Solutions HW#1.

1)

$$\begin{pmatrix} 2 & -1 & 2 \\ -1 & -1 & 3 \\ 3 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 2 \\ 0 & -3/2 & 4 \\ 0 & 3/2 & -5 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 2 \\ 0 & -3/2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2u - v + 2w = 2 \\ -3/2\, v + 4w = 2 \\ -w = 0. \end{cases}$$

$\Rightarrow w = 0.$

$\Rightarrow -3/2\, v = 2 - 4w = 2 \Rightarrow v = -4/3.$

$\Rightarrow 2u = 2 + v - 2w \Rightarrow u = 1/3$ ▨

2) $$\begin{cases} a + 2b - c = 3 \\ a - 2 - c = 1 \\ 1 + 2b + c = 2 \end{cases} \Rightarrow \begin{cases} a + 2b - c = 3 \\ a - c = 3 \\ 2b + c = 1. \end{cases}$$

Solving the system we get $\begin{cases} a = 4 \\ b = 0 \\ c = 1. \end{cases}$ ▨

3) Let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then

$$AD = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} ax & by \\ az & bw \end{pmatrix},$$

and

$$DA = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax & ay \\ bz & bw \end{pmatrix}$$

$\Rightarrow$ $AD = DA$ if and only if

$$AD - DA = \begin{pmatrix} 0 & (a-b)y \\ (a-b)z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $a \neq b$, equation above implies.

$$y = z = 0.$$

(b) If $a = b$, we have

$$AD - DA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \forall A \in \mathbb{R}^{2\times 2}$$

$$\Rightarrow \quad AD = DA$$

Alternatively, $D = a \cdot I_2$, then

$$\begin{aligned} A \cdot D = A \cdot a \cdot I_2 &= a \cdot A \cdot I_2 \\ &= a \cdot I_2 \cdot A \\ &= D \cdot A. \end{aligned}$$

(c) $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Then,

$$D \cdot A = \begin{pmatrix} a_{11} \cdot a & a_{12} \cdot a & a_{13} \cdot a \\ a_{21}\, b & a_{22}\, b & a_{23}\, b \\ a_{31}\, c & a_{32}\, c & a_{33}\, c \end{pmatrix}$$

and

$$A \cdot D = \begin{pmatrix} a \cdot a_{11} & b \cdot a_{12} & c \cdot a_{13} \\ a \cdot a_{21} & b \cdot a_{22} & c \cdot a_{23} \\ a \cdot a_{31} & b \cdot a_{32} & c \cdot a_{33} \end{pmatrix}$$

$$\Rightarrow AD - DA = 0 \underset{a \neq b \neq c \neq 0}{\Longleftrightarrow} a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0.$$

d) If $a \neq b = c$, the computation above shows that A should have the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \quad \boxtimes$$

4) (a) $AB = BA$, then

$$\begin{aligned} (A+B)^2 &= (A+B)(A+B) \\ &= A(A+B) + B(A+B) \\ &= A^2 + \underbrace{AB + BA}_{2AB} + B^2 \end{aligned}$$

(b) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$; $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\boxtimes$

5) (a) i) $tr(A) = 4$

ii) $tr(A) = 0.$

(b) $$tr(A+B) = \sum_{i=1}^{n} a_{ii} + b_{ii}$$

$$= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$$

$$= tr(A) + tr(B).$$

(c) $$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\Rightarrow (AB)_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki}$$

Then, $$tr(AB) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}$$

On the other hand, $(BA)_{ii} = \sum_{k=1}^{n} b_{ik} a_{ki}$;

and $$tr(BA) = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki}$$

$$\underset{\substack{i \to k \\ k \to i}}{=} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}$$

Consequently,

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

(d) $\mathrm{tr}(C) = \mathrm{tr}(AB - BA)$

$$\underset{\uparrow \atop (b)}{=} \mathrm{tr}(AB) - \mathrm{tr}(BA)$$

$$\underset{\uparrow \atop (c)}{=} 0.$$

(e) Same proof! ▨