (2)$$

From (1) and (2), we have

$$b = 0 \text{ and } a = -\frac{1}{3}c$$

Then $\quad p(x) = c\left(x^2 - \frac{1}{3}\right)$ ☒